General Relativity

Spacetime, Geometry and Gravitation

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Preface

These comprehensive notes provide a rigorous, self-contained, and pedagogically motivated exposition of the Special and General Theories of Relativity. Starting from the foundational spacetime concepts and symmetry principles of Special Relativity, the text develops differential geometry, tensor analysis, spacetime curvature, Einstein's field equations, geodesic congruences, and the Raychaudhuri equation. Detailed physical motivations, step-by-step derivations, conceptual insights, and illustrative diagrams accompany each topic.

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Disclaimer: These are pedagogical study notes. While I have strived for accuracy, any errors in the calculations, typos, or physical interpretations remain entirely my own. I am actively revising and correcting the notes; the latest version can always be found on the project page at lunar-photon.github.io.

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Chapter 1

Foundations of Spacetime and Special Relativity

Physical Insight & Motivation: Course References and Suggested Literature

These notes synthesize the geometric and field-theoretic foundations of General Relativity, drawing upon key treatises in the literature:

1. **L. D. Landau and E. M. Lifshitz**, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2, Butterworth-Heinemann). Exceptional for its elegant variational principles, geometric symmetry arguments, and relativistic electrodynamics.
2. **T. Padmanabhan**, *Gravitation: Foundations and Frontiers* (Cambridge University Press). Renowned for deep physical insights, thermodynamic connections, and rigorous foundational perspectives.
3. **S. M. Carroll**, *Spacetime and Geometry: An Introduction to General Relativity* (Cambridge University Press). A modern, pedagogically lucid introduction to differential geometry, curvature, and gravitational dynamics.
4. **R. M. Wald**, *General Relativity* (University of Chicago Press). The definitive reference for rigorous mathematical formulations, global spacetime structure, and advanced causal analysis.

1.1 Events, Spacetime, and Reference Frames

1.1.1 The Concept of an Event

The fundamental primitive in the description of relativistic physics is an **event**.

Definition: Event

An **event** is an idealized physical occurrence that happens at a definite, unique point in space and at a precise instant in time. Mathematically and geometrically, an event is an abstract point p residing on a 4-dimensional differentiable manifold $\mathcal{M}$, termed **spacetime**.

To describe events quantitatively, an observer assigns coordinate labels to each point $p \in \mathcal{M}$:

$$p \xmapsto{\text{chart}} x^a(p) = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \vec{x}), \tag{1.1}$$

where index $a \in \{0, 1, 2, 3\}$ labels the four spacetime dimensions, with $x^0 = ct$ denoting the temporal coordinate and x^μ ($\mu \in \{1, 2, 3\}$) denoting the spatial coordinates.

Physical Insight & Motivation: Geometric Points vs. Coordinate Labels

It is critical to distinguish between the intrinsic **geometric entity** (the event $p \in \mathcal{M}$) and its **coordinate representation** $x^a(p)$:

- The **event** p possesses an invariant, observer-independent existence in nature (e.g., the emission of a photon, the collision of two particles).
- The **coordinate assignment** $x^a(p)$ is merely an extrinsic label, dependent entirely upon the observer's chosen reference frame and coordinate system.

1.1.2 Spacetime and Reference Frames

Definition: Spacetime

Spacetime is the collection (continuum) of all possible events, mathematically modeled as a 4-dimensional smooth manifold $\mathcal{M}$ equipped with a pseudo-Riemannian metric tensor g_{ab} of Lorentzian signature.

A **reference frame** corresponds physically to a realization of an observer equipped with a measuring grid of spatial measuring rods and synchronized clocks.

Regarding reference frames, nature presents several profound foundational principles:

(i) **Absence of Global Inertial Frames in Nature:** Strictly speaking, ideal global inertial frames do not exist in the physical universe. Gravity is universally attractive, unshieldable, and omnipresent. Because mass-energy curves spacetime, tidal gravitational forces cannot be completely eliminated over extended regions.

(ii) **Democracy of Reference Frames:** Nature does not favor or assign a privileged ontological status to any particular frame of reference. The laws of nature must hold universally regardless of the coordinate systems or states of motion employed by observers.

(iii) **Mathematical Simplicity of Inertial Frames in Flat Spacetime:** In Special Relativity, we assign a special operational status to **inertial reference frames** (non-accelerating, non-rotating frames) because the mathematical formulation of physics is maximally simple:

- No fictitious forces (e.g., Coriolis, centrifugal) arise.
- The spacetime metric takes the constant Minkowski form:

$$\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1). \tag{1.2}$$

- The connection coefficients (Christoffel symbols) vanish identically: $\Gamma^a{}_{bc} = 0$.

1.2 A Conceptual Bridge to General Relativity

Foundational Question: Does Local Reduction to SR Solve the Einstein Field Equations?

In General Relativity, the Equivalence Principle asserts that *locally, curved spacetime reduces to Special Relativity*. Does this imply that the Einstein Field Equations (EFE) are locally "solved" merely by adopting Local Inertial Coordinates (LIC)?
Answer: Even in Local Inertial Coordinates (LIC), the Riemann tensor and Ricci scalar generally remain nonzero. The Equivalence Principle only guarantees that, at a particular point, one can choose coordinates such that the metric takes the form of the Minkowski metric,

$$g_{\mu\nu}(p) = \eta_{\mu\nu},$$

and its first derivatives vanish,

$$\partial_\alpha g_{\mu\nu}(p) = 0.$$

However, the second derivatives of the metric generally do not vanish. These are related to the Riemann curvature tensor, so spacetime can still be intrinsically curved even in LIC. The locally inertial coordinates are therefore only a local construction. In a sufficiently small neighborhood of the chosen point, the metric differs from the Minkowski metric only by small corrections. In Riemann normal coordinates, for example,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}(p)x^\alpha x^\beta + \cdots .$$

Thus, adopting LIC does *not* solve the Einstein Field Equations. It only makes the metric locally Minkowskian at a point and eliminates the first derivatives of the metric there. The EFE still determine the spacetime curvature and, consequently, how the metric varies from point to point in response to the matter and energy distribution:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}.$$

Therefore, LIC make spacetime *locally* resemble Special Relativity, but they do not make the spacetime globally flat or eliminate its curvature. The physics described by the EFE is coordinate-independent; changing coordinates changes only the components used to describe the same underlying geometry.

Physical Insight & Motivation: Resolution: Curvature vs. Coordinate Effects

Let $p \in \mathcal{M}$ be an arbitrary event in a curved spacetime. According to the **Einstein Equivalence Principle**:

1. One can always establish a **Local Inertial Frame** (Riemann Normal Coordinates) centered at p such that:

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \qquad \partial_\lambda g_{\mu\nu}(p) = 0 \implies \Gamma^\mu{}_{\nu\lambda}(p) = 0. \tag{1.3}$$

   At the point p, all first-order gravitational effects (coordinate accelerations) vanish, and the laws of mechanics locally match those of Special Relativity.

2. **Crucial Distinction:** The second derivatives of the metric $\partial_\alpha\partial_\beta g_{\mu\nu}(p)$ **cannot** be eliminated by any coordinate transformation. These irreducible second derivatives constitute the **Riemann Curvature Tensor** $R^\rho{}_{\sigma\mu\nu}(p) \neq 0$.

3. The Einstein Field Equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.4}$$

are second-order differential equations governing the intrinsic curvature of spacetime sourced by the stress-energy tensor $T_{\mu\nu}$. While choosing LIC eliminates the Christoffel symbols at a single event, the non-zero curvature $R^{\rho}{}_{\sigma\mu\nu}$ and non-trivial stress-energy $T_{\mu\nu}$ remain fully active. Thus, LIC simplifies local kinematics, but does not trivialize or bypass the dynamical field equations of gravity.

1.3 The Postulates of Special Relativity

Special Relativity is built upon two foundational postulates:

Postulate / Principle: Postulate 1: The Principle of Relativity

All physical laws are **covariant** (form-invariant) across all inertial reference frames. No physical experiment (mechanical, electromagnetic, or otherwise) performed entirely within an inertial frame can detect its absolute uniform velocity.

- **Form Independence:** A physical law expressed as an abstract equation holds in all inertial frames:

$$\vec{F} = m\vec{a} \quad \xrightarrow{\text{primed frame}} \quad \vec{F}' = m\vec{a}'. \tag{1.5}$$

The specific coordinate expression depends on the choice of coordinates (e.g., $\vec{a} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}$), but the algebraic structural relationship between physical quantities remains invariant.

- **Demand for Tensorial Formulation:** The requirement of covariance demands that theoretical physics be formulated in a universal geometric language—namely, **tensor calculus on manifolds**.

Postulate / Principle: Postulate 2: Invariance and Upper Bound of Interaction Speed

There exists a universal, finite upper limit for the speed of propagation of interactions and signals in nature, which is identical to the speed of light in vacuum, c. The speed of light c has the same invariant value in all inertial reference frames, independent of the motion of the emitting source or the observer.

Key Concept: Paradigm Shift: Demise of Instantaneous Action-at-a-Distance

In Newtonian mechanics, interactions were assumed to act instantaneously across space (e.g., Newton's gravitational law $F = Gm_1m_2/r^2$, corresponding to Poisson's equation $\nabla^2\Phi = 4\pi G\rho$).
The second postulate forces a radical paradigm shift:

- **No instantaneous interactions:** Changes in a source cannot propagate instantaneously to a distant point. Action-at-a-distance must be replaced by **local field theories** where disturbances propagate at finite speeds $\leq c$.
- **Limitation on Observation:** An observer at event P cannot observe the entire

spatial hypersurface $t = \text{const}$ simultaneously. Information about the universe reaches the observer exclusively along or inside their **past light cone**.

Foundational Question: Why Can No Inertial Frame Move at the Speed of Light?

Can we establish an inertial reference frame attached to a photon ($v = c$)?

Physical Insight & Motivation: Impossibility of a Photon Rest Frame

The existence of a reference frame traveling at $v = c$ is ruled out by foundational consistency:

1. **Direct Contradiction with Postulate 2:** If an observer could travel at c, light traveling alongside them would appear at rest ($v_{\text{light}} = 0$) in their frame. This directly contradicts the postulate that light propagates at speed c in *every* inertial frame.
2. **Singularity of Lorentz Boosts:** As $v \to c$, the Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2} \to \infty$. The coordinate transformation matrix becomes singular and non-invertible.
3. **Degeneracy of Intervals:** For a photon, the proper time interval is identically zero ($d\tau = 0$). Hence, there is no physical ticking clock that can serve as the time parameter for a lightlike observer.

1.4 Consequences of the Postulates

1.4.1 Relativity of Simultaneity

A direct and profound consequence of the constancy of the speed of light is that **simultaneity is not absolute**; it depends on the observer's state of motion.

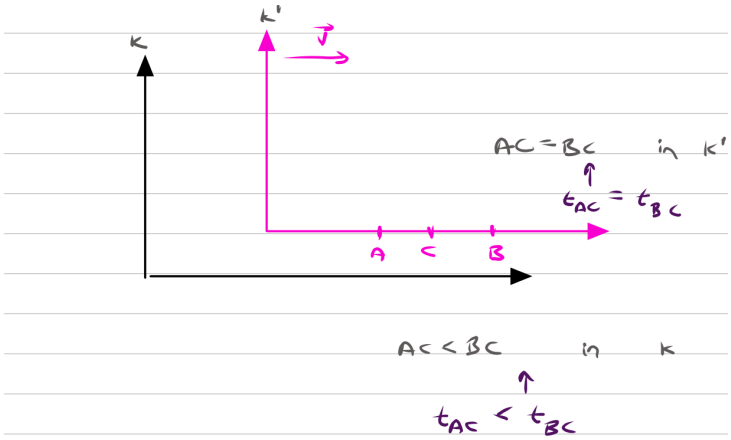


Figure 1.1: Relativity of simultaneity thought experiment. A light source at midpoint C emits signals toward detectors A and B. In frame K', the events are simultaneous ($t'_{AC} = t'_{BC}$). In frame K, relative motion causes the signals to arrive at different times ($t_{AC} < t_{BC}$).

Consider the classic thought experiment illustrated in Figure 1.1:

- Let frame K' move with velocity $\vec{v}$ relative to frame K along the positive x-axis.

- In frame K', a light pulse is emitted from a point C situated precisely halfway between two detectors A and B (so distance $AC = BC$).
- Because the speed of light is c in all directions within K', the light waves reach A and B at identical times:

$$t'_{AC} = t'_{BC} \quad \Longrightarrow \quad \text{Events are simultaneous in } K'. \tag{1.6}$$

- Now observe the same physical process from frame K: during the signal transit, detector A moves *toward* the emitted wavefront, while detector B moves *away* from it.
- Since light propagates at speed c in frame K as well, the light wavefront reaches detector A before it reaches detector B:

$$t_{AC} < t_{BC} \quad \Longrightarrow \quad \text{Events are NOT simultaneous in } K. \tag{1.7}$$

Key Concept: Absolute Simultaneity vs. Spacetime Coincidence

Two spatially separated events that occur at the same time in one reference frame are **not simultaneous** in other moving frames.
The only notion of simultaneity that is absolute and agreed upon by all observers is **local spacetime coincidence**—events that happen at the exact same spatial location and at the exact same instant ($x_1^a = x_2^a$).

1.4.2 The Spacetime Interval

Let two infinitesimally close events be separated by coordinate differentials $\mathrm{d}x^a = (c\,\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ in a Cartesian coordinate system.

Definition: Infinitesimal Spacetime Interval

The infinitesimal **spacetime interval** $\mathrm{d}s^2$ between two neighboring events $P(ct, \vec{x})$ and $P'(c(t + \mathrm{d}t), \vec{x} + \mathrm{d}\vec{x})$ is defined by:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b, \tag{1.8}$$

where η_{ab} is the Minkowski metric tensor in Cartesian coordinates:

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.9}$$

Physical Insight & Motivation: Signature Convention Note

In these notes, we adopt the **mostly plus** metric signature $(-, +, +, +)$, standard in General Relativity and modern differential geometry (Wald, Carroll). If one adopts the alternative **mostly minus** signature $(+, -, -, -)$ (common in particle physics and Landau & Lifshitz), the overall sign of $\mathrm{d}s^2$ is reversed: $\mathrm{d}s^2 = +c^2\mathrm{d}t^2 - |\mathrm{d}\vec{x}|^2$.

1.5 Universal Invariance of the Spacetime Interval

We now present the rigorous proof of the universal invariance of the interval $\mathrm{d}s^2$, following the elegant geometric derivation by **Landau and Lifshitz**.

1.5.1 Invariance for Light Propagation

Consider two events connected by the propagation of a light signal:

- In reference frame K, the distance traversed by light during time $\mathrm{d}t$ is $|\mathrm{d}\vec{x}| = c\,\mathrm{d}t$, so:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = 0. \tag{1.10}$$

- In another inertial frame K', by Postulate 2, light also travels with speed c, hence $|\mathrm{d}\vec{x}'| = c\,\mathrm{d}t'$, giving:

$$\mathrm{d}s'^2 = -c^2\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 = 0. \tag{1.11}$$

Thus, for any two events connected by a light signal:

$$\mathrm{d}s^2 = 0 \quad \Longleftrightarrow \quad \mathrm{d}s'^2 = 0. \tag{1.12}$$

1.5.2 Relation Between Infinitesimal Intervals

Since $\mathrm{d}s^2$ and $\mathrm{d}s'^2$ are quadratic differentials of the same order of smallness, $\mathrm{d}s^2$ can be expressed in general as a power series in $\mathrm{d}s'^2$:

$$\mathrm{d}s^2 = c_0 + C\,\mathrm{d}s'^2 + \mathcal{O}((\mathrm{d}s'^2)^2). \tag{1.13}$$

Because $\mathrm{d}s^2 = 0$ when $\mathrm{d}s'^2 = 0$, the constant term must vanish: $c_0 = 0$. In the infinitesimal limit, higher-order terms are neglected, yielding the linear relation:

$$\mathrm{d}s^2 = C\,\mathrm{d}s'^2, \tag{1.14}$$

where the proportionality factor C could, a priori, depend on the spacetime coordinates and the relative velocity $\vec{v}$ between the frames.

1.5.3 Constraints from Spacetime Symmetries

We constrain the factor C by appealing to the fundamental symmetries of spacetime:

1. **Homogeneity of Spacetime:** Space and time are homogeneous—all points in spacetime are physically equivalent. Spacetime physics is invariant under arbitrary spacetime translations $x^a \to x^a + a^a$. Consequently, C cannot depend on the coordinate locations x^a or x'^a.

2. **Isotropy of Space:** Space is isotropic—there are no preferred spatial directions. Spacetime physics is invariant under spatial rotations. Therefore, C cannot depend on the *direction* of the relative velocity vector $\vec{v}$, but only on its magnitude $v = |\vec{v}|$:

$$C = C(v). \tag{1.15}$$

1.5.4 The Three-Frame Group Consistency Argument

To determine the functional form of $C(v)$, consider three arbitrary inertial frames K, K_1, and K_2:

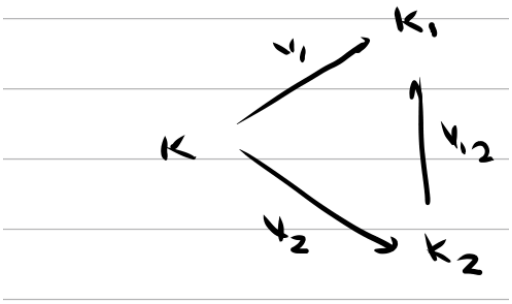


Figure 1.2: Three-frame composition configuration. Frame K observes frames K_1 and K_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$. The relative velocity between K_1 and K_2 is $\vec{v}_{12}$.

Let frame K_1 move with velocity $\vec{v}_1$ relative to K, and frame K_2 move with velocity $\vec{v}_2$ relative to K. Let $\vec{v}_{12}$ denote the relative velocity of K_2 with respect to K_1.

Applying Equation (1.14) between the respective frame pairs:

$$\mathrm{d}s^2 = C(v_1)\,\mathrm{d}s_1^2, \tag{1.16}$$

$$\mathrm{d}s^2 = C(v_2)\,\mathrm{d}s_2^2, \tag{1.17}$$

$$\mathrm{d}s_1^2 = C(v_{12})\,\mathrm{d}s_2^2. \tag{1.18}$$

Substituting Equation (1.18) into Equation (1.16):

$$\mathrm{d}s^2 = C(v_1)\left[C(v_{12})\,\mathrm{d}s_2^2\right] = C(v_1)C(v_{12})\,\mathrm{d}s_2^2. \tag{1.19}$$

Comparing this directly with Equation (1.17):

$$C(v_2)\,\mathrm{d}s_2^2 = C(v_1)C(v_{12})\,\mathrm{d}s_2^2 \quad \Longrightarrow \quad \frac{C(v_2)}{C(v_1)} = C(v_{12}). \tag{1.20}$$

Physical Insight & Motivation: The Invariance Deduction

Analyze the structural dependencies of Equation (1.20):

- The left-hand side $\frac{C(v_2)}{C(v_1)}$ depends **solely** on the scalar speeds $v_1 = |\vec{v}_1|$ and $v_2 = |\vec{v}_2|$, completely independent of the angle θ between the velocity vectors $\vec{v}_1$ and $\vec{v}_2$.
- The right-hand side $C(v_{12})$ depends on the magnitude of the relative velocity $v_{12} = |\vec{v}_{12}|$. By relativistic velocity addition, v_{12} explicitly depends on the relative angle θ between $\vec{v}_1$ and $\vec{v}_2$:

$$v_{12} = \sqrt{\frac{(\vec{v}_1 - \vec{v}_2)^2 - \frac{1}{c^2}(\vec{v}_1 \times \vec{v}_2)^2}{\left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}\right)^2}}. \tag{1.21}$$

- For Equation (1.20) to hold for all arbitrary orientations of $\vec{v}_1$ and $\vec{v}_2$, the function $C(v)$ cannot possess any non-trivial dependence on velocity!
- Therefore, $C(v)$ must be a universal constant identically equal to unity:

$$C(v) \equiv 1 \quad \Longrightarrow \quad C_1 = C_2 = C_{12} = 1. \tag{1.22}$$

Postulate / Principle: Theorem: Universal Invariance of the Spacetime Interval

For **any** two infinitesimally separated events in spacetime, the spacetime interval is strictly invariant between all inertial reference frames:

$$\mathrm{d}s^2 = \mathrm{d}s'^2. \tag{1.23}$$

1.6 Causal Classification of Intervals and Spacetime Curves

1.6.1 Classification of Spacetime Intervals

Due to the Lorentzian signature $(-,+,+,+)$, the squared spacetime interval $\mathrm{d}s^2$ can take negative, zero, or positive values. This divides spacetime intervals into three distinct invariant categories:

1. **Timelike Interval ($\mathrm{d}s^2 < 0$):**

$$c^2\mathrm{d}t^2 > |\mathrm{d}\vec{x}|^2 \iff \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} < c. \tag{1.24}$$

   The events can be causally connected by a physical signal or massive particle traveling at speed $v < c$. An inertial frame always exists in which the two events occur at the *same spatial point* ($\mathrm{d}\vec{x}' = 0$).

2. **Null / Lightlike Interval ($\mathrm{d}s^2 = 0$):**

$$c^2\mathrm{d}t^2 = |\mathrm{d}\vec{x}|^2 \iff \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c. \tag{1.25}$$

   The events can be connected only by a massless particle (such as a photon) moving at the speed of light c. The collection of all null rays emanating from an event forms the **light cone**.

3. **Spacelike Interval ($\mathrm{d}s^2 > 0$):**

$$c^2\mathrm{d}t^2 < |\mathrm{d}\vec{x}|^2 \iff \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} > c. \tag{1.26}$$

   The events cannot be causally connected by any physical interaction. An inertial frame always exists in which the two events occur *simultaneously* ($\mathrm{d}t' = 0$).

1.6.2 Are Coordinates x^a Vectors?

Foundational Question: Is x^a a 4-Vector or a Set of Scalar Fields?

Is the coordinate quadruple $x^a = (ct, x, y, z)$ a true four-vector in spacetime?

Physical Insight & Motivation: Geometric Clarification: Coordinates vs. Tangent Vectors

- In differential geometry and General Relativity, the coordinates x^a are **not components of a vector**. Rather, each x^a is a **scalar coordinate function** (chart mapping) $x^a : \mathcal{M} \to \mathbb{R}$ assigning real numbers to events.

- In general curvilinear coordinates (e.g., spherical coordinates (ct, r, θ, ϕ)) or on a curved manifold, the collection (x^0, x^1, x^2, x^3) does not transform linearly as a vector. Manifold points cannot be added or scaled.
- What **is** a genuine vector is the **infinitesimal coordinate differential** $\mathrm{d}x^a$, or the **tangent vector** $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ along a parameterized curve $x^a(\lambda)$. Under a general coordinate transformation $x^a \to x'^a(x)$, differentials transform linearly via the Jacobian:
$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^b}\mathrm{d}x^b. \tag{1.27}$$
- Only in the very special case of flat affine Minkowski spacetime with global Cartesian coordinates can one identify a "position vector" from a designated origin to a point.

1.7 Worldlines, Proper Time, Time Dilation, and Length Contraction

1.7.1 Worldlines

Definition: Worldline

The continuous path traced out by a particle as it traverses the 4-dimensional spacetime continuum is called its **worldline**. Mathematically, a worldline is a smooth curve $\gamma : \mathbb{R} \to \mathcal{M}$ parameterized by a scalar parameter λ:
$$x^a = x^a(\lambda), \quad \lambda \in [\lambda_1, \lambda_2]. \tag{1.28}$$

For physical material particles (which have non-zero rest mass), the tangent vector is everywhere timelike:
$$\eta_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} < 0. \tag{1.29}$$

1.7.2 Proper Time

Definition: Proper Time τ

The **proper time** τ of a particle is the time measured by an ideal standard clock attached to the particle and carried along its worldline—i.e., the time recorded in the particle's **instantaneous comoving rest frame** K' (where $\mathrm{d}\vec{x}' = 0$).

In the instantaneous rest frame K', the spacetime interval is:
$$\mathrm{d}s'^2 = -c^2\mathrm{d}\tau^2 + |\mathrm{d}\vec{x}'|^2 = -c^2\mathrm{d}\tau^2. \tag{1.30}$$

By the universal invariance of the spacetime interval ($\mathrm{d}s^2 = \mathrm{d}s'^2$), we equate this to the interval in an arbitrary laboratory frame K where the particle moves with coordinate velocity $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$:
$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + |\mathrm{d}\vec{x}|^2 = -c^2\mathrm{d}t^2\left(1 - \frac{|\mathrm{d}\vec{x}/\mathrm{d}t|^2}{c^2}\right) = -c^2\mathrm{d}t^2\left(1 - \frac{v^2}{c^2}\right). \tag{1.31}$$

Equating $-c^2 \mathrm{d}\tau^2 = -c^2 \mathrm{d}t^2 \left(1 - \frac{v^2}{c^2}\right)$ immediately yields:

$$\mathrm{d}\tau = \mathrm{d}t \sqrt{1 - \frac{v^2}{c^2}} = \frac{\mathrm{d}t}{\gamma(v)}, \tag{1.32}$$

where $\gamma(v) \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$ is the **Lorentz factor**.

1.7.3 Time Dilation

Since $v < c$ for all massive bodies, the Lorentz factor satisfies $\gamma(v) \geq 1$, which implies:

$$\mathrm{d}\tau \leq \mathrm{d}t. \tag{1.33}$$

Key Concept: Time Dilation: Moving Clocks Run Slow

The elapsed proper time $\mathrm{d}\tau$ recorded by a moving clock is always **strictly less** than the coordinate time $\mathrm{d}t$ recorded by synchronized clocks in the stationary frame:

$$\Delta t = \gamma \, \Delta\tau \geq \Delta\tau. \tag{1.34}$$

A moving clock runs slower relative to clocks at rest.

For arbitrary motion along a curved worldline with time-varying velocity $\vec{v}(t)$, the total accumulated proper time between events $A(t_1)$ and $B(t_2)$ is obtained by integrating along the worldline:

$$\tau_{AB} = \int_{t_1}^{t_2} \sqrt{1 - \frac{v(t)^2}{c^2}} \, \mathrm{d}t = \int_A^B \frac{\mathrm{d}t}{\gamma(v(t))}. \tag{1.35}$$

1.7.4 Length Contraction via Lorentz Differentials and Invariant Intervals

Having established time dilation, we now derive the relativistic contraction of lengths. By analyzing the differential relations between inertial coordinates and exploiting the universal invariance of the spacetime interval, we can uncover a deep structural duality between time dilation and length contraction.

Consider two inertial frames: a frame K with coordinates (t, x) and a frame K' with coordinates (τ, x'), where K' moves with uniform velocity v relative to K along the x-axis. The coordinates are related by the standard Lorentz boost:

$$\tau = \gamma \left(t - \frac{v}{c^2} x\right), \tag{1.36}$$

$$x' = \gamma \left(x - vt\right), \tag{1.37}$$

where $\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$ is the Lorentz factor. Evaluating the relevant coordinate differentials immediately gives:

$$\mathrm{d}\tau = \gamma \left(\mathrm{d}t - \frac{v}{c^2} \mathrm{d}x\right), \tag{1.38}$$

$$\mathrm{d}x' = \gamma \left(\mathrm{d}x - v \, \mathrm{d}t\right). \tag{1.39}$$

A Unified View: Time Dilation vs. Length Contraction

Recall that the infinitesimal spacetime interval (for motion along the x-axis, $\mathrm{d}y = \mathrm{d}z = 0$) is invariant between K and K':

$$-c^2 \mathrm{d}t^2 + \mathrm{d}x^2 = -c^2 \mathrm{d}\tau^2 + \mathrm{d}x'^2. \tag{1.40}$$

1. Time Dilation Revisited: For time dilation, we consider an object (or clock) at rest in the moving frame K'. Because the clock remains at a fixed position in its comoving rest frame, its spatial differential in that frame vanishes identically:

$$\mathrm{d}x' = \gamma\left(\mathrm{d}x - v\,\mathrm{d}t\right), \tag{1.41}$$

$$0 = \gamma\left(\mathrm{d}x - v\,\mathrm{d}t\right) \implies \mathrm{d}x = v\,\mathrm{d}t. \tag{1.42}$$

Substituting $\mathrm{d}x' = 0$ and $\mathrm{d}x = v\,\mathrm{d}t$ into the invariant spacetime interval (1.40):

$$\begin{aligned} -c^2\mathrm{d}t^2 + \mathrm{d}x^2 &= -c^2\mathrm{d}\tau^2 + \mathrm{d}x'^2 \\ -c^2\mathrm{d}t^2 + v^2\mathrm{d}t^2 &= -c^2\mathrm{d}\tau^2 \\ -c^2\left(1 - \frac{v^2}{c^2}\right)\mathrm{d}t^2 &= -c^2\mathrm{d}\tau^2 \\ \mathrm{d}\tau &= \sqrt{1 - \frac{v^2}{c^2}}\,\mathrm{d}t. \end{aligned} \tag{1.43}$$

2. Derivation of Length Contraction: Similarly, for length contraction, consider a measuring rod at rest in frame K, so that its rest length (proper length) is $\mathrm{d}x \equiv L_0$. An observer in frame K' is in relative motion with velocity v and measures the length of this rod as $\mathrm{d}x' \equiv L$.

To measure the spatial length of an object in motion, the observer must record the positions of both ends **simultaneously in the observer's own reference frame**. This operational requirement means that the time differential in the observer's frame must vanish:

$$\mathrm{d}\tau = \gamma\left(\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x\right), \tag{1.44}$$

$$0 = \gamma\left(\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x\right) \implies \mathrm{d}t = \frac{v}{c^2}\mathrm{d}x. \tag{1.45}$$

We now evaluate the invariant spacetime interval (1.40) with $\mathrm{d}\tau = 0$:

$$\begin{aligned} \mathrm{d}x'^2 &= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 \\ &= -c^2\left(\frac{v}{c^2}\mathrm{d}x\right)^2 + \mathrm{d}x^2 \\ &= -\frac{v^2}{c^2}\mathrm{d}x^2 + \mathrm{d}x^2 \\ &= \left(1 - \frac{v^2}{c^2}\right)\mathrm{d}x^2. \end{aligned} \tag{1.46}$$

Taking the positive square root:

$$\mathrm{d}x' = \sqrt{1 - \frac{v^2}{c^2}}\,\mathrm{d}x = \frac{\mathrm{d}x}{\gamma}. \tag{1.47}$$

Since $\mathrm{d}x = L_0$ is the proper length measured in the rod's rest frame and $\mathrm{d}x' = L$ is the length measured by the moving observer:

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{v^2}{c^2}} \leq L_0. \tag{1.48}$$

Physical Insight & Motivation: Frame Duality: Where Are the Boundary Conditions Imposed?

Here, we must emphasize the fundamental physical reason why this derivation works and why the two derivations are exact structural duals:

- **In the time dilation part:** We set $\mathrm{d}x' = 0$ in the **object's rest frame** to evaluate the intrinsic proper time interval $\mathrm{d}\tau$ ticked off by a clock carried by the object. A clock is localized at a spatial point; isolating its ticking requires sampling that single spatial location ($\mathrm{d}x' = 0$) in its rest frame.
- **In the length contraction part:** We set $\mathrm{d}\tau = 0$ in the **observer's frame** to evaluate the spatial length $\mathrm{d}x'$ as measured simultaneously by the observer. Length is an extended spatial separation; measuring it requires determining both endpoints at the exact same instant ($\mathrm{d}\tau = 0$) according to the observer's synchronized clocks.
- **Calculations in Different Frames of Reference:** Both calculations are happening in different frames of reference! Time dilation imposes the physical boundary condition in the object's rest frame ($\mathrm{d}x' = 0$), whereas length contraction imposes the measurement condition in the observer's frame ($\mathrm{d}\tau = 0$).
- **Origin in the Relativity of Simultaneity:** The condition $\mathrm{d}\tau = 0$ in the observer's frame implies that in the object's frame, the two measurements occur at different times:
$$\mathrm{d}t = \frac{v}{c^2}\mathrm{d}x \neq 0. \tag{1.49}$$
The observer samples the front and back of the rod at different moments of rest-frame time! It is precisely this negative temporal interval $-c^2\mathrm{d}t^2 = -\frac{v^2}{c^2}\mathrm{d}x^2$ in the invariant interval $-c^2\mathrm{d}t^2 + \mathrm{d}x^2$ that reduces the spatial separation, giving $\mathrm{d}x'^2 = (1 - v^2/c^2)\mathrm{d}x^2$.

Key Concept: Length Contraction: Moving Rods Are Shortened

The length L of a moving body measured along its direction of motion is always **strictly contracted** relative to its proper length L_0:

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{v^2}{c^2}} \leq L_0. \tag{1.50}$$

Spatial dimensions oriented perpendicular to the direction of relative motion are completely unaffected:

$$\mathrm{d}y' = \mathrm{d}y, \qquad \mathrm{d}z' = \mathrm{d}z. \tag{1.51}$$

Physical Insight & Motivation: Summary of Conventions Established

Throughout this text, we maintain the following standard conventions:

- **Metric Signature:** $(-,+,+,+)$ (mostly plus).
- **Spacetime Indices:** Latin lowercase letters $a, b, c, \ldots$ or Greek lowercase letters $\mu, \nu, \rho, \cdots \in \{0, 1, 2, 3\}$.
- **Spatial Indices:** Latin indices $i, j, k \in \{1, 2, 3\}$.

- **Einstein Summation Convention:** Repeated upper and lower indices are automatically summed over their range: $A_a B^a \equiv \sum_{a=0}^{3} A_a B^a$.
- **Speed of Light:** Explicitly preserved as c where foundational clarity is needed, or set to $c = 1$ in natural units.

Chapter 2

Relativistic Kinematics, Accelerated Observers, and Causal Structure

2.1 Accelerated Observers and the Radar Method

Foundational Question: Can Accelerated Frames Be Obtained by Setting $v \to v(t)$?

Can we derive coordinate transformations to an accelerated reference frame simply by replacing the constant velocity v in the Lorentz transformation formulas with a time-dependent velocity $v(t)$?

Physical Insight & Motivation: Failure of Naive Velocity Substitution

The naive substitution $v \to v(t)$ fails fundamentally for several reasons:

1. **Curvature of Coordinate Lines:** An accelerated frame is inherently non-inertial. The spatial hypersurfaces of constant time and the coordinate grid lines become non-orthogonal and curved.

2. **Metric Modification:** The metric in an accelerated frame no longer takes the Minkowski form $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. For instance, for a uniformly accelerated observer with proper acceleration g, the metric in Rindler coordinates is:

$$\mathrm{d}s^2 = -\left(1 + \frac{gx}{c^2}\right)^2 c^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{2.1}$$

3. **The Invariance Principle:** The constancy of the speed of light in all directions is a much stronger and more restrictive physical constraint than mere Galilean kinematics. To set up valid coordinates for a general moving or accelerating observer, one must employ operational protocols based on light signaling—specifically, the **Einstein-Poincaré radar method**.

2.1.1 The Radar Coordinate Protocol

Consider an observer moving along an arbitrary timelike worldline γ parameterized by their own proper time τ, as shown in Figure 2.1.

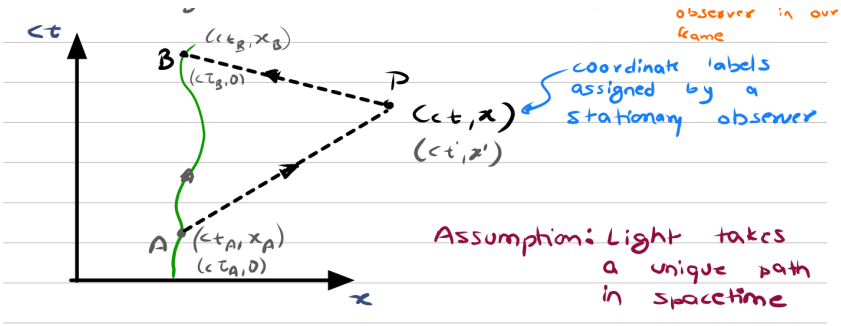


Figure 2.1: Radar coordinate assignment for an arbitrary observer. A light signal emitted at proper time τ_A bounces off event P and returns to the observer at proper time τ_B. The observer assigns coordinates (ct', x') based on these emission and reception times.

Let the 4-velocity of the observer be:

- $U'^a = (c, 0, 0, 0)$ in the observer's instantaneous comoving frame.
- $U^a = \gamma(v)(c, \vec{v})$ in the background laboratory inertial frame K.

To assign spacetime coordinates (ct', x') to an arbitrary event P in spacetime, the moving observer executes the following radar protocol:

1. The observer emits a radar light signal at event A along their worldline at proper time τ_A, with laboratory coordinates (ct_A, x_A) and comoving coordinates $(c\tau_A, 0)$.
2. The light ray propagates along a null trajectory to event P, where it reflects.
3. The reflected light ray returns along a null trajectory to the observer, arriving at event B at proper time τ_B, with laboratory coordinates (ct_B, x_B) and comoving coordinates $(c\tau_B, 0)$.

Definition: Radar Coordinates

Assuming light takes unique null paths in spacetime with invariant speed c, the moving observer defines the time and spatial distance of event P via the half-sum and half-difference of the proper times:

$$t' \equiv \frac{1}{2}(\tau_A + \tau_B), \qquad x' \equiv \frac{c}{2}(\tau_B - \tau_A). \tag{2.2}$$

Equivalently, the emission and reception proper times are related to (t', x') by:

$$\tau_A = t' - \frac{x'}{c}, \qquad \tau_B = t' + \frac{x'}{c}. \tag{2.3}$$

2.1.2 Connecting Laboratory and Moving Frame Coordinates

In the laboratory inertial frame K, light propagates with speed c. Therefore, the null intervals from $A \to P$ and $P \to B$ satisfy:

$$c(t - t_A) = x - x_A \quad \Longrightarrow \quad x - ct = x_A - ct_A, \tag{2.4}$$

$$c(t_B - t) = x_B - x \quad \Longrightarrow \quad x + ct = x_B + ct_B. \tag{2.5}$$

The trajectory of the observer's worldline in the laboratory frame is parameterized by proper time τ:

$$x^0(\tau) = ct(\tau) \equiv cf_0(\tau), \qquad x^1(\tau) = x(\tau) \equiv f_1(\tau). \tag{2.6}$$

Evaluating the laboratory coordinates at the emission event A ($\tau = \tau_A$) and reception event B ($\tau = \tau_B$), and substituting Equation (2.3):

$$x - ct = f_1(\tau_A) - cf_0(\tau_A) = f_1\left(t' - \frac{x'}{c}\right) - cf_0\left(t' - \frac{x'}{c}\right), \tag{2.7}$$

$$x + ct = f_1(\tau_B) + cf_0(\tau_B) = f_1\left(t' + \frac{x'}{c}\right) + cf_0\left(t' + \frac{x'}{c}\right). \tag{2.8}$$

Combining these relations into a single master equation:

$$x \pm ct = f_1\left(t' \pm \frac{x'}{c}\right) \pm cf_0\left(t' \pm \frac{x'}{c}\right). \tag{2.9}$$

2.2 Derivation of the Standard Lorentz Transformation

2.2.1 Uniform Inertial Motion

We now specialize Equation (2.9) to the case where the observer moves with a **constant velocity** v along the x-axis.

From the 4-velocity and time dilation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \implies x(t) = vt. \tag{2.10}$$

Since $\mathrm{d}t = \gamma \mathrm{d}\tau$, integrating from $t = 0$ ($\tau = 0$) gives:

$$t(\tau) = f_0(\tau) = \gamma\tau, \tag{2.11}$$

$$x(\tau) = f_1(\tau) = vt(\tau) = \gamma v\tau. \tag{2.12}$$

Substituting these explicit linear functions into Equation (2.9):

$$\begin{aligned} x \pm ct &= \gamma v\left(t' \pm \frac{x'}{c}\right) \pm c\gamma\left(t' \pm \frac{x'}{c}\right) \\ &= \gamma(v \pm c)\left(t' \pm \frac{x'}{c}\right) \\ &= \gamma\left(1 \pm \frac{v}{c}\right)(x' \pm ct'). \end{aligned} \tag{2.13}$$

Physical Insight & Motivation: The Bondi k-Calculus Factor

Notice the remarkable identity:

$$\gamma\left(1 \pm \frac{v}{c}\right) = \frac{1 \pm v/c}{\sqrt{1 - v^2/c^2}} = \frac{1 \pm v/c}{\sqrt{(1 - v/c)(1 + v/c)}} = \sqrt{\frac{1 \pm v/c}{1 \mp v/c}}. \tag{2.14}$$

Defining Bondi's Doppler factor $k \equiv \sqrt{\frac{1+v/c}{1-v/c}}$, the light-cone coordinate transformations take the beautifully symmetric form:

$$x + ct = k(x' + ct'), \qquad x - ct = k^{-1}(x' - ct'). \tag{2.15}$$

Adding and subtracting these two equations yields the spatial and temporal coordinate transformations:

$$x = \frac{1}{2}\left[(x + ct) + (x - ct)\right] = \frac{1}{2}\left[k(x' + ct') + k^{-1}(x' - ct')\right]$$

$$= \frac{1}{2}(k + k^{-1})x' + \frac{c}{2}(k - k^{-1})t', \tag{2.16}$$

$$ct = \frac{1}{2}\left[(x + ct) - (x - ct)\right] = \frac{1}{2}\left[k(x' + ct') - k^{-1}(x' - ct')\right]$$

$$= \frac{1}{2}(k - k^{-1})x' + \frac{c}{2}(k + k^{-1})t'. \tag{2.17}$$

Evaluating the coefficients:

$$\frac{1}{2}(k + k^{-1}) = \frac{1}{2}\left(\sqrt{\frac{1 + v/c}{1 - v/c}} + \sqrt{\frac{1 - v/c}{1 + v/c}}\right) = \frac{(1 + v/c) + (1 - v/c)}{2\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma, \tag{2.18}$$

$$\frac{1}{2}(k - k^{-1}) = \frac{1}{2}\left(\sqrt{\frac{1 + v/c}{1 - v/c}} - \sqrt{\frac{1 - v/c}{1 + v/c}}\right) = \frac{(1 + v/c) - (1 - v/c)}{2\sqrt{1 - v^2/c^2}} = \frac{v/c}{\sqrt{1 - v^2/c^2}} = \gamma\frac{v}{c}. \tag{2.19}$$

Postulate / Principle: Standard Lorentz Transformation Equations

In Cartesian coordinates, the forward and inverse Lorentz boost transformations between inertial frames K and K' moving at relative speed v along the x-axis are:

$$\begin{aligned} x' &= \gamma(x - vt), & & x = \gamma(x' + vt'), \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right), & \Longleftrightarrow \quad & t = \gamma\left(t' + \frac{vx'}{c^2}\right), \\ y' &= y, & & y = y', \\ z' &= z, & & z = z'. \end{aligned} \tag{2.20}$$

2.3 Linearity, Integrability, and Deep Physical Connections

2.3.1 Why Are Lorentz Transformations Linear?

Foundational Question: What Is the Physical Origin of Linearity?

Why must coordinate transformations between inertial frames be strictly linear?

Physical Insight & Motivation: Foundational Reasons for Linearity

1. **Spacetime Homogeneity:** If the transformation $x'^a(x)$ were non-linear, the derivatives $\frac{\partial x'^a}{\partial x^b}$ would depend on x^b. This would mean that physical laws, lengths, and clock rates would vary depending on *where* and *when* an experiment is performed, violating the translation invariance (homogeneity) of flat spacetime.

2. **Preservation of Inertial (Straight) Trajectories:** Newton's first law and relativistic free motion assert that particles with no forces acting on them move along straight worldlines ($\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} = 0$). Only affine (linear plus constant shift) transformations map straight lines to straight lines in general.

2.3.2 Integrability of the Spacetime Interval

Because the transformation between inertial frames is linear, the differential invariance $\mathrm{d}s^2 = \mathrm{d}s'^2$ can be integrated immediately between any two finite events $(t_1, \vec{x}_1)$ and $(t_2, \vec{x}_2)$:

$$-c^2(t_1 - t_2)^2 + |\vec{x}_1 - \vec{x}_2|^2 = -c^2(t_1' - t_2')^2 + |\vec{x}_1' - \vec{x}_2'|^2. \tag{2.21}$$

If non-linear coordinates are introduced (such as polar coordinates or curvilinear accelerated frames), the coordinate differentials acquire space-dependent coefficients, transforming the metric from its constant Cartesian form to a general Riemannian metric $g_{\mu\nu}(x)\,\mathrm{d}x^\mu \mathrm{d}x^\nu$.

2.3.3 Deep Questions: Hawking Radiation, Unruh Effect, and Energy Conservation

Foundational Question: Why Does Non-Linearity Produce Radiation for Accelerated Observers?

Why do inertial observers see a vacuum state, while accelerated observers perceive thermal Hawking/Unruh radiation?

Physical Insight & Motivation: Mode Linearity vs. Bogoliubov Mixing

- **Inertial Frames and Linearity:** In Quantum Field Theory (QFT), linear Lorentz transformations map positive-frequency plane waves ($e^{-i\omega t + i\vec{k}\cdot\vec{x}}$) purely into positive-frequency plane waves in the primed frame. Consequently, the vacuum state $|0\rangle$ is invariant across all inertial observers.
- **Non-Linearity and the Unruh Effect:** For an accelerated observer (Rindler frame), the coordinate relationship between Minkowski time t and proper time τ is exponential:

$$t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right), \qquad x = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right). \tag{2.22}$$

  This non-linear mapping mixes positive and negative frequency modes via non-trivial **Bogoliubov coefficients**. As a result, the Minkowski vacuum appears to the accelerated observer as a thermal bath of particles at the **Unruh temperature**:

$$T_{\text{Unruh}} = \frac{\hbar g}{2\pi c k_B}. \tag{2.23}$$

  An identical mechanism operating near the event horizon of a black hole produces **Hawking radiation**.

Foundational Question: Why Is P^0 Interpreted as Energy if Energy Is Not Conserved for Non-Inertial Observers?

If energy is frame-dependent and not conserved in non-inertial frames, why do we identify P^0 as energy?

Physical Insight & Motivation: Noether's Theorem and Spacetime Symmetries

According to **Noether's Theorem**, conservation of energy is not an intrinsic property of matter alone, but the direct consequence of a continuous spacetime symmetry: **time-translation invariance**.

- In Minkowski spacetime, the vector field $\xi = \partial_t$ is a **timelike Killing vector field** ($\nabla_{(a}\xi_{b)} = 0$). The conserved quantity along any geodesic is $E = -P_a\xi^a = P^0 c$.
- In a non-inertial frame or a time-dependent curved spacetime (such as the expanding FLRW universe), there is no global timelike Killing vector field. Energy is therefore **not conserved globally**. P^0 represents the instantaneous energy measured by an observer whose 4-velocity aligns with the temporal coordinate vector.

2.4 The Rapidity Parameter and Hyperbolic Geometry

2.4.1 Rapidity Formulation

A Lorentz boost is mathematically isomorphic to a **hyperbolic rotation** of the (ct, x) spacetime plane.

Definition: Rapidity χ

The **rapidity** $\chi \in (-\infty, +\infty)$ is defined such that:

$$\tanh\chi \equiv \beta = \frac{v}{c}. \tag{2.24}$$

Using standard hyperbolic trigonometric identities, the Lorentz factor γ and $\gamma\beta$ become:

$$\cosh\chi = \frac{1}{\sqrt{1-\tanh^2\chi}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma, \tag{2.25}$$

$$\sinh\chi = \gamma\beta = \frac{v/c}{\sqrt{1-v^2/c^2}}. \tag{2.26}$$

Note that $\cosh^2\chi - \sinh^2\chi = 1$ identically for all χ.

In terms of rapidity χ, the Lorentz boost matrix takes the elegant rotational form:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \tag{2.27}$$

and its inverse:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}. \tag{2.28}$$

2.4.2 Eigenvalues and Light-Cone Invariance

In light-cone combinations $x \pm ct$, the boost becomes diagonal:

$$x' \pm ct' = (\cosh\chi \mp \sinh\chi)(x \pm ct) = e^{\mp\chi}(x \pm ct). \tag{2.29}$$

The invariant spacetime interval is visibly preserved:

$$(x' + ct')(x' - ct') = e^{-\chi}(x + ct) \cdot e^{+\chi}(x - ct) = x^2 - c^2t^2. \tag{2.30}$$

2.4.3 The Non-Relativistic / Galilean Limit

To recover classical Newtonian/Galilean mechanics, we take the non-relativistic limit where the dimensionless parameter $\beta = \frac{v}{c} \to 0$ (or equivalently $c \to \infty$):

$$\lim_{v/c\to 0} \gamma = 1, \qquad \lim_{v/c\to 0} \frac{vx}{c^2} = 0. \tag{2.31}$$

The Lorentz transformation equations reduce smoothly to the **Galilean transformations**:

$$x' = x - vt, \qquad t' = t, \qquad y' = y, \qquad z' = z. \tag{2.32}$$

2.5 Null Coordinates and Conformal Decoupling

2.5.1 Definition and Geometry of Null Coordinates

Definition: Null / Light-Cone Coordinates

For 2-dimensional Minkowski spacetime (ct, x), the **null coordinates** (or light-cone coordinates) are defined by:

$$u \equiv ct - x, \qquad v \equiv ct + x. \tag{2.33}$$

Inversely:

$$ct = \frac{1}{2}(v + u), \qquad x = \frac{1}{2}(v - u). \tag{2.34}$$

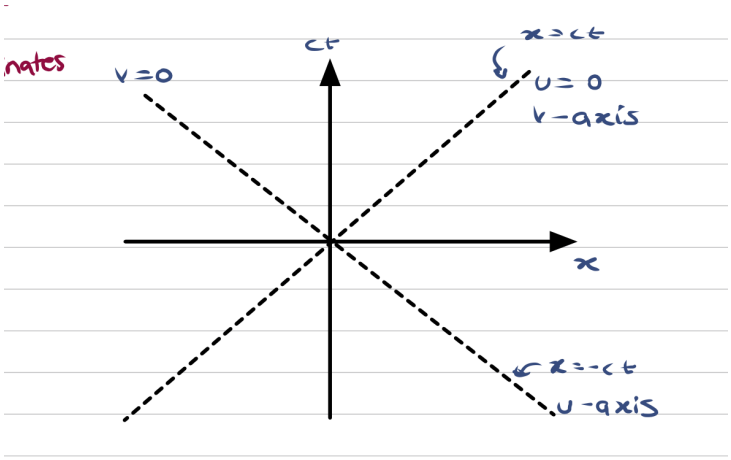


Figure 2.2: Null coordinates (u, v) in Minkowski spacetime. The line $u = 0$ corresponds to $x = ct$ (the v-axis, outgoing light rays), while $v = 0$ corresponds to $x = -ct$ (the u-axis, ingoing light rays).

In terms of null coordinates, the Minkowski metric takes the off-diagonal form:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 = -(c\,\mathrm{d}t - \mathrm{d}x)(c\,\mathrm{d}t + \mathrm{d}x) = -\mathrm{d}u\,\mathrm{d}v. \tag{2.35}$$

The diagonal components vanish identically: $g_{uu} = 0$ and $g_{vv} = 0$, reflecting the fact that the coordinate axes themselves are null rays (trajectories of light).

2.5.2 Conformal Decoupling (No Mixing of u and v)

Substituting null coordinates into the general transformation master equation (2.9):

$$v = f_1\left(\frac{v'}{c}\right) + cf_0\left(\frac{v'}{c}\right) \equiv V(v'), \tag{2.36}$$

$$u = f_1\left(\frac{u'}{c}\right) - cf_0\left(\frac{u'}{c}\right) \equiv U(u'). \tag{2.37}$$

Key Concept: Chiral Decoupling of Light-Cone Modes

Under any coordinate transformation compatible with the constancy of the speed of light:

$$u = U(u'), \qquad v = V(v'). \tag{2.38}$$

There is **strictly no mixing** between the left-moving coordinate u and the right-moving coordinate v. This chiral decoupling is the foundation of 2D conformal field theory and the causal structure of spacetime horizons.

2.6 Relativistic Velocity Addition and Aberration

2.6.1 Relativistic Velocity Addition Formula

Let a particle move with velocity $\vec{u} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ in frame K, and velocity $\vec{u}' = \frac{\mathrm{d}\vec{x}'}{\mathrm{d}t'}$ in frame K' (where K' moves with boost speed v along the x-axis of K).

Taking differentials of the Lorentz transformation equations:

$$\mathrm{d}x' = \gamma(\mathrm{d}x - v\,\mathrm{d}t), \tag{2.39}$$

$$\mathrm{d}t' = \gamma\left(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x\right), \tag{2.40}$$

$$\mathrm{d}y' = \mathrm{d}y, \tag{2.41}$$

$$\mathrm{d}z' = \mathrm{d}z. \tag{2.42}$$

Dividing the spatial differentials by the time differential $\mathrm{d}t'$:

$$u'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\gamma(\mathrm{d}x - v\,\mathrm{d}t)}{\gamma\left(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x\right)} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - v}{1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}, \tag{2.43}$$

$$u'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\gamma\left(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}, \tag{2.44}$$

$$u'_z = \frac{\mathrm{d}z'}{\mathrm{d}t'} = \frac{\mathrm{d}z}{\gamma\left(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x\right)} = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}. \tag{2.45}$$

Inverting the transformation ($v \to -v$, exchanging primed and unprimed quantities):

Postulate / Principle: Relativistic Velocity Addition Theorem

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}, \qquad u_y = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}, \qquad u_z = \frac{u'_z}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}. \tag{2.46}$$

Physical Insight & Motivation: Invariance of the Speed of Light Under Velocity Addition

If the moving object is a photon traveling along the x-axis in K' ($u'_x = c$):

$$u_x = \frac{c+v}{1+\frac{vc}{c^2}} = \frac{c(1+v/c)}{1+v/c} = c. \tag{2.47}$$

The speed of light remains identically c, confirming the absolute consistency of the kinematic structure.

2.6.2 Relativistic Aberration of Light

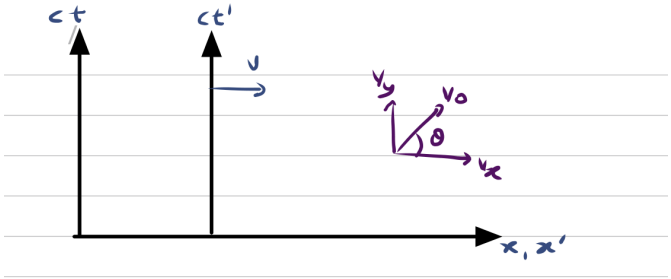


Figure 2.3: Relativistic velocity transformation and aberration. A particle (or light ray) moving with speed v_0 at an angle θ relative to the boost direction appears at a different angle θ' in the moving frame.

Consider a particle or signal moving in the xy-plane with speed v_0 at an angle θ in frame K, and speed v'_0 at angle θ' in frame K':

$$u_x = v_0\cos\theta, \qquad u_y = v_0\sin\theta, \tag{2.48}$$

$$u'_x = v'_0\cos\theta', \qquad u'_y = v'_0\sin\theta'. \tag{2.49}$$

Substituting into the velocity transformation equations:

$$v_0\cos\theta = \frac{v'_0\cos\theta' + v}{1+\frac{vv'_0\cos\theta'}{c^2}}, \tag{2.50}$$

$$v_0\sin\theta = \frac{v'_0\sin\theta'}{\gamma\left(1+\frac{vv'_0\cos\theta'}{c^2}\right)}. \tag{2.51}$$

Dividing the two equations gives the general angular transformation:

$$\tan\theta = \frac{v'_0\sin\theta'}{\gamma\left(v'_0\cos\theta' + v\right)}. \tag{2.52}$$

Postulate / Principle: Relativistic Aberration Formula for Light

For a **photon** ($v_0 = v'_0 = c$), the aberration formula simplifies to:

$$\tan\theta = \frac{\sin\theta'}{\gamma\left(\cos\theta' + \frac{v}{c}\right)} = \gamma^{-1}\frac{\sin\theta'}{\cos\theta' + \beta}. \tag{2.53}$$

Equivalently, expressing $\cos\theta$ directly:

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}. \tag{2.54}$$

2.7 General Lorentz Boost in an Arbitrary Direction

When the boost velocity vector $\vec{v}$ is oriented in an arbitrary spatial direction (not necessarily along the x-axis), we decompose the spatial position vector $\vec{x}$ into components parallel and perpendicular to $\vec{v}$:

$$\vec{x} = \vec{x}_\parallel + \vec{x}_\perp, \tag{2.55}$$

where defining the unit vector $\hat{n} \equiv \frac{\vec{v}}{v}$ and $\vec{\beta} \equiv \frac{\vec{v}}{c}$:

$$\vec{x}_\parallel \equiv (\vec{x}\cdot\hat{n})\hat{n} = \frac{(\vec{x}\cdot\vec{v})}{v^2}\vec{v} = \frac{(\vec{x}\cdot\vec{\beta})}{\beta^2}\vec{\beta}, \qquad \vec{x}_\perp \equiv \vec{x} - \vec{x}_\parallel. \tag{2.56}$$

Under the boost:

1. The perpendicular spatial components remain invariant:

$$\vec{x}'_\perp = \vec{x}_\perp. \tag{2.57}$$

2. The parallel spatial component transforms with the standard 1D Lorentz formula:

$$\vec{x}'_\parallel = \gamma(\vec{x}_\parallel - \vec{v}t). \tag{2.58}$$

3. The temporal coordinate transforms with the projection $\vec{v}\cdot\vec{x}$:

$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right) \iff x'^0 = \gamma(x^0 - \vec{\beta}\cdot\vec{x}). \tag{2.59}$$

Recombining the spatial components:

$$\begin{aligned}\vec{x}' &= \vec{x}'_\perp + \vec{x}'_\parallel \\ &= \vec{x}_\perp + \gamma(\vec{x}_\parallel - \vec{v}t) \\ &= (\vec{x} - \vec{x}_\parallel) + \gamma\vec{x}_\parallel - \gamma\vec{v}t \\ &= \vec{x} + (\gamma - 1)\vec{x}_\parallel - \gamma\vec{v}t \\ &= \vec{x} + (\gamma - 1)\frac{(\vec{x}\cdot\vec{v})}{v^2}\vec{v} - \gamma\vec{v}t.\end{aligned} \tag{2.60}$$

Postulate / Principle: General 3D Lorentz Boost Vector Formula

In terms of the dimensionless velocity vector $\vec{\beta} = \vec{v}/c$ and $x^0 = ct$:

$$x'^0 = \gamma\left(x^0 - \vec{\beta}\cdot\vec{x}\right), \tag{2.61}$$

$$\vec{x}' = \vec{x} + \left(\frac{\gamma - 1}{\beta^2}\right)(\vec{x}\cdot\vec{\beta})\vec{\beta} - \gamma\vec{\beta}x^0. \tag{2.62}$$

2.8 Causality, Light Cones, and Spacetime Ordering

Foundational Question: Does Relativity of Simultaneity Break Causality?

Since the temporal order of spatially separated events can be reversed by changing reference frames ($\Delta t > 0$ while $\Delta t' < 0$), does Special Relativity violate causality? Is spacetime an ordered set? Do we need an absolute time-ordering operator?

Physical Insight & Motivation: Invariance of Temporal Ordering for Causal Events

Causality is **strictly preserved** in Special Relativity due to the Lorentzian light-cone structure:

1. **Causally Connected Events ($\mathrm{d}s^2 \leq 0$):** Let events 1 and 2 be causally connectable (separated by a timelike or null interval):

$$c^2\Delta t^2 \geq |\Delta\vec{x}|^2 \implies \left|\frac{\Delta x}{\Delta t}\right| \leq c. \tag{2.63}$$

Under a Lorentz transformation, the new time interval is:

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = \gamma\Delta t\left(1 - \frac{v}{c}\frac{\Delta x}{c\Delta t}\right). \tag{2.64}$$

Because $|v/c| < 1$ for any physical frame and $\left|\frac{\Delta x}{c\Delta t}\right| \leq 1$ for any causal signal:

$$\frac{v}{c}\frac{\Delta x}{c\Delta t} < 1 \implies 1 - \frac{v}{c}\frac{\Delta x}{c\Delta t} > 0. \tag{2.65}$$

Since $\gamma > 0$, the sign of $\Delta t'$ is **strictly identical** to the sign of Δt:

$$\Delta t > 0 \iff \Delta t' > 0. \tag{2.66}$$

All observers in all reference frames agree on the temporal sequence of cause and effect!

2. **Causally Disconnected Events ($\mathrm{d}s^2 > 0$):** For spacelike separated events, $|\Delta x| > c|\Delta t|$. The temporal order *can* be inverted by choosing an appropriate boost frame ($v > c^2\Delta t/\Delta x$). However, because no physical particle or signal can travel faster than c, event 1 could never have influenced event 2 in the first place. Temporal reversal for spacelike events creates zero causal paradoxes.

3. **Spacetime as a Partially Ordered Set (Poset):** Spacetime $\mathcal{M}$ is not totally ordered (there is no universal linear scalar clock), but it is a rigorous **partially ordered set** under the causal relation:

$$p \prec q \iff q \in J^+(p), \tag{2.67}$$

where $J^+(p)$ is the causal future (interior and boundary of the future lightcone) of event p. No external absolute time-ordering operator is needed.

Chapter 3

The Lorentz Group, Relativistic Dynamics, and Hyperbolic Velocity Space

3.1 Tensorial Representation of the Lorentz Group

3.1.1 Matrix Formulation of General Boosts

In four-dimensional spacetime, any coordinate transformation between inertial reference frames is a linear mapping:

$$x'^i = L^i{}_j x^j = \Lambda^i{}_j x^j, \qquad i,j \in \{0,1,2,3\}. \tag{3.1}$$

For a general Lorentz boost with arbitrary velocity $\vec{v} = v\hat{n}$ (where $\hat{n} = (n^1, n^2, n^3)$ is a spatial unit vector and $\vec{\beta} = \frac{\vec{v}}{c} = \beta\hat{n}$), the matrix elements $L^i{}_j$ are partitioned into temporal (0) and spatial ($\alpha, \beta \in \{1,2,3\}$) components:

$$x'^0 = L^0{}_0 x^0 + L^0{}_\alpha x^\alpha, \tag{3.2}$$

$$x'^\alpha = L^\alpha{}_0 x^0 + L^\alpha{}_\beta x^\beta. \tag{3.3}$$

Comparing with the general vector boost formula derived in Chapter 2:

$$L^0{}_0 = \gamma, \tag{3.4}$$

$$L^0{}_\alpha = -\gamma\beta n_\alpha = -\gamma\frac{v_\alpha}{c}, \tag{3.5}$$

$$L^\alpha{}_0 = -\gamma\beta n^\alpha = -\gamma\frac{v^\alpha}{c}, \tag{3.6}$$

$$L^\alpha{}_\beta = \delta^\alpha_\beta + (\gamma-1)n^\alpha n_\beta = \delta^\alpha_\beta + (\gamma-1)\frac{v^\alpha v_\beta}{v^2}. \tag{3.7}$$

In explicit 4×4 matrix representation:

$$L = \begin{pmatrix} \gamma & -\gamma\beta n_1 & -\gamma\beta n_2 & -\gamma\beta n_3 \\ -\gamma\beta n^1 & 1+(\gamma-1)(n^1)^2 & (\gamma-1)n^1n^2 & (\gamma-1)n^1n^3 \\ -\gamma\beta n^2 & (\gamma-1)n^2n^1 & 1+(\gamma-1)(n^2)^2 & (\gamma-1)n^2n^3 \\ -\gamma\beta n^3 & (\gamma-1)n^3n^1 & (\gamma-1)n^3n^2 & 1+(\gamma-1)(n^3)^2 \end{pmatrix}. \tag{3.8}$$

3.1.2 Defining Invariance of the Lorentz Group

The defining geometric property of the Lorentz group $\mathrm{O}(1,3)$ is that it preserves the Minkowski metric tensor $\eta = \mathrm{diag}(-1,+1,+1,+1)$:

$$L^T \eta L = \eta \iff \eta_{ij} L^i{}_k L^j{}_l = \eta_{kl}. \tag{3.9}$$

Taking the determinant of both sides:

$$\det(L^T \eta L) = \det(\eta) \implies (\det L)^2(-1) = -1 \implies \det L = \pm 1. \tag{3.10}$$

Furthermore, evaluating the 00-component of Equation (3.9):

$$\eta_{ij} L^i{}_0 L^j{}_0 = -(L^0{}_0)^2 + \sum_{\alpha=1}^{3} (L^\alpha{}_0)^2 = -1 \implies (L^0{}_0)^2 = 1 + \sum_{\alpha=1}^{3} (L^\alpha{}_0)^2 \geq 1. \tag{3.11}$$

This divides $\mathrm{O}(1,3)$ into four topologically disconnected components:

1. **Proper Orthochronous Lorentz Group $\mathrm{SO}^+(1,3)$:** $\det L = +1$ and $L^0{}_0 \geq +1$. This connected subgroup contains the identity and describes all physically realizable continuous rotations and boosts.
2. **Improper / Non-Orthochronous Components:** Connected via the discrete symmetry operations:
   - **Parity Inversion (P):** $P = \mathrm{diag}(+1,-1,-1,-1)$, $\det P = -1, P^0{}_0 = +1$.
   - **Time Reversal (T):** $T = \mathrm{diag}(-1,+1,+1,+1)$, $\det T = -1, T^0{}_0 = -1$.
   - **Total Inversion (PT):** $PT = \mathrm{diag}(-1,-1,-1,-1)$, $\det(PT) = +1, (PT)^0{}_0 = -1$.

Foundational Question: Why Do We Care About Discrete Symmetries and CPT?

Lorentz transformations between inertial observers belong to the continuous subgroup $\mathrm{SO}^+(1,3)$. Why, then, are discrete transformations (P, T, and C) of such foundational importance in physics?

Physical Insight & Motivation: CPT Theorem and Quantum Field Theory

- **Discrete Transformations in Nature:** While continuous Lorentz symmetry $\mathrm{SO}^+(1,3)$ is exact in all known fundamental interactions, discrete symmetries are broken: the weak interaction violates Parity (P) maximally (Wu experiment, 1956) and violates CP (Cronin–Fitch, 1964).
- **The CPT Theorem:** The celebrated **CPT Theorem** (proved by Schwinger, Lüders, and Pauli) states that *any local, Lorentz-invariant quantum field theory described by a Hermitian Hamiltonian must be strictly invariant under the combined product CPT*, where C is charge conjugation, P is parity inversion, and T is time reversal.
- **Physical Consequences:** The exactness of CPT guarantees that particles and antiparticles have identical masses, lifetimes, and opposite quantum numbers, cementing the deep synergy between Special Relativity and quantum mechanics.

3.1.3 Inverse Lorentz Transformations

The inverse transformation matrix $M^i{}_j \equiv (L^{-1})^i{}_j$ satisfies:

$$L^i{}_j M^j{}_k = \delta^i_k. \tag{3.12}$$

Using the metric condition $L^T \eta L = \eta \implies L^{-1} = \eta^{-1} L^T \eta$, the index-lowered inverse is:

$$M^i{}_j = (L^{-1})^i{}_j = L_j{}^i = \eta^{ik} L^l{}_k \eta_{lj}. \tag{3.13}$$

Physically, $M(\vec{v}) = L(-\vec{v})$.

3.2 Non-Commutativity of Boosts and Thomas Precession

3.2.1 Composition of Successive Non-Collinear Boosts

Consider three inertial frames related by two successive boosts:

$$x \xrightarrow{L(\vec{v}_1)} x' \xrightarrow{L(\vec{v}_2)} x'' \implies x''^a = L^a{}_b(\vec{v}_2) L^b{}_c(\vec{v}_1) x^c. \tag{3.14}$$

Foundational Question: Do Successive Boosts Commute?

Does $L(\vec{v}_2)L(\vec{v}_1) = L(\vec{v}_1)L(\vec{v}_2)$ for two arbitrary velocity vectors?

If $\vec{v}_1$ and $\vec{v}_2$ are parallel (collinear), boosts commute and add according to the collinear velocity addition rule. However, if $\vec{v}_1$ and $\vec{v}_2$ are **non-collinear**, boosts do **not** commute!

Physical Insight & Motivation: Why Expand to Second Order?

To reveal non-commutativity, a first-order expansion in $\vec{\beta}$ is insufficient because first-order terms commute: $\vec{\beta}_1 + \vec{\beta}_2 = \vec{\beta}_2 + \vec{\beta}_1$. The non-commutative structure arises strictly at second order $\mathcal{O}(\beta^2)$ through cross terms like $\vec{\beta}_1 \times \vec{\beta}_2$.

3.2.2 Second-Order Perturbative Expansion

Expanding the boost matrix $L(\vec{v})$ up to second order in $\vec{\beta} = \vec{v}/c$:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \frac{1}{2}\beta^2 + \mathcal{O}(\beta^4), \tag{3.15}$$

$$\gamma - 1 \approx \frac{1}{2}\beta^2, \tag{3.16}$$

$$L^0{}_0 \approx 1 + \frac{1}{2}\beta^2, \qquad L^0{}_\alpha = L^\alpha{}_0 \approx -\beta_\alpha, \qquad L^\alpha{}_\beta \approx \delta^\alpha_\beta + \frac{1}{2}\beta^\alpha \beta_\beta. \tag{3.17}$$

Applying the composite boost $L(\vec{v}_2)L(\vec{v}_1)$ to a spacetime position $(x^0, \vec{x})$:

$$x''^0 \approx \left[1 + \frac{1}{2}(\vec{\beta}_1 + \vec{\beta}_2)^2\right] x^0 - (\vec{\beta}_1 + \vec{\beta}_2) \cdot \vec{x}, \tag{3.18}$$

$$\vec{x}'' \approx \vec{x} - (\vec{\beta}_1 + \vec{\beta}_2) x^0 + \frac{1}{2}\left[\vec{\beta}_1(\vec{\beta}_1 \cdot \vec{x}) + \vec{\beta}_2(\vec{\beta}_2 \cdot \vec{x})\right] + \vec{\beta}_2(\vec{\beta}_1 \cdot \vec{x}). \tag{3.19}$$

Now reverse the order of application $L(\vec{v}_1)L(\vec{v}_2)$:

$$\vec{x}''_{\text{rev}} \approx \vec{x} - (\vec{\beta}_1 + \vec{\beta}_2) x^0 + \frac{1}{2}\left[\vec{\beta}_1(\vec{\beta}_1 \cdot \vec{x}) + \vec{\beta}_2(\vec{\beta}_2 \cdot \vec{x})\right] + \vec{\beta}_1(\vec{\beta}_2 \cdot \vec{x}). \tag{3.20}$$

Taking the difference between the two ordering operations:

$$\delta\vec{x} \equiv \vec{x}'' - \vec{x}''_{\text{rev}} = \vec{\beta}_2(\vec{\beta}_1 \cdot \vec{x}) - \vec{\beta}_1(\vec{\beta}_2 \cdot \vec{x}). \tag{3.21}$$

Using the standard vector identity $(\vec{A} \times \vec{B}) \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{A}(\vec{B} \cdot \vec{C})$:

$$\delta\vec{x} = (\vec{\beta}_1 \times \vec{\beta}_2) \times \vec{x} = \frac{1}{c^2}(\vec{v}_1 \times \vec{v}_2) \times \vec{x}. \tag{3.22}$$

Key Concept: Thomas Precession / Wigner Rotation

Comparing Equation (3.22) with an infinitesimal spatial rotation $\delta\vec{x} = \vec{\Omega} \times \vec{x}$, we identify the induced spatial rotation angle:

$$\vec{\Omega} = \frac{1}{2c^2}(\vec{v}_1 \times \vec{v}_2). \tag{3.23}$$

The composition of two non-collinear Lorentz boosts is **not a pure boost**, but a boost combined with a spatial rotation (the **Wigner rotation**).
For a particle undergoing accelerated motion with velocity $\vec{v}$ and instantaneous acceleration $\vec{a}$, this kinematic rotation generates an intrinsic precession of its rest frame called **Thomas Precession**:

$$\vec{\omega}_T = \frac{\gamma^2}{\gamma+1}\frac{\vec{a}\times\vec{v}}{c^2} \approx \frac{1}{2c^2}(\vec{a}\times\vec{v}). \tag{3.24}$$

In atomic physics, Thomas precession contributes an essential relativistic correction of $-1/2$ to the electron's spin-orbit interaction energy, correctly resolving the factor-of-two discrepancy in atomic fine structure.

3.3 4-Velocity and Relativistic Dynamics

3.3.1 Definition and Invariant Normalization of 4-Velocity

In Newtonian mechanics, the velocity $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ uses coordinate time t, which is not a Lorentz scalar. In relativity, we parameterize worldlines using the invariant proper time τ:

Definition: 4-Velocity u^i

The **4-velocity** u^i of a massive particle is the derivative of its spacetime position $x^i(\tau)$ with respect to proper time τ:

$$u^i \equiv \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}x^i}{\mathrm{d}t} = (\gamma c, \gamma\vec{v}), \tag{3.25}$$

with covariant components lowered by η_{ij}:

$$u_i = \eta_{ij}u^j = (-\gamma c, \gamma\vec{v}). \tag{3.26}$$

Evaluating the Lorentz-invariant scalar magnitude of u^i:

$$u^i u_i = \eta_{ij}u^i u^j = -\gamma^2c^2 + \gamma^2|\vec{v}|^2 = -\gamma^2c^2\left(1 - \frac{v^2}{c^2}\right) = -c^2. \tag{3.27}$$

Postulate / Principle: Fundamental 4-Velocity Invariant

For any physical massive particle, the 4-velocity has a universal invariant norm:

$$u^i u_i \equiv -c^2. \tag{3.28}$$

The 4-velocity is strictly timelike at every point along the worldline.

3.3.2 Relationships Between Components

From Equation (3.27), the temporal component u^0 is uniquely constrained by the spatial components u^α:

$$-(u^0)^2 + u^\alpha u_\alpha = -c^2 \implies u^0 = \sqrt{c^2 + u^\alpha u_\alpha}. \tag{3.29}$$

The ordinary 3-velocity components v^α can be reconstructed directly from the 4-velocity:

$$v^\alpha = c\frac{u^\alpha}{u^0} = u^\alpha \left(1 + \frac{u^\mu u_\mu}{c^2}\right)^{-1/2}. \tag{3.30}$$

Inverting for the 3-speed v:

$$u^0 = \gamma c \implies v = c\sqrt{1 - \frac{c^2}{(u^0)^2}}. \tag{3.31}$$

3.4 Invariant Relative Velocity Between Moving Frames

3.4.1 Geometric Formulation

Consider three reference frames S_0, S_1, and S_2, as depicted in Figure 3.1.

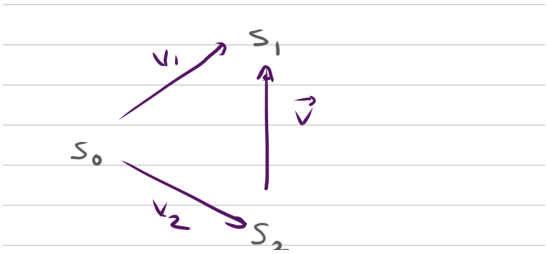


Figure 3.1: Three inertial reference frames S_0, S_1, S_2. Frame S_0 observes frames S_1 and S_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$. The vector $\vec{v}$ represents the relative velocity of S_2 measured in the rest frame of S_1.

Let the 4-velocities of S_1 and S_2 in the laboratory frame S_0 be:

$$u_1^i = (\gamma_1 c, \gamma_1 \vec{v}_1), \qquad u_2^i = (\gamma_2 c, \gamma_2 \vec{v}_2). \tag{3.32}$$

Now evaluate these 4-velocities in the comoving rest frame of S_1:

$$u_1'^i = (c, \vec{0}), \qquad u_2'^i = (\gamma c, \gamma \vec{v}), \tag{3.33}$$

where v is the relative speed of S_2 as seen by S_1, and $\gamma = (1 - v^2/c^2)^{-1/2}$.

3.4.2 Derivation of the Invariant Relative Velocity Formula

Because the Minkowski scalar product of two 4-vectors is strictly frame-invariant:

$$u_1^i u_{i2} = u_1'^i u_{i2}'. \tag{3.34}$$

Evaluating the left-hand side in frame S_0:

$$u_1^i u_{i2} = \eta_{ij} u_1^i u_2^j = -\gamma_1\gamma_2 c^2 + \gamma_1\gamma_2(\vec{v}_1 \cdot \vec{v}_2). \tag{3.35}$$

Evaluating the right-hand side in frame S_1:

$$u_1'^i u_{i2}' = \eta_{ij}(c, \vec{0})^i (\gamma c, \gamma \vec{v})^j = -\gamma c^2. \tag{3.36}$$

Equating both expressions:

$$-\gamma c^2 = -\gamma_1\gamma_2 c^2 + \gamma_1\gamma_2(\vec{v}_1\cdot\vec{v}_2) \implies \gamma = \gamma_1\gamma_2\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right). \tag{3.37}$$

To express the relative speed v explicitly, square both sides of Equation (3.37):

$$\frac{1}{1-v^2/c^2} = \gamma_1^2\gamma_2^2\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2 \implies 1-\frac{v^2}{c^2} = \frac{(1-v_1^2/c^2)(1-v_2^2/c^2)}{\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2}. \tag{3.38}$$

Solving for v^2/c^2:

$$\begin{aligned}
\frac{v^2}{c^2} &= 1-\frac{(1-v_1^2/c^2)(1-v_2^2/c^2)}{\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2} \\
&= \frac{\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2 - \left(1-\frac{v_1^2+v_2^2}{c^2}+\frac{v_1^2v_2^2}{c^4}\right)}{\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2} \\
&= \frac{\left(1-\frac{2\vec{v}_1\cdot\vec{v}_2}{c^2}+\frac{(\vec{v}_1\cdot\vec{v}_2)^2}{c^4}\right) - \left(1-\frac{v_1^2+v_2^2}{c^2}+\frac{v_1^2v_2^2}{c^4}\right)}{\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2} \\
&= \frac{\frac{v_1^2+v_2^2-2\vec{v}_1\cdot\vec{v}_2}{c^2} - \frac{v_1^2v_2^2-(\vec{v}_1\cdot\vec{v}_2)^2}{c^4}}{\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2}.
\end{aligned} \tag{3.39}$$

Recognizing $v_1^2+v_2^2-2\vec{v}_1\cdot\vec{v}_2 = |\vec{v}_1-\vec{v}_2|^2$ and using Lagrange's identity $v_1^2v_2^2-(\vec{v}_1\cdot\vec{v}_2)^2 = |\vec{v}_1\times\vec{v}_2|^2$:

Postulate / Principle: Invariant Relativistic Relative Speed Theorem

The relative speed v between two inertial frames moving with velocities $\vec{v}_1$ and $\vec{v}_2$ relative to a common frame is:

$$v^2 = \frac{|\vec{v}_1-\vec{v}_2|^2 - \frac{|\vec{v}_1\times\vec{v}_2|^2}{c^2}}{\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)^2}. \tag{3.40}$$

3.5 The Hyperbolic Geometry of Velocity Space (Lobachevsky Space)

3.5.1 Infinitesimal Metric on Velocity Space

Foundational Question: Is the Relativistic Velocity Space Compact?

Because all physical velocities are bounded by the speed of light ($|\vec{v}| < c$), does the manifold of relativistic velocities form a compact space (such as a closed 3-ball)?

Physical Insight & Motivation: Non-Compactness of Velocity Space

While the coordinate domain of velocities $\{\vec{v} \in \mathbb{R}^3 : |\vec{v}| < c\}$ is bounded in Euclidean coordinates, the **intrinsic geometric distance** between two velocities near c diverges to infinity. As $v \to c$, rapidity $\chi \to \infty$. Thus, velocity space is **topologically non-compact and complete**.

Let $\vec{v}_1 = \vec{v}$ and $\vec{v}_2 = \vec{v} + \mathrm{d}\vec{v}$ be two infinitesimally neighboring velocities. Applying the relative speed formula:

$$\mathrm{d}\ell_v^2 = \frac{|\mathrm{d}\vec{v}|^2 - \frac{|\vec{v}\times\mathrm{d}\vec{v}|^2}{c^2}}{\left(1 - \frac{\vec{v}\cdot(\vec{v}+\mathrm{d}\vec{v})}{c^2}\right)^2}. \tag{3.41}$$

Since $\mathrm{d}\ell_v^2$ is already second order in differentials, expanding the denominator to leading order gives:

$$\left(1 - \frac{v^2 + \vec{v}\cdot\mathrm{d}\vec{v}}{c^2}\right)^2 \approx \left(1 - \frac{v^2}{c^2}\right)^2 \left(1 - \frac{2\vec{v}\cdot\mathrm{d}\vec{v}}{c^2(1 - v^2/c^2)}\right) \longrightarrow \left(1 - \frac{v^2}{c^2}\right)^2. \tag{3.42}$$

Using the identity $|\vec{v} \times \mathrm{d}\vec{v}|^2 = v^2|\mathrm{d}\vec{v}|^2 - (\vec{v}\cdot\mathrm{d}\vec{v})^2$:

$$\begin{aligned} |\mathrm{d}\vec{v}|^2 - \frac{|\vec{v}\times\mathrm{d}\vec{v}|^2}{c^2} &= |\mathrm{d}\vec{v}|^2 - \frac{v^2|\mathrm{d}\vec{v}|^2 - (\vec{v}\cdot\mathrm{d}\vec{v})^2}{c^2} \\ &= \left(1 - \frac{v^2}{c^2}\right)|\mathrm{d}\vec{v}|^2 + \frac{(\vec{v}\cdot\mathrm{d}\vec{v})^2}{c^2}. \end{aligned} \tag{3.43}$$

Therefore, the invariant metric on velocity space is:

$$\mathrm{d}\ell_v^2 = \frac{\left(1 - \frac{v^2}{c^2}\right)|\mathrm{d}\vec{v}|^2 + \frac{(\vec{v}\cdot\mathrm{d}\vec{v})^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^2}. \tag{3.44}$$

3.5.2 Transformation to Spherical Coordinates and Rapidity

In spherical polar coordinates (v, θ, ϕ) in velocity space:

$$|\mathrm{d}\vec{v}|^2 = \mathrm{d}v^2 + v^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{3.45}$$

$$\vec{v}\cdot\mathrm{d}\vec{v} = v\,\mathrm{d}v. \tag{3.46}$$

Substituting into the line element:

$$\begin{aligned} \mathrm{d}\ell_v^2 &= \frac{\left(1 - \frac{v^2}{c^2}\right)\left[\mathrm{d}v^2 + v^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] + \frac{v^2\,\mathrm{d}v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^2} \\ &= \frac{\mathrm{d}v^2}{\left(1 - \frac{v^2}{c^2}\right)^2} + \frac{v^2}{1 - \frac{v^2}{c^2}}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \end{aligned} \tag{3.47}$$

Now transform from speed v to the **rapidity parameter** χ via $v = c\tanh\chi$:

$$\mathrm{d}v = c\,\mathrm{sech}^2\chi\,\mathrm{d}\chi = c(1 - \tanh^2\chi)\,\mathrm{d}\chi = c\left(1 - \frac{v^2}{c^2}\right)\mathrm{d}\chi \implies \frac{\mathrm{d}v}{1 - v^2/c^2} = c\,\mathrm{d}\chi, \tag{3.48}$$

$$\frac{v^2}{1 - v^2/c^2} = \frac{c^2\tanh^2\chi}{\mathrm{sech}^2\chi} = c^2\sinh^2\chi. \tag{3.49}$$

Postulate / Principle: Lobachevsky Metric of Relativistic Velocity Space

In terms of rapidity χ, the metric on the space of relativistic velocities is:

$$\mathrm{d}\ell_v^2 = c^2\left[\mathrm{d}\chi^2 + \sinh^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]. \tag{3.50}$$

This is precisely the metric of a **3-dimensional hyperbolic space (Lobachevsky space $\mathbb{H}^3$)** with constant negative sectional curvature $K = -1/c^2$.

Key Concept: Geometrization of Velocity Space

Geometrizing the velocity parameters provides deep structural insights into relativity:

- Relativistic velocity addition corresponds to geodesic parallel transport in Lobachevsky space $\mathbb{H}^3$.
- The non-commutativity of non-collinear boosts is nothing other than the **curvature holonomy (angular defect)** of geodesic triangles in hyperbolic space.
- In the non-relativistic limit $c \to \infty$, the curvature $K = -1/c^2 \to 0$, and Lobachevsky velocity space flattens to standard Euclidean velocity space $\mathbb{R}^3$ where velocities commute.

3.6 4-Acceleration and Orthogonality

3.6.1 Definition of 4-Acceleration

Definition: 4-Acceleration a^i

The **4-acceleration** a^i of a particle is defined as the proper-time derivative of its 4-velocity:

$$a^i \equiv \frac{\mathrm{d}u^i}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2}. \tag{3.51}$$

3.6.2 The Fundamental Orthogonality Theorem

Differentiating the universal 4-velocity normalization condition $u^iu_i = -c^2$ with respect to proper time τ:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u^iu_i) = \frac{\mathrm{d}}{\mathrm{d}\tau}(-c^2) = 0. \tag{3.52}$$

Using the metric compatibility and Leibniz rule:

$$\eta_{ij}\frac{\mathrm{d}u^i}{\mathrm{d}\tau}u^j + \eta_{ij}u^i\frac{\mathrm{d}u^j}{\mathrm{d}\tau} = 2\eta_{ij}a^iu^j = 2a^iu_i = 0. \tag{3.53}$$

Postulate / Principle: Orthogonality of 4-Velocity and 4-Acceleration

For any physical worldline, the 4-acceleration is always strictly orthogonal to the 4-velocity in the Minkowski metric:

$$a^iu_i \equiv 0. \tag{3.54}$$

3.6.3 Comoving Rest Frame Representation

In the particle's instantaneous comoving rest frame, the 4-velocity is purely temporal:

$$u'^i = (c, \vec{0}) \implies u'_i = (-c, \vec{0}). \tag{3.55}$$

Applying the orthogonality condition $a'^i u'_i = 0$:

$$-ca'^0 + \vec{a}' \cdot \vec{0} = 0 \implies a'^0 = 0. \tag{3.56}$$

Therefore, in the instantaneous rest frame, the 4-acceleration is **purely spatial**:

$$a'^i = (0, \vec{a}_0), \tag{3.57}$$

where $\vec{a}_0 = \frac{\mathrm{d}^2\vec{x}'}{\mathrm{d}t'^2}$ is the **proper acceleration** felt by the particle (measured by an onboard accelerometer).

The invariant scalar magnitude of the 4-acceleration is:

$$a^i a_i = a'^i a'_i = |\vec{a}_0|^2 \geq 0 \quad \text{(strictly spacelike)}. \tag{3.58}$$

Thus, while the 4-velocity of a massive particle is always timelike, its 4-acceleration is always strictly **spacelike**.

Chapter 4

Variational Principles, Hamilton–Jacobi Theory, and Relativistic Particle Mechanics

4.1 Why Naive 4-Vector Potential Forces Fail

4.1.1 The Inconsistency of Scalar Forces in Flat Spacetime

In non-relativistic Newtonian mechanics, the motion of a particle of mass m in a conservative scalar potential $\Phi(\vec{x})$ is governed by:

$$ma^\alpha = -\partial^\alpha \Phi = -\frac{\partial \Phi}{\partial x^\alpha}, \qquad \alpha \in \{1,2,3\}. \tag{4.1}$$

Foundational Question: Can We Promote Newton's Law Directly to 4-Vectors?

Can we generalize Newtonian gravitational or potential mechanics to Special Relativity simply by promoting the 3-vector index α to a 4-vector index $i \in \{0,1,2,3\}$:

$$ma^i \stackrel{?}{=} -\partial^i \Phi = -\eta^{ij}\frac{\partial \Phi}{\partial x^j}? \tag{4.2}$$

Physical Insight & Motivation: Proof of Inconsistency

Let us test the mathematical and physical consistency of Equation (4.2) against the fundamental orthogonality identity of relativistic kinematics ($a^i u_i \equiv 0$):

1. Contract both sides of Equation (4.2) with the 4-velocity u_i:
$$ma^i u_i = -u_i \partial^i \Phi = -\eta_{ij} u^j \eta^{ik} \partial_k \Phi = -u^k \partial_k \Phi. \tag{4.3}$$
2. By the chain rule, $u^k \partial_k \Phi = \frac{\mathrm{d}x^k}{\mathrm{d}\tau}\frac{\partial \Phi}{\partial x^k} = \frac{\mathrm{d}\Phi}{\mathrm{d}\tau}$, which is the total rate of change of the potential along the particle's worldline:
$$ma^i u_i = -\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}. \tag{4.4}$$
3. But by the universal 4-velocity normalization $u^i u_i = -c^2$, the left-hand side vanishes identically ($a^i u_i = 0$). This imposes the severe constraint:
$$\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} \equiv 0 \quad \text{along all physical worldlines!} \tag{4.5}$$

4. If $\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} = 0$ for every particle regardless of its velocity and trajectory, the scalar field Φ must be constant everywhere ($\partial_i \Phi = 0$). Hence:

$$ma^i = 0 \implies \text{No interaction or force can exist!} \tag{4.6}$$

Key Concept: Foundational Lesson: Nature of Relativistic Interactions

A simple scalar field Φ in flat Minkowski spacetime cannot act as a universal force field. This foundational obstacle reveals why relativistic physics requires richer geometric structures:

- **Gauge Theories (Electromagnetism):** Forces couple via a 4-vector gauge potential A_i producing an antisymmetric field strength tensor $F_{ij} = \partial_i A_j - \partial_j A_i$. The Lorentz 4-force $F^i = qF^i{}_j u^j$ automatically satisfies $F^i u_i = qF_{ij}u^j u^i \equiv 0$ due to the antisymmetry $F_{ij} = -F_{ji}$.
- **Gravitation (General Relativity):** Gravity cannot be a static scalar field on Minkowski space; instead, the **metric tensor g_{ij} itself** is the dynamical gravitational field, and particles follow force-free geodesics in curved spacetime.

4.2 The Principle of Least Action in Classical Mechanics

To construct the correct relativistic mechanics of particles and fields, we must rely on the **Principle of Stationary (Least) Action**.

4.2.1 The Action Functional

Definition: Action Functional

For a mechanical system described by generalized coordinates $q(t)$ and generalized velocities $\dot{q}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}$, the **action** S is the functional:

$$S[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, \mathrm{d}t, \tag{4.7}$$

where $L(q, \dot{q}, t)$ is the Lagrangian of the system.

Under an arbitrary infinitesimal variation of the trajectory $q(t) \to q(t) + \delta q(t)$ holding the initial and final configurations fixed:

$$\delta q(t_1) = 0, \qquad \delta q(t_2) = 0. \tag{4.8}$$

The first variation of the action is:

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right). \tag{4.9}$$

Using $\delta \dot{q} = \delta\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\delta q)$ and integrating by parts:

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q}\delta q + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \delta q \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t\, \delta q \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2}. \end{aligned} \tag{4.10}$$

Since the boundary term vanishes by virtue of $\delta q(t_1) = \delta q(t_2) = 0$, demanding $\delta S = 0$ for all arbitrary variations $\delta q(t)$ yields the **Euler–Lagrange Equations of Motion**:

Postulate / Principle: Euler–Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{4.11}$$

The canonical conjugate momentum is defined as $p \equiv \frac{\partial L}{\partial \dot{q}}$.

4.3 On-Shell Variation and the Hamilton–Jacobi Equation

4.3.1 Varying the Endpoints (On-Shell Action)

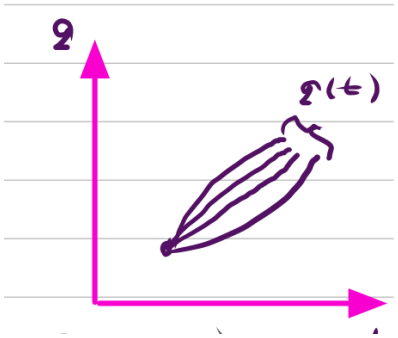


Figure 4.1: On-shell action variation with a variable endpoint $q(t)$. The bundle of trajectories satisfies the classical Euler–Lagrange equations of motion, so the variation of the action is governed entirely by the boundary terms.

Consider the action evaluated along a true classical path that satisfies the Euler–Lagrange equations (an **on-shell** trajectory), but regarded as a function of the upper boundary point (q, t):

$$S(q,t) = \int_{t_1}^{t} L(q_{\mathrm{cl}}(t'), \dot{q}_{\mathrm{cl}}(t'), t')\,\mathrm{d}t'. \tag{4.12}$$

When we vary the endpoint $q(t) \to q(t) + \delta q(t)$, the bulk integral vanishes because the equations of motion are satisfied everywhere along the path. The remaining variation comes exclusively from the boundary:

$$\delta S = [p\,\delta q]_{t_1}^{t} = p(t)\delta q(t). \tag{4.13}$$

This establishes the fundamental relationship:

$$p = \frac{\partial S}{\partial q}. \tag{4.14}$$

4.3.2 Derivation of the Hamilton–Jacobi Equation

Now compute the total time derivative of the on-shell action $S(q,t)$:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = L = \frac{\partial S}{\partial q}\dot{q} + \frac{\partial S}{\partial t}. \tag{4.15}$$

Substituting $p = \frac{\partial S}{\partial q}$:

$$L = p\dot{q} + \frac{\partial S}{\partial t} \implies (p\dot{q} - L) + \frac{\partial S}{\partial t} = 0. \tag{4.16}$$

Recalling that the Legendre transform defines the Hamiltonian $H(q, p, t) \equiv p\dot{q} - L$:

Postulate / Principle: Hamilton–Jacobi Partial Differential Equation

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{4.17}$$

Physical Insight & Motivation: Comparative Summary of Mechanical Formulations

- **Newtonian / Lagrangian Mechanics:** N second-order ordinary differential equations (ODEs) in configuration space $\{q^i\}$.
- **Hamiltonian Mechanics:** $2N$ coupled first-order ODEs in phase space $\{q^i, p_i\}$.
- **Hamilton–Jacobi Theory:** A single first-order non-linear partial differential equation (PDE) for the action function $S(q, \alpha_i, t)$. It constructs a canonical transformation to a frame where all new coordinates and momenta are constants of motion ($\dot{Q}_i = 0, \dot{P}_i = 0$), transforming dynamical evolution into geometry.

4.4 Relativistic Action for a Free Particle

4.4.1 Constructing the Invariant Action

In Special Relativity, the action S of a free particle must satisfy two physical criteria:

1. It must be a **Lorentz scalar** (invariant under all Poincaré transformations).
2. It must be proportional to the only scalar quantity characterizing the particle's trajectory: the **proper time** τ.

Thus, the action must take the form:

$$S = \alpha \int \mathrm{d}\tau = \alpha \int \mathrm{d}t \sqrt{1 - \frac{v^2}{c^2}} = \int L \, \mathrm{d}t, \tag{4.18}$$

where α is a constant characterizing the particle, and the relativistic Lagrangian is:

$$L = \alpha \sqrt{1 - \frac{v^2}{c^2}}. \tag{4.19}$$

4.4.2 Determining the Coupling Constant via the Non-Relativistic Limit

To determine the constant α, expand the Lagrangian in the non-relativistic limit $v/c \ll 1$:

$$L = \alpha \left(1 - \frac{v^2}{2c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right)\right) = \alpha - \frac{\alpha v^2}{2c^2}. \tag{4.20}$$

Because adding a constant to a Lagrangian does not affect the Euler–Lagrange equations of motion, the velocity-dependent term must match the Newtonian kinetic energy $T = \frac{1}{2}mv^2$:

$$-\frac{\alpha v^2}{2c^2} = \frac{1}{2}mv^2 \implies \alpha = -mc^2. \tag{4.21}$$

Postulate / Principle: Relativistic Free Particle Lagrangian and Action

The Lagrangian and action for a free particle of rest mass m are:

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} = -\frac{mc^2}{\gamma}, \tag{4.22}$$

$$S = -mc^2\int_A^B \mathrm{d}\tau = -mc\int_A^B \sqrt{-\eta_{ab}\,\mathrm{d}x^a\mathrm{d}x^b}. \tag{4.23}$$

Physical Insight & Motivation: Principle of Maximal Proper Time

Because $\alpha = -mc^2 < 0$, *minimizing* the action S is mathematically equivalent to **maximizing the elapsed proper time** τ along the worldline between two fixed events:

$$\delta S = 0 \iff \delta\int \mathrm{d}\tau = 0 \quad \text{(Maximal Proper Time)}. \tag{4.24}$$

Free inertial particles in spacetime follow paths that maximize the time measured by their own clocks!

4.5 Covariant Variation and the Geodesic Equation

We now perform a fully covariant variation of the relativistic action:

$$S = -mc\int \sqrt{-\eta_{ab}\,\mathrm{d}x^a\mathrm{d}x^b}. \tag{4.25}$$

Under an arbitrary variation $x^a \to x^a + \delta x^a$:

$$\begin{aligned}\delta S &= -mc\int \frac{\delta(-\eta_{ab}\,\mathrm{d}x^a\mathrm{d}x^b)}{2\sqrt{-\eta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j}} \\ &= -mc\int \frac{-\eta_{ab}\,\delta(\mathrm{d}x^a)\,\mathrm{d}x^b}{\sqrt{-\eta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j}}.\end{aligned} \tag{4.26}$$

Using $\delta(\mathrm{d}x^a) = \mathrm{d}(\delta x^a)$ and recognizing the denominator as $c\,\mathrm{d}\tau$:

$$\begin{aligned}\delta S &= mc\int \frac{\eta_{ab}\,\mathrm{d}(\delta x^a)\,\mathrm{d}x^b}{c\,\mathrm{d}\tau} \\ &= m\int \eta_{ab}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\mathrm{d}(\delta x^a) \\ &= m\int u_a\,\mathrm{d}(\delta x^a).\end{aligned} \tag{4.27}$$

Integrating by parts:

$$\delta S = [mu_a\delta x^a]_i^f - m\int \frac{\mathrm{d}u_a}{\mathrm{d}\tau}\delta x^a\,\mathrm{d}\tau. \tag{4.28}$$

For variations with fixed endpoints ($\delta x^a(i) = \delta x^a(f) = 0$), the boundary term vanishes, yielding the **relativistic equation of motion**:

Postulate / Principle: Free Geodesic Equation in Minkowski Spacetime

$$\frac{\mathrm{d}u_a}{\mathrm{d}\tau} = 0 \iff \frac{\mathrm{d}u^a}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} = 0. \tag{4.29}$$

A free relativistic particle moves with constant 4-velocity along straight timelike lines in flat spacetime.

4.6 Relativistic 4-Momentum, Mass-Shell Constraint, and Hamilton–Jacobi Equation

4.6.1 Canonical and 4-Momentum

From the parameterized Lagrangian $\mathcal{L}(x, \dot{x}) = -mc\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}$, the canonical momentum conjugate to x^a is:

$$P_a \equiv -\frac{\partial \mathcal{L}}{\partial \dot{x}^a} = mc\frac{\eta_{ab}\dot{x}^b}{\sqrt{-\eta_{cd}\dot{x}^c\dot{x}^d}} = mc\frac{u_a}{c} = mu_a. \tag{4.30}$$

In contravariant form, the **4-momentum** P^a is:

$$P^a = mu^a = (\gamma mc,\, \gamma m\vec{v}) = \left(\frac{E}{c},\, \vec{p}\right). \tag{4.31}$$

where:

- Relativistic spatial momentum: $\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$.
- Total relativistic energy: $E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}}$.

4.6.2 The Mass-Shell Constraint

Evaluating the Lorentz-invariant norm of the 4-momentum:

$$P_aP^a = \eta_{ab}P^aP^b = -(P^0)^2 + |\vec{p}|^2 = -\frac{E^2}{c^2} + |\vec{p}|^2. \tag{4.32}$$

Since $P_aP^a = m^2u_au^a = -m^2c^2$:

$$-\frac{E^2}{c^2} + |\vec{p}|^2 = -m^2c^2. \tag{4.33}$$

Postulate / Principle: Einstein's Relativistic Energy-Momentum Dispersion Relation

$$E^2 = |\vec{p}|^2c^2 + m^2c^4. \tag{4.34}$$

For a particle at rest ($\vec{p} = 0$), this simplifies to Einstein's rest energy formula:

$$E_0 = mc^2. \tag{4.35}$$

Physical Insight & Motivation: Dirac Constraint Language

In the Hamiltonian formulation of constrained systems (Dirac's theory), the mass-shell condition,

$$\mathcal{H} \equiv P_a P^a + m^2 c^2 \approx 0, \tag{4.36}$$

is a **first-class primary constraint**. It reflects the underlying reparameterization invariance ($\tau \to \tau'(\tau)$) of the particle's worldline action.

Chapter 5

Relativistic Dynamics, Hamilton–Jacobi Theory, and Uniformly Accelerated Observers

5.1 Relativistic Energy, Canonical Formalism, and Observer-Dependent Quantities

5.1.1 Energy in Terms of 4-Momentum and Low-Velocity Expansion

In the previous chapter, we established that the relativistic 4-momentum of a free particle with rest mass m is given by:

$$P^i = mu^i = (\gamma mc, \gamma m\vec{v}) = \left(\frac{E}{c}, \vec{p}\right), \tag{5.1}$$

where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is the Lorentz factor, and $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ is the 3-velocity.

From the temporal component of the 4-momentum:

$$P^0 = m\frac{\mathrm{d}x^0}{\mathrm{d}\tau} = \gamma m\frac{c\,\mathrm{d}t}{\mathrm{d}t} = \gamma mc = \frac{E}{c}. \tag{5.2}$$

Multiplying by c yields the total relativistic energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{5.3}$$

Physical Insight & Motivation: Non-Relativistic Limit of Relativistic Energy

To understand how Einstein's relativistic energy connects to classical Newtonian mechanics, we perform a binomial series expansion for small velocities ($v \ll c$):

$$\begin{aligned} E &= mc^2\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \mathcal{O}\left(\frac{v^6}{c^6}\right)\right] \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots \end{aligned} \tag{5.4}$$

The physical interpretation is profound:

- **Rest Energy ($E_0 = mc^2$):** The leading term is constant and independent of motion. It represents the intrinsic mass-energy content of the particle at rest.

- **Newtonian Kinetic Energy ($T_{\mathbf{NR}} = \frac{1}{2}mv^2$):** The second term precisely reproduces Newtonian kinetic energy.
- **Relativistic Corrections ($\frac{3}{8}mv^4/c^2$):** Higher-order terms quantify the departure from classical mechanics as the speed approaches c.

5.1.2 The Vanishing Covariant Hamiltonian

Let us now investigate the canonical Hamiltonian corresponding to the covariant free particle Lagrangian:

$$L(x,u) = -mc\sqrt{-\eta_{ij}u^i u^j}. \tag{5.5}$$

Recall that the conjugate 4-momentum is:

$$P_i = \frac{\partial L}{\partial u^i} = \frac{mc\eta_{ij}u^j}{\sqrt{-\eta_{ab}u^a u^b}} = mu_i \quad \text{(on-shell where } u^a u_a = -c^2\text{)}. \tag{5.6}$$

The canonical Hamiltonian obtained via Legendre transformation with respect to 4-velocity is:

$$H \equiv \frac{\partial L}{\partial u^a}u^a - L = P_a u^a - L. \tag{5.7}$$

Substituting the expressions for P_a and L:

$$\begin{aligned} H &= \left(\frac{mc\eta_{ab}u^b}{\sqrt{-\eta_{ij}u^i u^j}}\right)u^a - \left(-mc\sqrt{-\eta_{ij}u^i u^j}\right) \\ &= \frac{mc\eta_{ab}u^a u^b}{\sqrt{-\eta_{ij}u^i u^j}} + mc\sqrt{-\eta_{ij}u^i u^j} \\ &= -mc\sqrt{-\eta_{ij}u^i u^j} + mc\sqrt{-\eta_{ij}u^i u^j} \equiv 0. \end{aligned} \tag{5.8}$$

Key Concept: Why Does the Covariant Hamiltonian Vanish?

The vanishing of the canonical Hamiltonian ($H \equiv 0$) is not an algebraic accident; it is a fundamental theorem of variational calculus:

1. **Euler's Homogeneous Function Theorem:** The action $S = \int L\,\mathrm{d}\lambda$ is invariant under reparameterization of the worldline $\lambda \to \lambda'(\lambda)$. Consequently, the Lagrangian $L(x,\dot{x})$ is positively homogeneous of degree 1 in velocities: $L(x,k\dot{x}) = kL(x,\dot{x})$.
2. By Euler's theorem, $\frac{\partial L}{\partial \dot{x}^a}\dot{x}^a = L$, which implies that the Legendre transform $H = P_a\dot{x}^a - L \equiv 0$ vanishes identically.
3. Instead of generating dynamics through a non-zero Hamiltonian, the dynamics of reparameterization-invariant theories is governed by **Hamiltonian constraints**—in this case, the mass-shell constraint $\mathcal{H} = P^a P_a + m^2c^2 = 0$.

5.1.3 Observer-Dependent Energy as an Invariant Scalar

In Newtonian physics, kinetic energy depends on the chosen reference frame. How is this observer-dependence formalized geometrically in Special Relativity?

Definition: Energy Measured by an Arbitrary Observer

Consider an observer with timelike 4-velocity u^i_{obs} (satisfying $u^i_{\text{obs}} u_{i,\text{obs}} = -c^2$). The energy $E_{(\text{obs})}$ of a particle with 4-momentum P^i measured by this observer is defined by the invariant contraction:

$$E_{(\text{obs})} = -u^i_{\text{obs}} P_i = -\boldsymbol{u}_{\text{obs}} \cdot \boldsymbol{P}. \tag{5.9}$$

Proof. Let us verify this in the comoving (rest) frame of the observer:

- In the observer's rest frame, their 4-velocity is $u^i_{\text{obs}} = (c, 0, 0, 0)$.
- The particle's 4-momentum in this frame is $P_i = \eta_{ij} P^j = \left(-\frac{E}{c}, \vec{p}\right)$.
- Evaluating the scalar product:

$$-u^i_{\text{obs}} P_i = -u^0_{\text{obs}} P_0 - \vec{u}_{\text{obs}} \cdot \vec{p} = -c\left(-\frac{E}{c}\right) - 0 = E. \tag{5.10}$$

Because $-u^i_{\text{obs}} P_i$ is a tensorial scalar (contracted 4-vectors), it can be computed in **any** coordinate frame and will always equal the physical energy measured by that observer. □

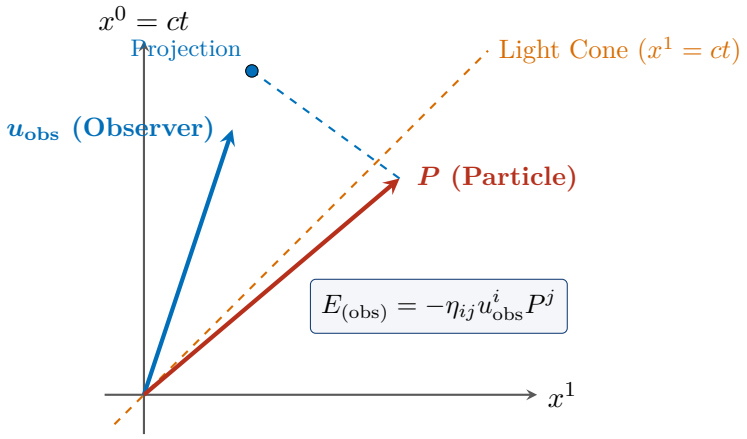


Figure 5.1: Geometric interpretation of energy: the energy of a particle measured by an observer is the invariant projection of the particle's 4-momentum $\boldsymbol{P}$ along the observer's timelike 4-velocity $\boldsymbol{u}_{\text{obs}}$.

5.2 The Relativistic Hamilton–Jacobi Equation

5.2.1 Derivation from the Mass-Shell Constraint

In analytical mechanics, the Hamilton–Jacobi equation provides a powerful formulation where the action $S(x)$ acts as a phase function whose gradient determines particle trajectories.

From Chapter 4, the on-shell variation of the action with respect to the endpoint coordinates yields the canonical 4-momentum:

$$P_i = \frac{\partial S}{\partial x^i}. \tag{5.11}$$

Substituting Equation (5.11) directly into the fundamental mass-shell identity:

$$P_i P^i = \eta^{ij} P_i P_j = -m^2 c^2, \tag{5.12}$$

we obtain the **Relativistic Hamilton–Jacobi Equation** for a free particle in flat spacetime:

Postulate / Principle: Relativistic Hamilton–Jacobi Equation

$$\eta^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} = -m^2c^2. \tag{5.13}$$

In terms of explicit time and space derivatives with $\eta^{ij} = \mathrm{diag}(-1,+1,+1,+1)$:

$$-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 + (\vec{\nabla}S)^2 = -m^2c^2, \tag{5.14}$$

where $(\vec{\nabla}S)^2 = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2$.

5.2.2 The Non-Relativistic Limit: Recovery of Classical Hamilton–Jacobi

In the non-relativistic regime, the total relativistic energy includes the dominant rest-mass energy mc^2:

$$E \approx mc^2 + E_{\text{classical}} \implies \frac{\partial S}{\partial t} = -E \approx -mc^2 - E_{\text{classical}}. \tag{5.15}$$

This motivates us to decompose the relativistic action $S(\vec{x},t)$ into a rest-mass phase factor and a non-relativistic action $s(\vec{x},t)$:

$$S(\vec{x},t) = -mc^2t + s(\vec{x},t). \tag{5.16}$$

Computing the partial derivatives of Equation (5.16):

$$\frac{\partial S}{\partial t} = -mc^2 + \frac{\partial s}{\partial t}, \qquad \vec{\nabla}S = \vec{\nabla}s. \tag{5.17}$$

Substituting these into the relativistic Hamilton–Jacobi PDE (5.14):

$$\begin{aligned} -\frac{1}{c^2}\left(-mc^2 + \frac{\partial s}{\partial t}\right)^2 + (\vec{\nabla}s)^2 &= -m^2c^2 \\ -\frac{1}{c^2}\left(m^2c^4 - 2mc^2\frac{\partial s}{\partial t} + \left(\frac{\partial s}{\partial t}\right)^2\right) + (\vec{\nabla}s)^2 &= -m^2c^2 \\ -m^2c^2 + 2m\frac{\partial s}{\partial t} - \frac{1}{c^2}\left(\frac{\partial s}{\partial t}\right)^2 + (\vec{\nabla}s)^2 &= -m^2c^2. \end{aligned} \tag{5.18}$$

The rest-mass energy term $-m^2c^2$ cancels identically on both sides:

$$2m\frac{\partial s}{\partial t} + (\vec{\nabla}s)^2 = \frac{1}{c^2}\left(\frac{\partial s}{\partial t}\right)^2. \tag{5.19}$$

Dividing through by $2m$:

$$\frac{\partial s}{\partial t} + \frac{1}{2m}(\vec{\nabla}s)^2 = \frac{1}{2mc^2}\left(\frac{\partial s}{\partial t}\right)^2. \tag{5.20}$$

Physical Insight & Motivation: Classical Limit ($c \to \infty$)

Taking the non-relativistic limit $c \to \infty$, the right-hand side of Equation (5.20) vanishes:

$$\lim_{c\to\infty}\frac{1}{2mc^2}\left(\frac{\partial s}{\partial t}\right)^2 = 0, \tag{5.21}$$

yielding exactly the classical **Hamilton–Jacobi equation for a free non-relativistic**

particle:

$$\frac{\partial s}{\partial t} + \frac{1}{2m}(\vec{\nabla} s)^2 = 0 \iff \frac{\partial s}{\partial t} + H(\vec{x}, \vec{\nabla} s) = 0. \tag{5.22}$$

In the presence of a potential $V(\vec{x})$, this generalizes smoothly to $\frac{\partial s}{\partial t} + \frac{1}{2m}(\vec{\nabla} s)^2 + V(\vec{x}) = 0$.

5.3 Kinematics of the Uniformly Accelerated Observer

5.3.1 Definition of Uniform Proper Acceleration

Definition: Uniform Proper Acceleration

An observer undergoes **uniform acceleration** if the magnitude of their 4-acceleration $a^i = \frac{\mathrm{d}u^i}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2}$ is strictly constant along their worldline:

$$a \equiv \sqrt{a^i a_i} = \sqrt{\eta_{ij} a^i a^j} = \text{constant} > 0. \tag{5.23}$$

Key Concept: Coordinate vs. Proper Acceleration

It is crucial to distinguish between:

- **Coordinate Acceleration ($\frac{\mathrm{d}^2 X}{\mathrm{d}T^2}$):** Acceleration measured in an inertial frame (cT, X). As the particle's speed approaches the speed of light ($v \to c$), relativistic inertia increases ($\gamma \to \infty$), and the coordinate acceleration must decelerate toward zero: $\frac{\mathrm{d}^2 X}{\mathrm{d}T^2} \to 0$.
- **Proper Acceleration (a):** Acceleration measured in the **instantaneous comoving inertial frame** (the reading of an onboard physical accelerometer). Under uniform proper acceleration, the physical force felt by the observer $F = ma$ remains constant for all time.

5.3.2 Derivation of the Proper Acceleration Formula

Consider rectilinear motion along the X-axis in Minkowski coordinates (cT, X, Y, Z) (with $Y = Z = 0$).

The 4-velocity of the observer is:

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = (\gamma c,\, \gamma v,\, 0,\, 0), \qquad v = \frac{\mathrm{d}X}{\mathrm{d}T}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{5.24}$$

Using the chain rule $\frac{\mathrm{d}}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}}{\mathrm{d}T}$, the 4-acceleration components are:

$$a^0 = \frac{\mathrm{d}u^0}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}}{\mathrm{d}T}(\gamma c) = c\gamma \frac{\mathrm{d}\gamma}{\mathrm{d}T}, \tag{5.25}$$

$$a^x = \frac{\mathrm{d}u^x}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}}{\mathrm{d}T}(\gamma v). \tag{5.26}$$

Let us compute the derivative of the Lorentz factor $\gamma(T)$:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}T} = \frac{\mathrm{d}}{\mathrm{d}T}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\frac{\mathrm{d}v}{\mathrm{d}T}\right) = \frac{v\gamma^3}{c^2}\frac{\mathrm{d}v}{\mathrm{d}T}. \tag{5.27}$$

Substituting Equation (5.27) into Equation (5.25):

$$a^0 = c\gamma\left(\frac{v\gamma^3}{c^2}\frac{\mathrm{d}v}{\mathrm{d}T}\right) = \frac{v\gamma^4}{c}\frac{\mathrm{d}v}{\mathrm{d}T}. \tag{5.28}$$

Next, expanding the spatial acceleration component Equation (5.26):

$$\begin{aligned}\frac{\mathrm{d}(\gamma v)}{\mathrm{d}T} &= \gamma\frac{\mathrm{d}v}{\mathrm{d}T} + v\frac{\mathrm{d}\gamma}{\mathrm{d}T} \\ &= \gamma\frac{\mathrm{d}v}{\mathrm{d}T} + v\left(\frac{v\gamma^3}{c^2}\frac{\mathrm{d}v}{\mathrm{d}T}\right) \\ &= \gamma\frac{\mathrm{d}v}{\mathrm{d}T}\left(1 + \frac{v^2\gamma^2}{c^2}\right).\end{aligned} \tag{5.29}$$

Using the identity $1 + \frac{v^2\gamma^2}{c^2} = 1 + \frac{v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2} = \gamma^2$:

$$\frac{\mathrm{d}(\gamma v)}{\mathrm{d}T} = \gamma^3\frac{\mathrm{d}v}{\mathrm{d}T}. \tag{5.30}$$

Therefore, the spatial component of the 4-acceleration is:

$$a^x = \gamma\frac{\mathrm{d}(\gamma v)}{\mathrm{d}T} = \gamma^4\frac{\mathrm{d}v}{\mathrm{d}T}. \tag{5.31}$$

Now, let us evaluate the invariant scalar $a^2 = \eta_{ij}a^i a^j = -(a^0)^2 + (a^x)^2$:

$$\begin{aligned}a^2 &= -\left(\frac{v\gamma^4}{c}\frac{\mathrm{d}v}{\mathrm{d}T}\right)^2 + \left(\gamma^4\frac{\mathrm{d}v}{\mathrm{d}T}\right)^2 \\ &= \gamma^8\left(\frac{\mathrm{d}v}{\mathrm{d}T}\right)^2\left(1 - \frac{v^2}{c^2}\right) \\ &= \gamma^8\left(\frac{\mathrm{d}v}{\mathrm{d}T}\right)^2\left(\frac{1}{\gamma^2}\right) \\ &= \gamma^6\left(\frac{\mathrm{d}v}{\mathrm{d}T}\right)^2.\end{aligned} \tag{5.32}$$

Taking the positive square root of Equation (5.32):

Postulate / Principle: Proper Acceleration Theorem

$$a = \gamma^3\frac{\mathrm{d}v}{\mathrm{d}T} = \frac{\mathrm{d}(\gamma v)}{\mathrm{d}T}. \tag{5.33}$$

In terms of proper time τ (since $\mathrm{d}T = \gamma\,\mathrm{d}\tau$), this simplifies directly to:

$$a = \gamma^2\frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{\mathrm{d}\chi}{\mathrm{d}\tau}, \tag{5.34}$$

where χ is the rapidity ($v = c\tanh\chi, \gamma = \cosh\chi$).

5.4 Foundational Geometric Insights on Vectors, Foliations, and Curvature

Before solving the trajectory equations, we pause to examine four foundational geometric principles that arise directly from studying non-inertial frames and relativistic field dynamics.

5.4.1 Tangent Spaces and the Meaning of Direction

Physical Insight & Motivation: Vector Directions Do Not Pre-Exist in Vacuum

In elementary Newtonian mechanics, space is Euclidean ($\mathbb{R}^3$), and vectors are thought of as arrows that can be freely transported anywhere.
In general relativistic geometry:

- A vector at an event $p \in \mathcal{M}$ lives strictly in the **tangent space** $T_p\mathcal{M}$.
- Vectors at different spacetime points $p \neq q$ belong to **different, distinct vector spaces** ($T_p\mathcal{M} \neq T_q\mathcal{M}$).
- One cannot compare or add vectors at different events without specifying a connection (rule for parallel transport along a curve).
- Thus, "direction" is a local geometric concept defined only within the local tangent space at a specific point on the manifold.

5.4.2 Local vs. Global Conservation Laws

Key Concept: Is Conservation of Stress-Energy Local or Global?

In curved spacetime, the conservation of energy and momentum is governed by the covariant divergence of the stress-energy tensor:

$$\nabla_\nu T^{\mu\nu} = 0. \tag{5.35}$$

1. **Local Conservation:** Equation (5.35) is a strictly **local** conservation law. It states that energy-momentum is conserved in an infinitesimal neighborhood of every event.
2. **Failure of Naive Global Conservation:** Expanding the covariant divergence reveals Christoffel connection terms:

   $$\partial_\nu T^{\mu\nu} + \Gamma^\mu_{\alpha\nu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\nu} T^{\mu\alpha} = 0. \tag{5.36}$$

   Because of the connection terms, $\partial_\nu T^{\mu\nu} \neq 0$. Consequently, integrating T^{00} over a spacelike hypersurface Σ does **not** yield a constant conserved total energy:

   $$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma T^{00} \sqrt{|g|}\, \mathrm{d}^3x \neq 0 \quad \text{(in general)}. \tag{5.37}$$

   The physical reason is that matter exchanges energy and momentum with the dynamical gravitational field itself, and gravity does not possess a local, gauge-invariant stress-energy tensor.
3. **Requirement for Global Conservation (Killing Symmetries):** A true global conserved quantity exists if and only if spacetime admits an **isometry generated by a Killing vector field** ξ^μ ($\nabla_{(\mu}\xi_{\nu)} = 0$). In that case, the contracted current $J^\mu \equiv T^{\mu\nu}\xi_\nu$ is divergence-free ($\nabla_\mu J^\mu = 0$), yielding a globally conserved charge:

   $$Q = \int_\Sigma J^\mu \mathrm{d}\Sigma_\mu = \text{constant}. \tag{5.38}$$

5.4.3 Spacetime Foliations and Field Dynamics for Non-Inertial Observers

Key Concept: Spacetime Slicing: Inertial vs. Accelerated Observers

The dynamics of fields perceived by an observer depends essentially on how spacetime is sliced (foliated) into spatial hypersurfaces:

- **Inertial Observers:** Foliate Minkowski spacetime into flat, parallel spacelike hyperplanes of constant laboratory time: $T = \text{const}$.
- **Accelerated Observers:** Foliate spacetime into family of hyperplanes that are tilted relative to each other: $t = \text{const}$ (rays passing through the origin in the Rindler wedge).

Because the time coordinate t of the accelerated observer is non-trivially related to T, the definitions of positive and negative frequency modes in quantum field theory do not coincide. This foliation disparity is the geometric origin of the **Unruh effect**: the Minkowski vacuum state appears to a uniformly accelerated observer as a thermal bath of particles at temperature:

$$T_{\text{Unruh}} = \frac{\hbar a}{2\pi c k_B}. \tag{5.39}$$

5.4.4 Local Matter Sourcing vs. Global Gravitational Curvature

Physical Insight & Motivation: Can Attractive Gravity Exist Without Inhomogeneity? (MTW p. 40)

- **Local Sourcing:** According to Einstein's Field Equations, local matter distributions source **Ricci curvature**:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{5.40}$$

- **Vacuum Curvature:** In vacuum regions ($T_{\mu\nu} = 0$), the Ricci tensor vanishes ($R_{\mu\nu} = 0$), but tidal gravitational forces persist through the non-vanishing **Weyl conformal curvature tensor** $C_{\alpha\beta\gamma\delta}$ (e.g., around a black hole or in gravitational radiation).
- **Homogeneous Matter:** In a perfectly homogeneous universe (such as the FLRW metric), there is no localized attraction toward any center. Instead, gravity manifests globally as isotropic expansion or contraction of the entire spacetime manifold.

5.5 Integration of the Equations of Hyperbolic Motion

5.5.1 Solving for Velocity and Trajectory

We now integrate the equations of motion for an observer undergoing constant proper acceleration a.

From the proper acceleration relation Equation (5.33):

$$\frac{\mathrm{d}(\gamma v)}{\mathrm{d}T} = a. \tag{5.41}$$

Assuming the observer is instantaneously at rest at time $T = 0$ ($v(0) = 0$), we integrate directly:

$$\gamma v = aT. \tag{5.42}$$

From the relativistic momentum and energy relations:

$$p = m\gamma v = maT, \tag{5.43}$$

$$E = \sqrt{p^2c^2 + m^2c^4} = \sqrt{m^2a^2T^2c^2 + m^2c^4} = mc^2\sqrt{1 + \frac{a^2T^2}{c^2}}. \tag{5.44}$$

Since $v = \frac{c^2 p}{E}$:

$$v(T) = \frac{\mathrm{d}X}{\mathrm{d}T} = \frac{c^2(maT)}{mc^2\sqrt{1 + \frac{a^2T^2}{c^2}}} = \frac{aT}{\sqrt{1 + \frac{a^2T^2}{c^2}}}. \tag{5.45}$$

Physical Insight & Motivation: Alternative Algebraic Derivation

Alternatively, square Equation (5.42):

$$\gamma^2 v^2 = a^2T^2 \implies \frac{v^2}{1 - \frac{v^2}{c^2}} = a^2T^2. \tag{5.46}$$

Rearranging:

$$v^2 = a^2T^2\left(1 - \frac{v^2}{c^2}\right) \implies v^2\left(1 + \frac{a^2T^2}{c^2}\right) = a^2T^2 \implies v(T) = \frac{aT}{\sqrt{1 + \frac{a^2T^2}{c^2}}}. \tag{5.47}$$

Now, integrate $v(T) = \frac{\mathrm{d}X}{\mathrm{d}T}$ to obtain the spatial trajectory $X(T)$:

$$X(T) - X(0) = \int_0^T \frac{aT'}{\sqrt{1 + \frac{a^2T'^2}{c^2}}}\,\mathrm{d}T'. \tag{5.48}$$

Let $u = 1 + \frac{a^2T'^2}{c^2} \implies \mathrm{d}u = \frac{2a^2T'}{c^2}\mathrm{d}T'$:

$$\begin{aligned} X(T) - X(0) &= \frac{c^2}{2a}\int_1^{1+a^2T^2/c^2} u^{-1/2}\,\mathrm{d}u \\ &= \frac{c^2}{a}\left[\sqrt{1 + \frac{a^2T^2}{c^2}} - 1\right]. \end{aligned} \tag{5.49}$$

Setting the boundary condition such that at $T = 0$, $X(0) = \frac{c^2}{a}$:

$$X(T) = \frac{c^2}{a}\sqrt{1 + \frac{a^2T^2}{c^2}} = \sqrt{\frac{c^4}{a^2} + c^2T^2}. \tag{5.50}$$

5.5.2 The Hyperbolic Trajectory Equation

Squaring both sides of Equation (5.50):

$$X^2 = \frac{c^4}{a^2} + c^2T^2 \iff X^2 - c^2T^2 = \frac{c^4}{a^2}. \tag{5.51}$$

Postulate / Principle: Relativistic Hyperbolic Motion

In Minkowski spacetime, the worldline of a uniformly accelerated particle is a **hyperbola**:

$$X^2 - (cT)^2 = \left(\frac{c^2}{a}\right)^2. \tag{5.52}$$

Key geometric features:

1. **Light-Cone Asymptotes:** As $T \to \pm\infty$, the worldline asymptotically approaches the light-cone rays $X = \pm cT$.
2. **Turning Point / Closest Approach:** At $T = 0$, the particle is momentarily at rest at $X = \frac{c^2}{a}$.
3. **Rindler Horizon:** The null surface $X = cT$ forms a future event horizon for the accelerated observer; signals emitted from the region $X \leq cT$ (Left Wedge) can never reach the observer.

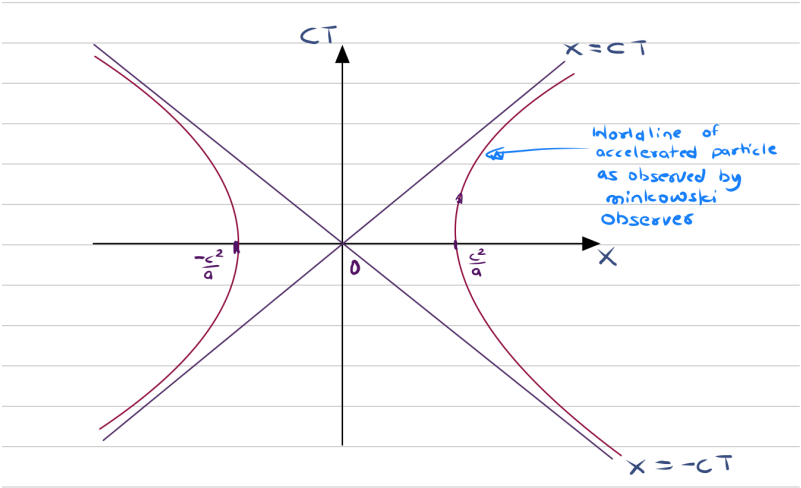


Figure 5.2: Hyperbolic worldline of a uniformly accelerated particle in Minkowski spacetime (cT, X). The asymptotes $X = \pm cT$ define the causal horizon of the accelerated observer, who remains entirely within the Right Rindler Wedge ($X > |cT|$).

5.6 Proper Time Parameterization and Radar Coordinates

5.6.1 Relationship Between Proper Time and Coordinate Time

The proper time τ experienced by the accelerated observer is given by:

$$\mathrm{d}\tau = \frac{\mathrm{d}T}{\gamma} = \sqrt{1 - \frac{v(T)^2}{c^2}}\,\mathrm{d}T. \tag{5.53}$$

Substituting $v(T) = \frac{aT}{\sqrt{1+a^2T^2/c^2}}$:

$$1 - \frac{v^2}{c^2} = 1 - \frac{\frac{a^2T^2}{c^2}}{1 + \frac{a^2T^2}{c^2}} = \frac{1}{1 + \frac{a^2T^2}{c^2}}. \tag{5.54}$$

Thus:

$$\mathrm{d}\tau = \frac{\mathrm{d}T}{\sqrt{1+\frac{a^2T^2}{c^2}}}. \tag{5.55}$$

Integrating from $T = 0$ (where $\tau = 0$):

$$\tau(T) = \int_0^T \frac{\mathrm{d}T'}{\sqrt{1+\left(\frac{aT'}{c}\right)^2}} = \frac{c}{a}\operatorname{arsinh}\left(\frac{aT}{c}\right). \tag{5.56}$$

Inverting Equation (5.56) to express the inertial coordinate time T as a function of proper time τ:

$$T(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right). \tag{5.57}$$

Substituting Equation (5.57) into Equation (5.50) for the spatial position $X(\tau)$:

$$\begin{aligned} X(\tau) &= \frac{c^2}{a}\sqrt{1+\sinh^2\left(\frac{a\tau}{c}\right)} \\ &= \frac{c^2}{a}\cosh\left(\frac{a\tau}{c}\right). \end{aligned} \tag{5.58}$$

Postulate / Principle: Parametric Hyperbolic Worldline Equations

$$cT(\tau) = \frac{c^2}{a}\sinh\left(\frac{a\tau}{c}\right), \qquad X(\tau) = \frac{c^2}{a}\cosh\left(\frac{a\tau}{c}\right). \tag{5.59}$$

These equations manifestly satisfy the hyperbola condition:

$$X^2 - (cT)^2 = \frac{c^4}{a^2}\left(\cosh^2\frac{a\tau}{c} - \sinh^2\frac{a\tau}{c}\right) = \frac{c^4}{a^2}. \tag{5.60}$$

5.6.2 Accelerated Radar Coordinates and Null Decompositions

To set up a complete curvilinear coordinate system (ct, x) attached to the accelerated observer, we employ light-cone (null) coordinates:

$$U = X - cT, \qquad V = X + cT. \tag{5.61}$$

Evaluating U and V along the hyperbolic worldline:

$$X - cT = \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) - \sinh\left(\frac{a\tau}{c}\right)\right] = \frac{c^2}{a}e^{-\frac{a\tau}{c}}, \tag{5.62}$$

$$X + cT = \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) + \sinh\left(\frac{a\tau}{c}\right)\right] = \frac{c^2}{a}e^{+\frac{a\tau}{c}}. \tag{5.63}$$

For an extended frame where x represents the spatial offset coordinate and t represents the master accelerated coordinate time, the radar-signaling protocol generalizes the null coordinates to:

$$X - cT = f_1\left(t - \frac{x}{c}\right) - cf_0\left(t - \frac{x}{c}\right) = \frac{c^2}{a}\exp\left[-\frac{a}{c}\left(t - \frac{x}{c}\right)\right], \tag{5.64}$$

$$X + cT = f_1\left(t + \frac{x}{c}\right) + cf_0\left(t + \frac{x}{c}\right) = \frac{c^2}{a}\exp\left[+\frac{a}{c}\left(t + \frac{x}{c}\right)\right]. \tag{5.65}$$

Multiplying and adding these relations yields the direct transformation between Minkowski laboratory coordinates (cT, X) and accelerated Rindler coordinates (ct, x):

$$cT = \frac{c^2}{a} e^{\frac{ax}{c^2}} \sinh\left(\frac{at}{c}\right), \tag{5.66}$$

$$X = \frac{c^2}{a} e^{\frac{ax}{c^2}} \cosh\left(\frac{at}{c}\right). \tag{5.67}$$

Physical Insight & Motivation: Towards the Rindler Metric and Curved Spacetime

Equations (5.66)–(5.67) map the entire Right Rindler Wedge ($X > |cT|$) to the flat coordinate domain $(t, x) \in \mathbb{R}^2$. Computing the pullback of the Minkowski line element $ds^2 = -c^2 dT^2 + dX^2 + dY^2 + dZ^2$ under this transformation gives:

$$ds^2 = e^{2ax/c^2}\left(-c^2 \mathrm{d}t^2 + \mathrm{d}x^2\right) + \mathrm{d}y^2 + \mathrm{d}z^2 = -\left(1 + \frac{a\bar{x}}{c^2}\right)^2 c^2 \mathrm{d}t^2 + \mathrm{d}\bar{x}^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{5.68}$$

This establishes the celebrated **Rindler metric**, which will form the foundation for our exploration of curved coordinates, gravitational redshift in accelerated frames, and the transition to Riemannian geometry in the subsequent chapters.

Chapter 6

Rindler Geometry, Causal Horizons, and the Geometrization of Relativistic Interactions

6.1 The Geometry and Metric of the Right Rindler Wedge

6.1.1 Curvilinear Transformation from Minkowski to Rindler Coordinates

In the previous chapter, we established that a uniformly accelerated observer with proper acceleration a follows a hyperbolic trajectory in Minkowski spacetime (cT, X, Y, Z). To construct a comprehensive reference frame for an accelerated laboratory, we extend this single worldline to a complete coordinate system (ct, x, y, z) covering the **Right Rindler Wedge** defined by the domain $X > |cT|$.

Definition: Rindler Coordinate Transformation

The transformation between Minkowski inertial coordinates (cT, X, Y, Z) and Rindler accelerated coordinates (ct, x, y, z) is:

$$cT = \frac{c^2}{a} e^{\frac{ax}{c^2}} \sinh\left(\frac{at}{c}\right), \tag{6.1}$$

$$X = \frac{c^2}{a} e^{\frac{ax}{c^2}} \cosh\left(\frac{at}{c}\right), \tag{6.2}$$

$$Y = y, \qquad Z = z. \tag{6.3}$$

Here:

- t is the **coordinate time** measured by a standard clock located at the reference origin $x = 0$.
- x is the **spatial coordinate** representing the position of a clock relative to the reference worldline.

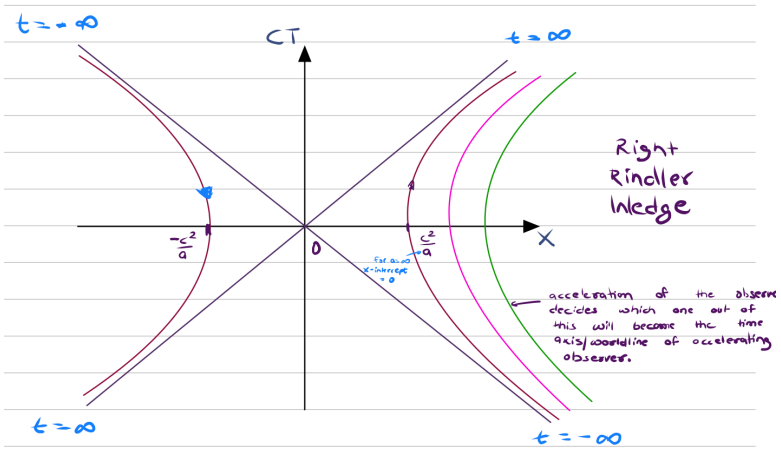


Figure 6.1: The Right Rindler Wedge ($X > |cT|$) in Minkowski spacetime. Hyperbolas of constant x represent the worldlines of stationary accelerated observers with different proper accelerations. The light cone rays $X = \pm cT$ form the asymptotes ($t \to \pm\infty$).

6.1.2 Derivation of the Rindler Metric

Let us compute the spacetime line element in the accelerated coordinate system (ct, x, y, z). Taking the differentials of Equations (6.1) and (6.2):

$$c\,\mathrm{d}T = e^{\frac{ax}{c^2}}\left[c\cosh\left(\frac{at}{c}\right)\mathrm{d}t + \sinh\left(\frac{at}{c}\right)\mathrm{d}x\right], \tag{6.4}$$

$$\mathrm{d}X = e^{\frac{ax}{c^2}}\left[c\sinh\left(\frac{at}{c}\right)\mathrm{d}t + \cosh\left(\frac{at}{c}\right)\mathrm{d}x\right]. \tag{6.5}$$

Now, evaluate the flat Minkowski line element $\mathrm{d}s^2 = -c^2\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2$:

$$\begin{aligned}
-(c\,\mathrm{d}T)^2 + \mathrm{d}X^2 &= e^{\frac{2ax}{c^2}}\left[-\left(c\cosh\frac{at}{c}\,\mathrm{d}t + \sinh\frac{at}{c}\,\mathrm{d}x\right)^2 + \left(c\sinh\frac{at}{c}\,\mathrm{d}t + \cosh\frac{at}{c}\,\mathrm{d}x\right)^2\right] \\
&= e^{\frac{2ax}{c^2}}\left[-c^2\cosh^2\left(\frac{at}{c}\right)\mathrm{d}t^2 - 2c\cosh\frac{at}{c}\sinh\frac{at}{c}\,\mathrm{d}t\,\mathrm{d}x - \sinh^2\left(\frac{at}{c}\right)\mathrm{d}x^2\right. \\
&\qquad \left. + c^2\sinh^2\left(\frac{at}{c}\right)\mathrm{d}t^2 + 2c\cosh\frac{at}{c}\sinh\frac{at}{c}\,\mathrm{d}t\,\mathrm{d}x + \cosh^2\left(\frac{at}{c}\right)\mathrm{d}x^2\right] \\
&= e^{\frac{2ax}{c^2}}\left[-c^2\left(\cosh^2\frac{at}{c} - \sinh^2\frac{at}{c}\right)\mathrm{d}t^2 + \left(\cosh^2\frac{at}{c} - \sinh^2\frac{at}{c}\right)\mathrm{d}x^2\right].
\end{aligned} \tag{6.6}$$

Using the hyperbolic identity $\cosh^2\theta - \sinh^2\theta \equiv 1$:

$$\mathrm{d}s^2 = e^{\frac{2ax}{c^2}}\left(-c^2\mathrm{d}t^2 + \mathrm{d}x^2\right) + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{6.7}$$

6.1.3 Standard Form via Proper Spatial Coordinate

We can define a new spatial coordinate $\bar{x}$ via the monotonic transformation:

$$1 + \frac{a\bar{x}}{c^2} \equiv e^{\frac{ax}{c^2}} \iff \mathrm{d}\bar{x} = e^{\frac{ax}{c^2}}\mathrm{d}x. \tag{6.8}$$

Substituting Equation (6.8) into Equation (6.7) yields the **standard Rindler metric**:

Postulate / Principle: Standard Rindler Metric

$$\mathrm{d}s^2 = -\left(1+\frac{a\bar{x}}{c^2}\right)^2 c^2\mathrm{d}t^2 + \mathrm{d}\bar{x}^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = g_{\mu\nu}(\bar{x})\,\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{6.9}$$

Physical Insight & Motivation: Physical Meaning of the Metric Components

1. **Proper Acceleration Gradient:** An observer held at a fixed spatial coordinate $\bar{x} = \text{const}$ experiences a proper acceleration:

$$a_{\text{proper}}(\bar{x}) = \frac{a}{1+\frac{a\bar{x}}{c^2}}. \tag{6.10}$$

   Clocks closer to the horizon ($\bar{x} \to -c^2/a$) must accelerate at higher rates, approaching infinity at the horizon itself.

2. **Gravitational Time Dilation / Redshift:** The proper time $\mathrm{d}\tau$ elapsed on a clock at fixed $\bar{x}$ is related to coordinate time $\mathrm{d}t$ by:

$$\mathrm{d}\tau = \sqrt{-g_{00}}\,\mathrm{d}t = \left(1+\frac{a\bar{x}}{c^2}\right)\mathrm{d}t. \tag{6.11}$$

   A clock at a higher position ($\bar{x}_2 > \bar{x}_1$) runs faster than a clock at a lower position:

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \frac{1+\frac{a\bar{x}_2}{c^2}}{1+\frac{a\bar{x}_1}{c^2}} \approx 1 + \frac{a(\bar{x}_2-\bar{x}_1)}{c^2} = 1+\frac{g\Delta h}{c^2}. \tag{6.12}$$

   This is Einstein's celebrated **gravitational redshift formula**, derived purely from Special Relativity and acceleration kinematics!

6.2 Causal Horizons, Foliations, and Microcausality

6.2.1 Spacetime Foliation and Horizon Structure

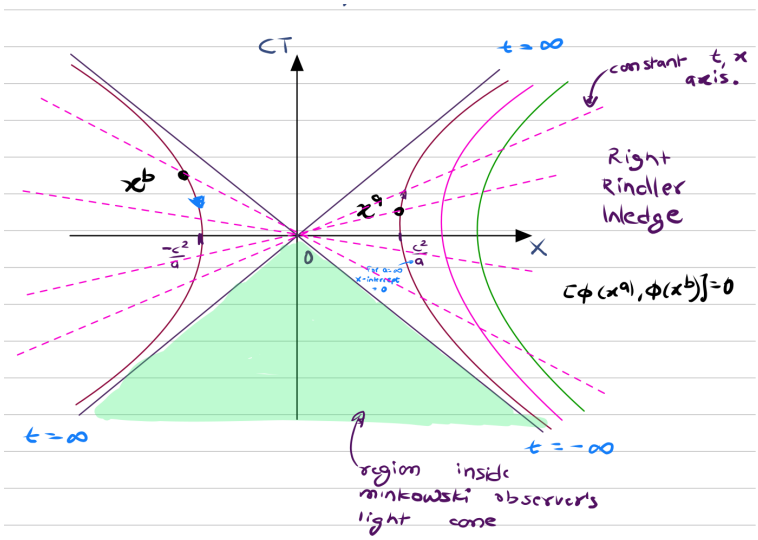


Figure 6.2: Spacetime structure of the Right Rindler Wedge ($X > |cT|$). Dashed magenta lines through the origin represent hypersurfaces of constant accelerated time $t = \text{const}$. The green shaded region is inside the past light cone. Events x^a and x^b are spacelike separated across the Rindler horizon, ensuring the field commutator vanishes: $[\phi(x^a), \phi(x^b)] = 0$.

Key Concept: Why an Accelerated Frame is a "Collection of Frames"

An accelerating reference frame cannot be described by a single inertial frame:

- **Foliation by Simultaneous Hypersurfaces:** The lines of simultaneity for the accelerated observer are straight rays radiating from the origin:

$$\frac{cT}{X} = \tanh\left(\frac{at}{c}\right) = \text{constant for each } t. \tag{6.13}$$

  These hyperplanes are tilted at continuously varying boost angles.

- **Born Rigidity and Bell's Spaceship Paradox:** To maintain a constant proper distance (Born rigidity) between two points $\bar{x}_1$ and $\bar{x}_2$, their proper accelerations must differ ($a_1 \neq a_2$). An accelerated rigid body is an entire congruence (collection) of worldlines with a spatial gradient of proper accelerations.

6.2.2 Causal Horizon and Information Loss

Key Concept: Acceleration Creates a Causal Horizon

The null surfaces $X = \pm cT$ act as event horizons for any observer confined to the Right Rindler Wedge:

1. **Future Horizon ($\mathcal{H}^+$):** The ray $X = cT$ ($T > 0$). Any signal emitted from the Left Wedge ($X < -|cT|$) can never cross into the Right Wedge to reach the accelerated observer.

2. **Information Barrier:** An accelerated observer is causally disconnected from half of Minkowski spacetime. Events beyond the horizon are permanently hidden.

3. **Microcausality:** For two events $x^a \in$ Right Wedge and $x^b \in$ Left Wedge, their spacetime interval is strictly spacelike ($s^2_{ab} > 0$). In quantum field theory, local operators at these events commute:

$$\left[\hat{\phi}(x^a),\, \hat{\phi}(x^b)\right] = 0. \tag{6.14}$$

4. **Thermal Density Matrix (Unruh Effect):** Because the observer must trace over all quantum states behind the horizon (the Left Wedge degrees of freedom), a pure Minkowski vacuum state $|0\rangle_{\text{Minkowski}}$ reduces to a mixed thermal density matrix $\hat{\rho}_{\text{Rindler}} = \frac{1}{Z} e^{-\hat{H}_R/k_B T_{\text{Unruh}}}$.

6.3 Systematic Construction of Relativistic Interactions

6.3.1 Why Potentials are More Fundamental Than Forces

In relativistic physics, force $F^i = \frac{\mathrm{d}P^i}{\mathrm{d}\tau}$ is a 4-vector, but trying to postulate force laws directly leads to severe algebraic inconsistencies (such as $a^i u_i = 0$).

Key Concept: The Primacy of the Potential / Action Principle

The fundamental description of interactions in modern theoretical physics must be formulated through **Lagrangian actions and gauge/metric potentials**:

- The equations of motion and force terms emerge automatically as on-shell Euler–Lagrange variations $\delta S = 0$.
- Constraints like 4-velocity orthogonality ($a^i u_i = 0$) are guaranteed by the symmetries of the action.

6.3.2 Derivative Expansion of the Relativistic Lagrangian

Consider an interacting relativistic particle with coordinates $x^a(\tau)$ and 4-velocity $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$. The most general reparameterization-invariant worldline action is:

$$S = -\int L(x^a, u^a)\, \mathrm{d}\tau. \tag{6.15}$$

Performing a systematic polynomial (derivative) expansion of $L(x^a, u^a)$ in powers of 4-velocity u^a:

$$L(x^a, u^a) = L_0(x) + c_1 u^a A_a(x) + c_2 u^a u^b g_{ab}(x) + \dots \tag{6.16}$$

Physical Insight & Motivation: Classification of Fundamental Relativistic Interactions

Each term in Equation (6.16) generates a distinct physical interaction:

- **Rank-0 (Scalar Field $\phi(x)$):** $L_0(x) = mc^2 + \lambda\phi(x)$. Couples to the particle's rest mass (Nordström scalar gravity, Higgs-type scalar interactions).
- **Rank-1 (Vector Gauge Field $A_a(x)$):** $u^a A_a(x)$. Couples to conserved 4-current

$J^a = qu^a$ (Maxwell's Electromagnetism).

- **Rank-2 (Symmetric Tensor Metric $g_{ab}(x)$):** $u^a u^b g_{ab}(x)$. Couples directly to the particle's 4-momentum and stress-energy (Einstein's General Relativity).

6.4 Dynamics of a Relativistic Particle in a Scalar Potential

6.4.1 The Scalar Interaction Action and Non-Relativistic Limit

Let us investigate the simplest non-trivial relativistic interaction: a particle coupled to a spacetime scalar field $\phi(x)$.

The action of the particle is:

$$\begin{aligned} S_{\text{particle}} &= -\int \left[mc^2 + \lambda\phi(x)\right] \mathrm{d}\tau \\ &= -\int \left[mc^2 + \lambda\phi(x)\right] \sqrt{1 - \frac{v^2}{c^2}}\, \mathrm{d}t. \end{aligned} \tag{6.17}$$

The mechanical Lagrangian $L_{\text{mech}}(t)$ defined such that $S = \int L_{\text{mech}}\, \mathrm{d}t$ is:

$$L_{\text{mech}} = -\left[mc^2 + \lambda\phi(\vec{x}, t)\right] \sqrt{1 - \frac{v^2}{c^2}}. \tag{6.18}$$

Physical Insight & Motivation: Non-Relativistic Expansion

Expanding L_{mech} for low velocities ($v \ll c$):

$$\begin{aligned} L_{\text{mech}} &\approx -\left(mc^2 + \lambda\phi\right)\left(1 - \frac{v^2}{2c^2} + \dots\right) \\ &= -mc^2 - \lambda\phi + \frac{1}{2}mv^2 + \frac{\lambda\phi v^2}{2c^2} + \dots \\ &= -mc^2 + \left[\frac{1}{2}mv^2 - \lambda\phi(\vec{x})\right] + \mathcal{O}\left(\frac{v^2}{c^2}\right). \end{aligned} \tag{6.19}$$

Discarding the constant rest energy $-mc^2$, we recover the standard Newtonian Lagrangian $L = T - V$:

$$L_{\text{classical}} = \frac{1}{2}mv^2 - V(\vec{x}), \qquad \text{where } V(\vec{x}) \equiv \lambda\phi(\vec{x}). \tag{6.20}$$

6.4.2 Complete Coupled Action (Particle + Dynamical Field)

To make the scalar field dynamical, we add the Klein–Gordon field action:

$$S_{\text{total}} = S_{\text{particle}} + S_\phi = -\int \left[mc^2 + \lambda\phi\right] \mathrm{d}\tau + \int \mathrm{d}^4x \left(\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}m_\phi^2\phi^2\right). \tag{6.21}$$

Physical Insight & Motivation: Physical Stability: Why the Minus Sign for Mass Term?

In the field Lagrangian density $\mathcal{L}_\phi = \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}m_\phi^2\phi^2$:

- The Hamiltonian density is $\mathcal{H}_\phi = \frac{1}{2c^2}(\partial_t\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m_\phi^2\phi^2 \geq 0$.
- The negative sign in the Lagrangian gives a strictly **positive-definite Hamiltonian**, ensuring vacuum stability (no tachyonic runaway modes).

The Euler–Lagrange variation $\frac{\delta S_{\text{total}}}{\delta\phi} = 0$ yields the sourced Klein–Gordon equation:

$$\Box\phi - m_\phi^2\phi = \lambda\rho_m(x) \xrightarrow{\text{static, massless}} \nabla^2\phi = 4\pi G\rho_m. \tag{6.22}$$

6.4.3 Covariant Worldline Variation of the Particle Trajectory

We now perform a detailed covariant variation $x^a(\tau) \to x^a(\tau) + \delta x^a(\tau)$ of the particle action:

$$S = -\int \left[mc^2 + \lambda\phi(x)\right] \mathrm{d}\tau. \tag{6.23}$$

The first variation is:

$$\delta S = -\int \left(mc^2 + \lambda\phi\right) \delta(\mathrm{d}\tau) - \lambda\int \delta\phi(x)\,\mathrm{d}\tau. \tag{6.24}$$

Recall that the proper time interval is defined by $c^2\mathrm{d}\tau^2 = -\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. Taking the variation:

$$2c^2\mathrm{d}\tau\delta(\mathrm{d}\tau) = -2\eta_{ij}\mathrm{d}x^i\delta(\mathrm{d}x^j) = -2\eta_{ij}\mathrm{d}x^i\mathrm{d}(\delta x^j). \tag{6.25}$$

Dividing by $2c^2\mathrm{d}\tau$:

$$\delta(\mathrm{d}\tau) = -\frac{\eta_{ij}\mathrm{d}x^i}{c^2\mathrm{d}\tau}\mathrm{d}(\delta x^j) = -\frac{u_j}{c^2}\mathrm{d}(\delta x^j). \tag{6.26}$$

Furthermore, the field variation along the worldline is:

$$\delta\phi(x) = \frac{\partial\phi}{\partial x^i}\delta x^i = (\partial_i\phi)\delta x^i. \tag{6.27}$$

Substituting these expressions into Equation (6.24):

$$\delta S = \int \left(mc^2 + \lambda\phi\right)\frac{u_i}{c^2}\mathrm{d}(\delta x^i) - \lambda\int(\partial_i\phi)\delta x^i\,\mathrm{d}\tau. \tag{6.28}$$

Integrating the first term by parts:

$$\int\left(m + \frac{\lambda\phi}{c^2}\right)u_i\,\mathrm{d}(\delta x^i) = \left[\left(m + \frac{\lambda\phi}{c^2}\right)u_i\delta x^i\right]_{\tau_1}^{\tau_2} - \int\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(m + \frac{\lambda\phi}{c^2}\right)u_i\right]\delta x^i\,\mathrm{d}\tau. \tag{6.29}$$

Setting fixed endpoint variations $\delta x^i(\tau_1) = \delta x^i(\tau_2) = 0$, the condition $\delta S = 0$ for arbitrary $\delta x^i(\tau)$ gives the **Euler–Lagrange Equations of Motion**:

Postulate / Principle: Relativistic Particle Equation of Motion in a Scalar Field

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(m + \frac{\lambda\phi}{c^2}\right)u_i\right] + \lambda\partial_i\phi = 0. \tag{6.30}$$

6.4.4 Canonical Momentum and Effective Variable Mass

From the boundary term of the variation, the on-shell canonical 4-momentum is:

$$P_i \equiv \left.\frac{\partial S}{\partial x^i}\right|_{\text{on-shell}} = \left(m + \frac{\lambda\phi(x)}{c^2}\right) u_i. \tag{6.31}$$

Computing the Lorentz norm of the 4-momentum:

$$P_i P^i = \eta^{ij} P_i P_j = \left(m + \frac{\lambda\phi}{c^2}\right)^2 u_i u^i = -c^2 \left(m + \frac{\lambda\phi(x)}{c^2}\right)^2. \tag{6.32}$$

Key Concept: Space-Dependent Effective Mass

In the presence of a scalar field, the particle behaves as if its rest mass is position-dependent:

$$m_{\text{eff}}(x) \equiv m + \frac{\lambda\phi(x)}{c^2}. \tag{6.33}$$

The mass-shell relation becomes:

$$P_i P^i = -m_{\text{eff}}^2(x) c^2. \tag{6.34}$$

6.5 Resolution of the Relativistic Force Paradox

6.5.1 Derivation of the 4-Acceleration

Let us now expand the total derivative in Equation (6.30):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(m + \frac{\lambda\phi}{c^2}\right) u_i\right] + \lambda\partial_i\phi = 0 \\ \left(m + \frac{\lambda\phi}{c^2}\right)\frac{\mathrm{d}u_i}{\mathrm{d}\tau} + \frac{\lambda}{c^2}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}u_i + \lambda\partial_i\phi = 0. \end{aligned} \tag{6.35}$$

Using $\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\partial\phi}{\partial x^j} = u^j\partial_j\phi$:

$$\left(m + \frac{\lambda\phi}{c^2}\right) a_i + \frac{\lambda}{c^2}(u^j\partial_j\phi)u_i + \lambda\partial_i\phi = 0. \tag{6.36}$$

Solving for the 4-acceleration $a^i = \frac{\mathrm{d}u^i}{\mathrm{d}\tau}$:

Postulate / Principle: 4-Acceleration in a Scalar Field

$$a^i = -\frac{\lambda}{m + \frac{\lambda\phi}{c^2}}\partial^i\phi - \frac{\lambda/c^2}{m + \frac{\lambda\phi}{c^2}}\left(u^j\partial_j\phi\right)u^i. \tag{6.37}$$

6.5.2 Exact Satisfaction of Orthogonality ($a^i u_i \equiv 0$)

Proof. Let us contract Equation (6.37) with u_i:

$$\begin{aligned} a^i u_i &= -\frac{\lambda}{m + \frac{\lambda\phi}{c^2}}(\partial^i\phi u_i) - \frac{\lambda/c^2}{m + \frac{\lambda\phi}{c^2}}(u^j\partial_j\phi)(u^i u_i) \\ &= -\frac{\lambda}{m + \frac{\lambda\phi}{c^2}}(u^j\partial_j\phi) - \frac{\lambda/c^2}{m + \frac{\lambda\phi}{c^2}}(u^j\partial_j\phi)(-c^2) \end{aligned}$$

$$= -\frac{\lambda}{m + \frac{\lambda\phi}{c^2}}(u^j\partial_j\phi) + \frac{\lambda}{m + \frac{\lambda\phi}{c^2}}(u^j\partial_j\phi) \equiv 0. \tag{6.38}$$

□

Key Concept: How the Action Principle Rescues Relativistic Dynamics

On Page 31, we showed that the naive force equation $ma^i = -\partial^i\Phi$ failed because $a^i u_i = 0$ forced $\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} = 0$.
Why does Equation (6.37) succeed?

- The rigorous variational derivation produced a **second, velocity-dependent projection term**:

$$a^i = -\frac{\lambda}{m + \frac{\lambda\phi}{c^2}}\mathcal{P}^i{}_j\partial^j\phi, \tag{6.39}$$

  where $\mathcal{P}^i{}_j \equiv \delta^i{}_j + \frac{1}{c^2}u^i u_j$ is the **orthogonal projection tensor** onto the 3-space perpendicular to u^i ($\mathcal{P}^i{}_j u_i \equiv 0$).

- The projection tensor automatically annihilates the component parallel to u^i, ensuring $a^i u_i \equiv 0$ holds identically for **any** arbitrary scalar field $\phi(x)$!

6.6 The Equivalence Principle and Emergence of Spacetime Metric

6.6.1 Universality of Free Fall and Coupling Constant

In the non-relativistic limit ($v \ll c$), the spatial 3-acceleration reduces to:

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\lambda\vec{\nabla}\phi \implies \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\left(\frac{\lambda}{m}\right)\vec{\nabla}\phi. \tag{6.40}$$

Postulate / Principle: Galileo's Universality of Free Fall / Weak Equivalence Principle

All test bodies fall with the exact same acceleration in a gravitational field, regardless of their mass, structure, or composition:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \text{independent of } m \implies \frac{\lambda}{m} = \text{universal constant } \ell. \tag{6.41}$$

Therefore, the gravitational coupling constant λ must be strictly proportional to inertial mass:

$$\lambda = m\ell \iff m_{\text{gravitational}} = m_{\text{inertial}}. \tag{6.42}$$

6.6.2 Cancellation of Mass and Geometrized Trajectories

Defining the dimensionless gravitational potential:

$$\bar{\Phi}(x) \equiv \ell\phi(x), \tag{6.43}$$

the acceleration equation (6.37) becomes:

$$\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = -\frac{m\ell}{m\left(1 + \frac{\ell\phi}{c^2}\right)}\partial^i\phi - \frac{m\ell/c^2}{m\left(1 + \frac{\ell\phi}{c^2}\right)}(u^j\partial_j\phi)u^i$$

$$= -\frac{1}{1+\frac{\bar{\Phi}}{c^2}}\partial^i\bar{\Phi} - \frac{1/c^2}{1+\frac{\bar{\Phi}}{c^2}}(u^j\partial_j\bar{\Phi})u^i. \tag{6.44}$$

Key Concept: Mass Independence: Trajectories are Pure Geometry

In Equation (6.44), the particle mass m has **completely vanished**! The trajectory of a particle in a gravitational field is a universal property of spacetime itself, independent of the mass or composition of the falling object.

6.6.3 The Conformal Metric Formulation (Nordström's Gravity)

Let us rewrite the particle action using $\lambda = m\ell$ and $\bar{\Phi} = \ell\phi$:

$$\begin{aligned} S &= -\int \left(mc^2 + m\bar{\Phi}(x)\right) \mathrm{d}\tau \\ &= -mc^2 \int \left(1 + \frac{\bar{\Phi}(x)}{c^2}\right) \mathrm{d}\tau \\ &= -mc \int \left(1 + \frac{\bar{\Phi}(x)}{c^2}\right) \sqrt{-\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j}. \end{aligned} \tag{6.45}$$

Can we absorb the interaction potential $\bar{\Phi}(x)$ directly into the geometry of spacetime? Define an **effective gravitational metric** $\mathrm{d}s^2_\Phi$:

$$S \equiv -mc \int \sqrt{-\mathrm{d}s^2_\Phi}, \tag{6.46}$$

where:

$$\mathrm{d}s^2_\Phi \equiv \left(1 + \frac{\bar{\Phi}(x)}{c^2}\right)^2 \eta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = g_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j. \tag{6.47}$$

In coordinate components:

$$\mathrm{d}s^2_\Phi = -\left(1 + \frac{\bar{\Phi}}{c^2}\right)^2 c^2\mathrm{d}t^2 + \left(1 + \frac{\bar{\Phi}}{c^2}\right)^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{6.48}$$

Postulate / Principle: The Geometrization Principle (Nordström / Einstein)

The motion of a particle under relativistic scalar gravity is mathematically identical to **free geodesic motion in a conformally flat curved spacetime**:

$$g_{ij}(x) = \Omega^2(x)\eta_{ij}, \qquad \Omega(x) \equiv 1 + \frac{\bar{\Phi}(x)}{c^2}. \tag{6.49}$$

Physical Insight & Motivation: Why Scalar Gravity Fails and Tensor Gravity (GR) is Required

Although Gunnar Nordström's conformally flat theory (1913) was the first mathematically consistent relativistic theory of gravitation satisfying the Equivalence Principle, it has one fatal flaw:

- **Light Trajectories are Null ($\mathrm{d}s^2 = 0$):** For photons, $\mathrm{d}s^2_\Phi = \Omega^2(x)\mathrm{d}s^2_{\text{Minkowski}} = 0 \iff \mathrm{d}s^2_{\text{Minkowski}} = 0$.

- Because null geodesics are conformally invariant, **scalar gravity predicts zero gravitational deflection of light**!
- The 1919 Eddington solar eclipse expedition measured a deflection of $\Delta\theta = \frac{4GM}{c^2 R_\odot}$, confirming **Einstein's rank-2 tensor General Relativity** where the metric is not conformally flat and both time and space components of the metric mediate gravitational bending.

Chapter 7

The Equivalence Principle, Local Inertial Frames, and the Geometry of the Metric Tensor

7.1 The Duality Between Interacting Particles and Curved Spacetime

7.1.1 Weak Gravitational Fields as Spacetime Metric Perturbations

In Chapter 6, we discovered a remarkable duality: the dynamics of a relativistic particle interacting with a gravitational potential $\Phi(x)$ in flat Minkowski spacetime can be recast identically as free geodesic motion in a modified background metric:

$$S = -mc \int \sqrt{-g_{\mu\nu}(x)\, \mathrm{d}x^\mu \mathrm{d}x^\nu}. \tag{7.1}$$

In the **weak-field limit** ($|\Phi| \ll c^2$), we expand the line element to first order in $\frac{\Phi}{c^2}$:

$$\begin{aligned} \mathrm{d}s_\Phi^2 &= -\left(1 + \frac{\Phi}{c^2}\right)^2 c^2 \mathrm{d}t^2 + \left(1 + \frac{\Phi}{c^2}\right)^2 |\mathrm{d}\vec{x}|^2 \\ &\approx -\left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 + \left(1 + \frac{2\Phi}{c^2}\right) |\mathrm{d}\vec{x}|^2. \end{aligned} \tag{7.2}$$

In Newtonian gravity, spatial gradients of the potential generate forces, while spatial metric perturbations $|\mathrm{d}\vec{x}|^2$ produce negligible corrections for non-relativistic particles ($v \ll c$). Hence, the effective Newtonian line element is:

$$\mathrm{d}s^2 \approx -\left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 + \mathrm{d}\vec{x}^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, \mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{7.3}$$

where $h_{\mu\nu}(x)$ is the linear metric perturbation whose dominant temporal component is:

$$h_{00} = -\frac{2\Phi(\vec{x})}{c^2}. \tag{7.4}$$

7.1.2 Comparison with Uniformly Accelerated Frames

Now consider the line element of an observer undergoing uniform proper acceleration a along the $\bar{x}$-axis, derived in Chapter 6:

$$\mathrm{d}s^2_{\text{accelerated}} = -\left(1 + \frac{a\bar{x}}{c^2}\right)^2 c^2 \mathrm{d}t^2 + \mathrm{d}\bar{x}^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

$$\approx -\left(1+\frac{2a\bar{x}}{c^2}\right)c^2\mathrm{d}t^2+\mathrm{d}\bar{x}^2+\mathrm{d}y^2+\mathrm{d}z^2. \tag{7.5}$$

Comparing the gravitational line element (7.3) with the accelerated line element (7.5):

$$\Phi(\bar{x})=a\bar{x} \implies \vec{\nabla}\Phi=a\,\hat{x} \iff \vec{a}_{\text{eff}}=-\vec{\nabla}\Phi. \tag{7.6}$$

Key Concept: Newtonian vs. Einsteinian Paradigms of Gravitation

- **Newtonian Description:** Gravity is an external physical interaction mediated by a scalar field $\Phi(\vec{x})$. Bodies accelerate in flat Euclidean space under a force $\vec{F}=-m\vec{\nabla}\Phi$.
- **Einsteinian Description:** Gravity is **not a force**. Spacetime is curved with metric $g_{\mu\nu}(x)$. Bodies simply follow natural, force-free geodesic trajectories ($\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}=0$).

7.1.3 Transforming Away Gravity: The Freely Falling Frame

Suppose the acceleration $a=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ is constant and uniform throughout a local region. The trajectory of a particle starting from rest is:

$$\bar{x}(t)=\frac{1}{2}\frac{m_g}{m_I}gt^2. \tag{7.7}$$

Let us define a new coordinate system $\xi(t)$ attached to a **freely falling observer**:

$$\xi=\bar{x}-\frac{1}{2}gt^2. \tag{7.8}$$

Differentiating with respect to time:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}=\frac{m_g}{m_I}g-g. \tag{7.9}$$

Thus,

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}=\left(\frac{m_g}{m_I}-1\right)g=0 \tag{7.10}$$

only because

$$m_g=m_I. \tag{7.11}$$

Postulate / Principle: Einstein's Equivalence Principle (Weak Version)

In a sufficiently small, localized region of spacetime, the physical effects of a uniform gravitational field are **completely indistinguishable** from the effects of being in a uniformly accelerated reference frame. Consequently:

$$\text{Gravity can be locally transformed away by choosing a freely falling frame!} \tag{7.12}$$

7.2 Algebraic Structure and Degrees of Freedom of the Metric

7.2.1 Symmetry of the Metric Tensor

The invariant spacetime line element is defined quadratic in the coordinate differentials:

$$\mathrm{d}s^2=g_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j. \tag{7.13}$$

Any rank-2 tensor g_{ij} can be uniquely decomposed into symmetric and antisymmetric parts:

$$g_{ij} = g_{(ij)} + g_{[ij]}, \qquad g_{(ij)} = \frac{1}{2}\left(g_{ij} + g_{ji}\right), \quad g_{[ij]} = \frac{1}{2}\left(g_{ij} - g_{ji}\right). \tag{7.14}$$

Contracting with the symmetric product of differentials $\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^j\mathrm{d}x^i$:

$$g_{[ij]}\,\mathrm{d}x^i\mathrm{d}x^j \equiv 0. \tag{7.15}$$

Because the antisymmetric part $g_{[ij]}$ makes zero contribution to physical proper intervals, the metric tensor of spacetime is defined to be **strictly symmetric**:

$$g_{ij}(x) = g_{ji}(x). \tag{7.16}$$

7.2.2 Counting Independent Metric Components

Definition: Independent Components in N Dimensions

In an N-dimensional spacetime manifold, a symmetric rank-2 tensor $g_{\mu\nu}$ possesses:

$$\mathcal{N}_{\text{metric}} = \frac{N(N+1)}{2} \quad \text{independent components.} \tag{7.17}$$

For physical 4-dimensional spacetime ($N = 4$):

$$\mathcal{N}_{\text{metric}} = \frac{4(4+1)}{2} = 10 \quad \text{independent metric functions } g_{\mu\nu}(x). \tag{7.18}$$

Key Concept: Why Flat Spacetime and Scalar Theories Cannot Capture True Gravity

- **Minkowski Metric ($\eta_{\mu\nu}$):** Has only 4 non-zero constant components $(-1, +1, +1, +1)$.
- **Scalar Gravitational Theory:** Has only 1 dynamical degree of freedom ($\phi(x)$) superimposed on flat spacetime (total 5 DOF).
- **General Relativity:** The metric tensor $g_{\mu\nu}(x)$ contains **10 dynamical spacetime functions**. Spacetime itself becomes an active, multi-component dynamical physical entity capable of shear, twist, expansion, and transverse-traceless gravitational waves (2 physical polarizations).

7.3 Construction of Local Inertial Coordinates (Riemann Normal Coordinates)

Can we always find a coordinate system in which gravity vanishes at an arbitrary event P? How much of the metric can be reduced to Minkowski space?

7.3.1 Taylor Series Coordinate Transformation

Let x^i be an arbitrary coordinate system around event P, with P situated at $x^i = 0$. We perform a general non-linear coordinate transformation $x^i \to x'^i(x)$:

$$x'^i(x) = B^i{}_k x^k + \frac{1}{2}C^i{}_{jk}x^j x^k + \frac{1}{6}D^i{}_{\ell mn}x^\ell x^m x^n + \mathcal{O}(x^4), \tag{7.19}$$

where without loss of generality the coefficients are totally symmetric in lower indices:

$$C^i{}_{jk} = C^i{}_{kj}, \qquad D^i{}_{\ell mn} = D^i{}_{(\ell mn)}. \tag{7.20}$$

Under this transformation, the contravariant inverse metric transforms as:

$$g'^{ij}(x') = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n} g^{mn}(x). \tag{7.21}$$

Evaluating the Jacobian matrix:

$$\frac{\partial x'^i}{\partial x^m} = B^i{}_m + C^i{}_{mk}x^k + \frac{1}{2}D^i{}_{m\ell n}x^\ell x^n + \dots \tag{7.22}$$

7.3.2 Order-by-Order Matching of Metric Components

Order 0: Setting the Metric to Flat Minkowski Space at P

At the origin P ($x = 0$):

$$g'^{ij}(P) = B^i{}_m B^j{}_n g^{mn}(P) \stackrel{!}{=} \eta^{ij}. \tag{7.23}$$

Let us count parameters in $N = 4$ dimensions:

- The matrix $B^i{}_m$ has $N^2 = 4^2 = 16$ free parameters.
- Demanding $g'_{ij}(P) = \eta_{ij}$ imposes $\frac{N(N+1)}{2} = 10$ independent algebraic constraints.
- Surplus degrees of freedom:

$$N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2} = \frac{4\times 3}{2} = 6. \tag{7.24}$$

Physical Insight & Motivation: Physical Meaning of the 6 Surplus Degrees of Freedom

The 6 remaining unconstrained parameters in $B^i{}_m$ correspond precisely to the **Lorentz Group** $\mathrm{SO}(1,3)$ at event P:

$$6 \text{ Degrees of Freedom} = 3 \text{ Spatial Rotations} + 3 \text{ Lorentz Boosts}. \tag{7.25}$$

This confirms that the local frame at P is determined up to an arbitrary choice of spatial orientation and inertial velocity!

Order 1: Eliminating All First Derivatives (Vanishing Christoffel Symbols)

Differentiating Equation (7.21) and evaluating at $x = 0$, we demand that all first derivatives of the metric vanish at P:

$$\partial_k g'_{ij}(P) \stackrel{!}{=} 0. \tag{7.26}$$

Let us count parameters:

- The coefficients $C^i{}_{jk}$ have N upper indices and $\frac{N(N+1)}{2}$ symmetric lower index pairs (j,k):

$$\mathcal{N}_{\text{coeffs}}(C) = N \times \frac{N(N+1)}{2} = \frac{N^2(N+1)}{2}. \tag{7.27}$$

  For $N = 4$: $\mathcal{N}_{\text{coeffs}} = 4 \times 10 = 40$ available parameters.

- The condition $\partial_k g'_{ij}(P) = 0$ requires setting N directional derivatives of $\frac{N(N+1)}{2}$ independent metric components to zero:

$$\mathcal{N}_{\text{constraints}} = N \times \frac{N(N+1)}{2} = \frac{N^2(N+1)}{2}. \tag{7.28}$$

  For $N = 4$: $\mathcal{N}_{\text{constraints}} = 4 \times 10 = 40$ equations.

Postulate / Principle: Existence of Local Inertial Frames

Because the number of available transformation coefficients ${C^i}_{jk}$ exactly equals the number of constraints ($40 = 40$), we can **always** uniquely choose ${C^i}_{jk}$ such that:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}, \qquad \partial_\lambda g_{\mu\nu}(P) = 0 \iff \Gamma^\lambda_{\mu\nu}(P) = 0. \tag{7.29}$$

In this coordinate system, called a **Local Inertial Frame (LIF)** or **Riemann Normal Coordinates**, all gravitational forces vanish at point P.

Order 2: Ineliminable Second Derivatives and Spacetime Curvature

Can we proceed to second derivatives and demand $\partial_k \partial_\ell g'_{ij}(P) \stackrel{!}{=} 0$? Let us count:

- The transformation coefficients ${D^i}_{\ell mn}$ are symmetric in 3 lower indices:

$$\mathcal{N}_{\text{coeffs}}(D) = N \times \binom{N+2}{3} = N \times \frac{N(N+1)(N+2)}{6} = \frac{N^2(N+1)(N+2)}{6}. \tag{7.30}$$

  For $N = 4$: $\mathcal{N}_{\text{coeffs}} = \frac{16\times5\times6}{6} = 80$ available parameters.

- The number of independent second derivatives $\partial_k \partial_\ell g'_{ij}$ is the product of symmetric derivative pairs $\frac{N(N+1)}{2}$ and symmetric metric components $\frac{N(N+1)}{2}$:

$$\mathcal{N}_{\text{constraints}} = \left[\frac{N(N+1)}{2}\right]^2. \tag{7.31}$$

  For $N = 4$: $\mathcal{N}_{\text{constraints}} = 10 \times 10 = 100$ equations.

Subtracting the available coefficients from the constraints:

$$\mathcal{N}_{\text{curvature}} = \left[\frac{N(N+1)}{2}\right]^2 - \frac{N^2(N+1)(N+2)}{6} = \frac{1}{12}N^2(N^2-1). \tag{7.32}$$

For 4-dimensional spacetime ($N = 4$):

$$\mathcal{N}_{\text{curvature}} = \frac{1}{12}(16)(15) = 20. \tag{7.33}$$

Key Concept: The Origin of the Riemann Curvature Tensor

- 80 of the 100 second derivatives of the metric can be transformed to zero by a clever choice of coordinates.
- The remaining **20 irreducible second derivatives cannot be eliminated by any coordinate transformation!**
- These 20 numbers constitute the **Riemann Curvature Tensor** ${R^\rho}_{\sigma\mu\nu}(P)$.
- Physical consequence: **Tidal gravitational forces and spacetime curvature are coordinate-independent physical reality.** While uniform acceleration can be transformed away locally, true tidal curvature ($R_{\mu\nu\rho\sigma} \neq 0$) can never be eliminated!

7.4 Physical Meaning and Consequences of Local Inertial Frames

Physical Insight & Motivation: What Does an Observer Observe Beyond the Local Patch?

- **Inside the Local Patch ($x \approx 0$):** Special Relativity holds to high precision ($g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2)$). Free particles move along straight lines, and local physics obeys standard SR conservation laws.
- **Beyond the Local Patch ($x > L_{\text{curvature}}$):** Second-order tidal effects become noticeable ($\Delta a \sim R^i{}_{0j0}x^j$). In cosmology, an astronomer observes distant galaxies receding in accordance with cosmic expansion, which cannot be transformed away globally.

Postulate / Principle: The Strong Equivalence Principle and Minimal Coupling Rule

Any physical law valid in Special Relativity can be promoted to General Relativity in curved spacetime via the **Minimal Coupling Prescription** (the comma-goes-to-semicolon rule):

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}(x), \qquad \partial_\mu \longrightarrow \nabla_\mu. \tag{7.34}$$

In a Local Inertial Frame, $\nabla_\mu \to \partial_\mu$ and $g_{\mu\nu} \to \eta_{\mu\nu}$, automatically recovering standard flat-space physics.

7.5 Tensor Transformation Laws: The Metric and Kronecker Delta

7.5.1 The Invariant Kronecker Delta Tensor

Definition: Kronecker Delta as a Rank-(1,1) Mixed Tensor

The Kronecker delta is defined by:

$$\delta^i{}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{7.35}$$

Its 4-dimensional trace is the dimension of spacetime:

$$\delta^i{}_i = \delta^0{}_0 + \delta^1{}_1 + \delta^2{}_2 + \delta^3{}_3 = 1 + 1 + 1 + 1 = 4. \tag{7.36}$$

Proof of Tensorial Invariance. Under an arbitrary coordinate transformation $x^i \to x'^i$:

$$\delta'^i{}_{j'} = \frac{\partial x'^i}{\partial x^a}\frac{\partial x^b}{\partial x'^j}\delta^a{}_b = \frac{\partial x'^i}{\partial x^a}\frac{\partial x^a}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta^i{}_j. \tag{7.37}$$

Hence, the Kronecker delta has the exact same components in **all coordinate systems**; it is an invariant isotropic tensor. □

7.5.2 Transformation Law of the Metric Tensor

Because the spacetime interval $\mathrm{d}s^2$ is an invariant scalar under coordinate transformations $x^a \to x'^c$:

$$\mathrm{d}s^2 = g_{ab}(x)\,\mathrm{d}x^a\mathrm{d}x^b = g'_{cd}(x')\,\mathrm{d}x'^c\mathrm{d}x'^d. \tag{7.38}$$

Expressing the old differentials in terms of the new coordinates:

$$\mathrm{d}x^a = \frac{\partial x^a}{\partial x'^c}\mathrm{d}x'^c, \qquad \mathrm{d}x^b = \frac{\partial x^b}{\partial x'^d}\mathrm{d}x'^d. \tag{7.39}$$

Substituting into the line element:

$$\mathrm{d}s^2 = g_{ab}(x)\left(\frac{\partial x^a}{\partial x'^c}\mathrm{d}x'^c\right)\left(\frac{\partial x^b}{\partial x'^d}\mathrm{d}x'^d\right) = \left(g_{ab}\frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}\right)\mathrm{d}x'^c\mathrm{d}x'^d. \tag{7.40}$$

Comparing both sides for arbitrary independent differentials $\mathrm{d}x'^c\mathrm{d}x'^d$:

Postulate / Principle: Tensor Transformation Law for the Metric

$$g'_{cd}(x') = \frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}g_{ab}(x). \tag{7.41}$$

This proves rigorously that g_{ab} is a **rank-(0,2) covariant tensor**.

The inverse metric g^{ab} is defined by the matrix inverse condition:

$$g_{ab}g^{bc} = \delta^c{}_a, \tag{7.42}$$

and transforms as a **rank-(2,0) contravariant tensor**:

$$g'^{cd}(x') = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b}g^{ab}(x). \tag{7.43}$$

Chapter 8

Tensor Densities, Hypersurface Geometry, and Metric Variations

8.1 The Levi-Civita Permutation Symbol and Its Contractions

8.1.1 Algebraic Definition of the Permutation Symbol

In $D = 4$ spacetime dimensions, the completely antisymmetric Levi-Civita permutation symbol e_{abcd} (also denoted $[abcd]$) is defined by:

$$e_{abcd} = \begin{cases} +1 & \text{if } (a,b,c,d) \text{ is an even permutation of } (0,1,2,3), \\ -1 & \text{if } (a,b,c,d) \text{ is an odd permutation of } (0,1,2,3), \\ 0 & \text{if any two indices are repeated.} \end{cases} \tag{8.1}$$

The contravariant symbol with upper indices is defined with the same numerical values in flat Euclidean indexing ($e^{0123} = +1$), but in Lorentzian spacetime signature $(-,+,+,+)$, lowering and raising indices with the Minkowski metric $\eta_{ab} = \text{diag}(-1,1,1,1)$ introduces a crucial relative minus sign:

$$e^{abcd} = \eta^{ai}\eta^{bj}\eta^{ck}\eta^{d\ell}e_{ijk\ell} = \eta^{00}\eta^{11}\eta^{22}\eta^{33}e_{0123} = (-1)(+1)(+1)(+1)(+1) = -1. \tag{8.2}$$

Hence, in Lorentzian signature:

$$e^{abcd} = -e_{abcd}. \tag{8.3}$$

8.1.2 Fundamental Contraction Identity (Generalized Kronecker Delta)

The fundamental product of two Levi-Civita permutation symbols is given by the determinant of Kronecker deltas:

$$e^{ik\ell m}e_{prst} = -\det\begin{pmatrix} \delta^i_p & \delta^k_p & \delta^\ell_p & \delta^m_p \\ \delta^i_r & \delta^k_r & \delta^\ell_r & \delta^m_r \\ \delta^i_s & \delta^k_s & \delta^\ell_s & \delta^m_s \\ \delta^i_t & \delta^k_t & \delta^\ell_t & \delta^m_t \end{pmatrix} \equiv -\delta^{ik\ell m}_{prst}, \tag{8.4}$$

where the overall minus sign reflects the Lorentzian metric signature.

8.1.3 Successive Index Contractions

By sequentially contracting pairs of indices, we obtain the standard contraction identities:

(1) Contraction over two index pairs: Setting $m = t$ and summing over $m = 0, 1, 2, 3$, followed by setting $\ell = s$:

$$e^{ik\ell m} e_{pr\ell m} = -2\left(\delta^i_p \delta^k_r - \delta^i_r \delta^k_p\right). \tag{8.5}$$

(2) Contraction over three index pairs: Setting $k = r$ in Equation (8.5):

$$e^{ik\ell m} e_{pk\ell m} = -2\left(\delta^i_p \delta^k_k - \delta^i_k \delta^k_p\right) = -2\left(4\delta^i_p - \delta^i_p\right) = -6\delta^i_p. \tag{8.6}$$

(3) Full contraction over all four indices: Setting $i = p$ in Equation (8.6):

$$e^{ik\ell m} e_{ik\ell m} = -6\delta^i_i = -6(4) = -24 = -4!. \tag{8.7}$$

8.1.4 Matrix Determinants via Levi-Civita Contractions

For any 4×4 matrix $A = (A_{ij})$, the determinant is algebraically expressed through index contraction with the permutation symbol:

$$e^{prst} A_{ip} A_{kr} A_{\ell s} A_{mt} = -(\det A)\, e_{ik\ell m}. \tag{8.8}$$

Contracting both sides with $e^{ik\ell m}$ yields:

$$e^{ik\ell m} e^{prst} A_{ip} A_{kr} A_{\ell s} A_{mt} = -(\det A)\left(e^{ik\ell m} e_{ik\ell m}\right) = -(\det A)(-24) = 24 \det A. \tag{8.9}$$

Therefore:

$$\det A = \frac{1}{24} e^{ik\ell m} e^{prst} A_{ip} A_{kr} A_{\ell s} A_{mt}. \tag{8.10}$$

8.2 The Invariant Volume Form and Tensor Densities

8.2.1 Transformation of the Volume Element

Under an arbitrary differentiable coordinate transformation $x^a \to x'^a(x)$, the differential coordinate volume transforms according to the Jacobian determinant:

$$\mathrm{d}^4 x' = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right| \mathrm{d}^4 x = J\, \mathrm{d}^4 x, \qquad J \equiv \det\left(\frac{\partial x'^a}{\partial x^b}\right). \tag{8.11}$$

Because $J \neq 1$ in general, the coordinate differential product $\mathrm{d}^4 x = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$ is **not** a coordinate-invariant scalar.

8.2.2 Transformation of the Metric Determinant

The covariant metric tensor transforms according to:

$$g'_{ab}(x') = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}(x). \tag{8.12}$$

Taking the determinant of both sides and using the matrix determinant product rule $\det(P^T M P) = (\det P)^2 \det M$:

$$\det\left(g'_{ab}\right) = \left[\det\left(\frac{\partial x}{\partial x'}\right)\right]^2 \det(g_{cd}). \tag{8.13}$$

Denoting $g \equiv \det(g_{ab}) < 0$ and $g' \equiv \det(g'_{ab}) < 0$:

$$g' = \left(J^{-1}\right)^2 g \implies -g' = \frac{1}{J^2}(-g). \tag{8.14}$$

Taking the positive square root on both sides:

$$\sqrt{-g'} = \frac{1}{J}\sqrt{-g} \iff J = \frac{\sqrt{-g}}{\sqrt{-g'}}. \tag{8.15}$$

8.2.3 Construction of the Invariant Spacetime Volume Form

Multiplying the transformation law of d^4x (8.11) by $\sqrt{-g'}$:

$$\sqrt{-g'}\,\mathrm{d}^4x' = \sqrt{-g'}\left(J\,\mathrm{d}^4x\right) = \sqrt{-g'}\left(\frac{\sqrt{-g}}{\sqrt{-g'}}\mathrm{d}^4x\right) = \sqrt{-g}\,\mathrm{d}^4x. \tag{8.16}$$

Postulate / Principle: Invariant Volume Element on Curved Spacetime

For any D-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g_{ab})$, the differential measure:

$$\mathrm{d}V \equiv \sqrt{-g}\,\mathrm{d}^Dx = \sqrt{|g|}\,\mathrm{d}x^0\mathrm{d}x^1\ldots\mathrm{d}x^{D-1} \tag{8.17}$$

is a strictly coordinate-invariant scalar volume element:

$$\mathrm{d}V' = \mathrm{d}V. \tag{8.18}$$

Physical Insight & Motivation: Example: Spherical Polar Coordinates in Flat Spacetime

In flat Minkowski spacetime expressed in spherical polar coordinates (ct, r, θ, ϕ):

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{8.19}$$

The metric matrix is diagonal:

$$g_{\mu\nu} = \mathrm{diag}\left(-1,\,1,\,r^2,\,r^2\sin^2\theta\right) \implies g = \det(g_{\mu\nu}) = -r^4\sin^2\theta. \tag{8.20}$$

Hence, the invariant volume measure is:

$$\mathrm{d}V = \sqrt{-g}\,\mathrm{d}x^0\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = c\,r^2\sin\theta\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{8.21}$$

reproducing the standard spatial volume element $\mathrm{d}V_3 = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.

8.2.4 The Covariant Levi-Civita Tensor (Volume Form Tensor)

In flat Cartesian coordinates, the permutation symbol e_{abcd} has constant components $\pm1, 0$. However, under general coordinate transformations $x \to x'$:

$$\frac{\partial x^m}{\partial x'^a}\frac{\partial x^n}{\partial x'^b}\frac{\partial x^i}{\partial x'^c}\frac{\partial x^j}{\partial x'^d}e_{mnij} = \det\left(\frac{\partial x}{\partial x'}\right)e'_{abcd} = J^{-1}e'_{abcd} = \frac{\sqrt{-g'}}{\sqrt{-g}}e'_{abcd}. \tag{8.22}$$

Equation (8.22) demonstrates that e_{abcd} is **not** a true tensor; it transforms with an extra Jacobian factor J^{-1}, classifying it as a **tensor density of weight** -1.

To construct a true coordinate tensor, we multiply by $\sqrt{-g}$:

$$\epsilon_{abcd} \equiv \sqrt{-g}\,e_{abcd}, \qquad \epsilon^{abcd} \equiv -\frac{1}{\sqrt{-g}}e^{abcd}. \tag{8.23}$$

Proof of Tensorial Transformation Law. Multiplying Equation (8.22) by $\sqrt{-g}$:

$$\frac{\partial x^m}{\partial x'^a}\frac{\partial x^n}{\partial x'^b}\frac{\partial x^i}{\partial x'^c}\frac{\partial x^j}{\partial x'^d}\left(\sqrt{-g}\,e_{mnij}\right) = \sqrt{-g'}\,e'_{abcd}. \tag{8.24}$$

Substituting the definition $\epsilon_{abcd} = \sqrt{-g}\,e_{abcd}$:

$$\epsilon'_{abcd}(x') = \frac{\partial x^m}{\partial x'^a}\frac{\partial x^n}{\partial x'^b}\frac{\partial x^i}{\partial x'^c}\frac{\partial x^j}{\partial x'^d}\epsilon_{mnij}(x). \tag{8.25}$$

Thus, ϵ_{abcd} is a genuine rank-4 totally antisymmetric covariant tensor on curved spacetime (the canonical Riemannian volume form). □

8.3 Hypersurface Geometry and Induced Metrics

8.3.1 Intrinsic Parameterization and Tangent Vectors

Consider an embedded $(D-1)$-dimensional submanifold (hypersurface) $\Sigma \subset \mathcal{M}$. Let y^μ $(\mu = 1, \ldots, D-1)$ denote intrinsic coordinates parameterizing the hypersurface:

$$x^a = x^a(y^1, y^2, \ldots, y^{D-1}). \tag{8.26}$$

The natural basis of tangent vectors spanning the tangent space $T_p\Sigma$ at any event on the hypersurface is given by:

$$e^a_\mu \equiv \frac{\partial x^a}{\partial y^\mu}. \tag{8.27}$$

Physical Insight & Motivation: Coordinate Tangent Vectors e^a_μ vs. Orthonormal Tetrads (Vielbeins)

It is essential to clarify the geometric nature of the tangent basis vectors $e^a_\mu = \frac{\partial x^a}{\partial y^\mu}$:

- The vectors e^a_μ are **coordinate basis vectors** determined solely by the choice of intrinsic parameterization y^μ of the hypersurface.
- Their mutual scalar products and norms are given by the induced metric components:

  $$g_{ab} e^a_\mu e^b_\nu = h_{\mu\nu}(y). \tag{8.28}$$

  The norm $g_{ab}e^a_\mu e^a_\mu$ is **not required to be normalized to** ± 1, and different tangent vectors e^a_μ, e^a_ν $(\mu \neq \nu)$ are **not required to be orthogonal** $(h_{\mu\nu} \neq 0)$.
- Because they lack orthonormality, the set $\{e^a_\mu\}$ is **not a tetrad (vielbein / frame field) yet**!
- **Promotion to a Tetrad:** If and only if we perform an orthonormalization procedure (such as Gram–Schmidt) to construct a set of $D-1$ orthonormal vectors $e^a_{(\hat{\mu})}$ satisfying $g_{ab}e^a_{(\hat{\mu})}e^b_{(\hat{\nu})} = \eta_{\hat{\mu}\hat{\nu}}$ (or $\delta_{\hat{\mu}\hat{\nu}}$), and combine them with the normalized unit normal vector $e^a_{(\hat{0})} \equiv n^a$, does the complete set of D vectors $\{e^a_{(\hat{A})}\}_{A=0}^{D-1}$ become a true **spacetime tetrad (vierbein)**.

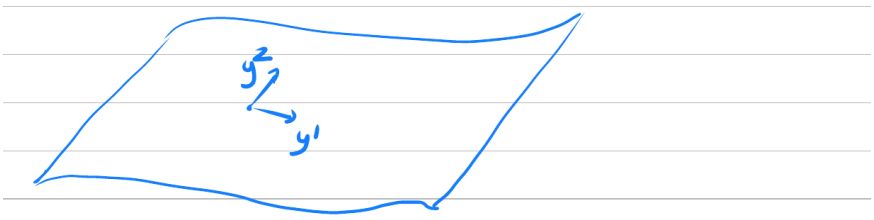


Figure 8.1: Geometric structure of an embedded 2D hypersurface Σ in spacetime. The intrinsic coordinates (y^1, y^2) define the tangent basis vectors $e^a_1 = \frac{\partial x^a}{\partial y^1}$ and $e^a_2 = \frac{\partial x^a}{\partial y^2}$, while the normal 1-form n_a satisfies $n_a e^a_\mu = 0$.

8.3.2 The Induced Metric (First Fundamental Form)

The spacetime line element $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\mathrm{d}x^b$ restricted to displacements along the hypersurface Σ becomes:

$$\mathrm{d}s^2|_\Sigma = g_{ab}(x(y))\left(\frac{\partial x^a}{\partial y^\mu}\mathrm{d}y^\mu\right)\left(\frac{\partial x^b}{\partial y^\nu}\mathrm{d}y^\nu\right) = \left(g_{ab}\frac{\partial x^a}{\partial y^\mu}\frac{\partial x^b}{\partial y^\nu}\right)\mathrm{d}y^\mu\mathrm{d}y^\nu. \tag{8.29}$$

Definition: The Induced Metric $h_{\mu\nu}$

The **induced metric** (or pullback metric / First Fundamental Form) on the hypersurface Σ is defined by:

$$h_{\mu\nu}(y) \equiv g_{ab}|_\Sigma \frac{\partial x^a}{\partial y^\mu}\frac{\partial x^b}{\partial y^\nu} = g_{ab}e^a_\mu e^b_\nu. \tag{8.30}$$

8.3.3 Unit Normal Vectors and Causal Classification

If the hypersurface is defined implicitly by a level-set scalar equation $f(x) = \text{const}$, the gradient 1-form is everywhere orthogonal to the tangent vectors:

$$n_a \propto \partial_a f \implies n_a e^a_\mu = n_a\frac{\partial x^a}{\partial y^\mu} = \frac{\partial f}{\partial y^\mu} = 0. \tag{8.31}$$

Normalizing n_a so that $n_a n^a = \epsilon$:

$$n_a = \frac{\partial_a f}{\sqrt{|g^{bc}\partial_b f\partial_c f|}}, \qquad \epsilon \equiv n_a n^a = \pm 1,\, 0. \tag{8.32}$$

Key Concept: Causal Classification of Hypersurfaces

- **Spacelike Hypersurface ($\epsilon = -1$):** The normal vector n^a is **timelike** ($n_a n^a = -1$). The induced metric $h_{\mu\nu}$ is positive-definite (Riemannian, signature $(+,+,+)$). Examples include constant-time Cauchy slices $t = \text{const}$ in cosmology and the ADM 3+1 formulation.
- **Timelike Hypersurface ($\epsilon = +1$):** The normal vector n^a is **spacelike** ($n_a n^a = +1$). The induced metric $h_{\mu\nu}$ is Lorentzian (signature $(-,+,+)$). Examples include the worldtube boundary of a star or domain walls.
- **Null Hypersurface ($\epsilon = 0$):** The normal vector n^a is **null** ($n_a n^a = 0$). The normal vector lies *inside* the tangent space ($n^a \in T\Sigma$). Examples include event horizons of black holes and Rindler causal horizons.

8.3.4 The Ambient Spacetime Projector h_{ab}

In ambient spacetime tensor notation, we define the **hypersurface projection tensor**:

$$h_{ab} \equiv g_{ab} - \epsilon\, n_a n_b. \tag{8.33}$$

Physical Insight & Motivation: Properties of the Projection Tensor

1. **Orthogonality to normal:**

$$h_{ab}n^a = (g_{ab} - \epsilon\, n_a n_b)\, n^a = n_b - \epsilon(n_a n^a)n_b = n_b - \epsilon(\epsilon)n_b = n_b - n_b = 0. \tag{8.34}$$

2. **Idempotence as a projection operator:** Raising one index $h^a{}_b \equiv g^{ac}h_{cb} = \delta^a{}_b - \epsilon\, n^a n_b$:

$$\begin{aligned} h^a{}_b h^b{}_c &= (\delta^a{}_b - \epsilon\, n^a n_b)\left(\delta^b{}_c - \epsilon\, n^b n_c\right) \\ &= \delta^a{}_c - \epsilon\, n^a n_c - \epsilon\, n^a n_c + \epsilon^2 (n_b n^b) n^a n_c \\ &= \delta^a{}_c - 2\epsilon\, n^a n_c + \epsilon(\epsilon) n^a n_c = \delta^a{}_c - \epsilon\, n^a n_c = h^a{}_c. \end{aligned} \tag{8.35}$$

3. **Connection to induced metric $h_{\mu\nu}$:** Because $n_a e^a_\mu = 0$, contracting h_{ab} with the tangent vectors reproduces the intrinsic metric identically:

$$e^a_\mu e^b_\nu h_{ab} = e^a_\mu e^b_\nu\, (g_{ab} - \epsilon\, n_a n_b) = g_{ab} e^a_\mu e^b_\nu - 0 = h_{\mu\nu}. \tag{8.36}$$

While $h_{\mu\nu}$ is an intrinsic $(D-1)$-dimensional tensor dependent on coordinate choices y^μ, h_{ab} is a full 4-dimensional spacetime tensor that projects any vector into the hypersurface tangent plane!

Foundational Question: Foundational Questions on Spacetime Foliations and Accelerated Horizons

1. **3+1 Slicing and Intersecting Foliations:** How does one treat spacetimes foliated into a family of constant-time hypersurfaces Σ_t parameterized by $\lambda = t$, and what happens when slicing hypersurfaces intersect? In the standard ADM 3+1 Hamiltonian formulation of General Relativity, spacetime is decomposed into a stack of non-intersecting spacelike Cauchy hypersurfaces Σ_t. If two level-set surfaces intersect (e.g., coordinate singularities or colliding shocks), the gradient $\partial_a f$ becomes multi-valued, signaling that the coordinate chart has broken down and must be covered by overlapping atlas patches.

2. **Exploring Beyond the Rindler Wedge via Foliation:** An observer undergoing constant proper acceleration is causally confined inside the Right Rindler Wedge ($X > |cT|$). Can we use the concept of hypersurface foliation to explore the spacetime region outside the wedge? Yes! While the accelerated observer's constant-time slices $t = \text{const}$ (lines passing through the origin) cover only the Right Wedge, one can analytically continue the foliation into the Future Wedge (expanding Milne universe), the Past Wedge (contracting Milne universe), and the Left Rindler Wedge, ultimately reconstructing the entire global manifold of Minkowski spacetime.

8.4 Metric Variations, Jacobi's Formula, and Derivative of the Inverse Metric

8.4.1 Jacobi's Formula for General Invertible Matrices

For any real, invertible $N \times N$ matrix $M(x)$ with non-zero determinant $\det M \neq 0$, the determinant can be expressed via the matrix logarithm identity:

$$\ln(\det M) = \mathrm{Tr}(\ln M). \tag{8.37}$$

Proof of Jacobi's Determinant Variation Formula. Consider an infinitesimal variation $M \to M + \delta M$. Taking the variation of the left-hand side:

$$\delta\big(\ln(\det M)\big) = \frac{\delta(\det M)}{\det M}. \tag{8.38}$$

Taking the variation of the right-hand side using the cyclic property of the trace:

$$\delta\big(\mathrm{Tr}(\ln M)\big) = \mathrm{Tr}\left(M^{-1}\delta M\right) = (M^{-1})_{ba}(\delta M)^{ab}. \tag{8.39}$$

Equating both variations yields **Jacobi's Formula**:

$$\delta(\det M) = (\det M)\,\mathrm{Tr}\left(M^{-1}\delta M\right). \tag{8.40}$$

□

8.4.2 Variation of the Spacetime Metric Determinant

Applying Jacobi's formula (8.40) to the spacetime metric tensor $g_{\mu\nu}$ with determinant $g \equiv \det(g_{\mu\nu})$ and inverse matrix $(g^{-1})^{\mu\nu} = g^{\mu\nu}$:

$$\delta g = g\, g^{ab}\delta g_{ab} \iff \frac{\delta g}{g} = g^{ab}\delta g_{ab}. \tag{8.41}$$

8.4.3 Variation of the Inverse Metric Tensor

The inverse metric g^{ab} is defined algebraically by:

$$g^{ab}g_{bc} = \delta^{a}{}_{c}. \tag{8.42}$$

Taking the infinitesimal variation of both sides:

$$\delta\left(g^{ab}g_{bc}\right) = \delta(\delta^{a}{}_{c}) = 0. \tag{8.43}$$

Applying the product rule:

$$(\delta g^{ab})g_{bc} + g^{ab}(\delta g_{bc}) = 0. \tag{8.44}$$

Contracting with g^{cd}:

$$\begin{aligned}(\delta g^{ab})g_{bc}g^{cd} &= -g^{ab}g^{cd}\delta g_{bc}\\ (\delta g^{ab})\delta^{d}{}_{b} &= -g^{am}g^{dn}\delta g_{mn}.\end{aligned} \tag{8.45}$$

Relabeling indices yields the fundamental relation:

Postulate / Principle: Variation of the Inverse Metric

$$\delta g^{ik} = -g^{im} g^{k\ell} \delta g_{m\ell}. \tag{8.46}$$

Contracting Equation (8.46) with g_{ik}:

$$g_{ik}\delta g^{ik} = -g_{ik} g^{im} g^{k\ell} \delta g_{m\ell} = -\delta^m{}_k g^{k\ell} \delta g_{m\ell} = -g^{m\ell} \delta g_{m\ell} = -g^{ab} \delta g_{ab}. \tag{8.47}$$

Hence, the determinant variation can equivalently be written in terms of the contravariant metric:

$$\frac{\delta g}{g} = -g_{ab} \delta g^{ab} \iff \delta g = -g\, g_{ab} \delta g^{ab}. \tag{8.48}$$

8.4.4 Variation of the Volume Measure Factor $\sqrt{-g}$

Using $\mathrm{d}(\sqrt{u}) = \frac{\mathrm{d}u}{2\sqrt{u}}$ with $u = -g$:

$$\delta\sqrt{-g} = \frac{-\delta g}{2\sqrt{-g}} = \frac{-g\left(g^{ab}\delta g_{ab}\right)}{2\sqrt{-g}} = \frac{1}{2}\sqrt{-g}\, g^{ab}\delta g_{ab}. \tag{8.49}$$

Expressed in terms of the variation of the contravariant metric δg^{ab}:

Postulate / Principle: Variation of the Invariant Volume Factor

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{ab}\delta g^{ab}. \tag{8.50}$$

This formula is the foundational cornerstone of the Einstein–Hilbert variational principle in General Relativity.

8.4.5 Partial Derivatives of the Inverse Metric

Because Jacobi's formula and the variation identity $\delta(g^{ab}g_{bc}) = 0$ hold for coordinate displacements $x^\ell \to x^\ell + \mathrm{d}x^\ell$ as well as functional variations, substituting $\delta \to \partial_\ell$ yields:

$$\partial_\ell g^{ik} = -g^{im} g^{kn} \partial_\ell g_{mn}. \tag{8.51}$$

Similarly, the spacetime gradient of the metric determinant satisfies:

$$\partial_\ell g = g\, g^{ab}\partial_\ell g_{ab} \implies \partial_\ell\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{ab}\partial_\ell g_{ab} = \sqrt{-g}\,\Gamma^\mu_{\mu\ell}. \tag{8.52}$$

8.5 Point Particle Dynamics in Curved Spacetime

8.5.1 4-Velocity Normalization in General Backgrounds

In an arbitrary curved spacetime metric $g_{ab}(x)$, the worldline of a massive point particle is parameterized by its proper time τ, with timelike tangent 4-velocity $u^a \equiv \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$. The invariant proper interval condition $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = g_{ab}(x)\,\mathrm{d}x^a\mathrm{d}x^b$ demands:

$$g_{ab}(x)u^a u^b = -c^2. \tag{8.53}$$

8.5.2 Covariant Action for a Relativistic Particle

Generalizing the flat-space action derived in Chapter 4 via the Minimal Coupling Prescription:

$$S[x^a(\lambda)] = -mc \int_{\lambda_1}^{\lambda_2} \sqrt{-g_{ab}(x) \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda}} \, \mathrm{d}\lambda = -mc \int_A^B \mathrm{d}\tau. \tag{8.54}$$

When varying the trajectory $x^a(\lambda) \to x^a(\lambda) + \delta x^a(\lambda)$, both the explicit coordinate differentials $\dot{x}^a$ and the spacetime-dependent metric coefficients $g_{ab}(x(\lambda))$ vary simultaneously:

$$\delta g_{ab}(x(\lambda)) = \partial_\mu g_{ab}(x) \, \delta x^\mu(\lambda). \tag{8.55}$$

In the next chapter, we perform this complete variation to derive the famous **Geodesic Equation** and the Christoffel connection symbols Γ^a_{bc}.

Chapter 9

The Geodesic Equation, Christoffel Symbols, and Affine Geometry

9.1 Variational Derivation of the Geodesic Equation

9.1.1 First Variation of the Square-Root Action

In Chapter 8, we established the covariant relativistic action for a point particle of rest mass m propagating through an arbitrary curved spacetime $(\mathcal{M}, g_{ab})$:

$$S[x^a(\lambda)] = -mc\int_{\lambda_1}^{\lambda_2}\sqrt{-g_{ab}(x)\,\dot{x}^a\dot{x}^b}\,\mathrm{d}\lambda = -mc\int_A^B \mathrm{d}\tau, \tag{9.1}$$

where $\dot{x}^a \equiv \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$, and $\mathrm{d}\tau = \frac{1}{c}\sqrt{-g_{ab}(x)\,\mathrm{d}x^a\mathrm{d}x^b}$ is the differential proper time along the worldline.

Let us evaluate the exact first variation δS under an arbitrary infinitesimal displacement of the worldline $x^a(\lambda) \to x^a(\lambda) + \delta x^a(\lambda)$:

$$\delta S = -mc\int \delta(\mathrm{d}\tau) = -mc\int \delta\left(\sqrt{-\frac{1}{c^2}g_{ab}(x)\,\mathrm{d}x^a\mathrm{d}x^b}\right). \tag{9.2}$$

Using $\delta\sqrt{u} = \frac{\delta u}{2\sqrt{u}}$ with $u = -g_{ab}(x)\,\mathrm{d}x^a\mathrm{d}x^b = c^2\mathrm{d}\tau^2$:

$$\begin{aligned}\delta(c\,\mathrm{d}\tau) &= \frac{\delta\left(-g_{ab}(x)\,\mathrm{d}x^a\mathrm{d}x^b\right)}{2c\,\mathrm{d}\tau}\\ &= \frac{-\delta g_{ab}(x)\,\mathrm{d}x^a\mathrm{d}x^b - 2g_{ab}(x)\,\mathrm{d}x^a\delta(\mathrm{d}x^b)}{2c\,\mathrm{d}\tau}.\end{aligned} \tag{9.3}$$

Because the variation δx^a is evaluated at the physical position of the particle:

$$\delta g_{ab}(x) = \partial_i g_{ab}(x)\,\delta x^i, \qquad \delta(\mathrm{d}x^b) = \mathrm{d}(\delta x^b). \tag{9.4}$$

Substituting these relations and dividing by $\mathrm{d}\tau$:

$$\begin{aligned}\delta(\mathrm{d}\tau) &= -\frac{1}{2c^2}\left(\partial_i g_{ab}\right)\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\delta x^i\,\mathrm{d}\tau - \frac{1}{c^2}g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta x^b)}{\mathrm{d}\tau}\,\mathrm{d}\tau\\ &= -\frac{1}{2c^2}\left(\partial_i g_{ab}\right)u^a u^b\delta x^i\,\mathrm{d}\tau - \frac{1}{c^2}g_{ab}u^a\frac{\mathrm{d}(\delta x^b)}{\mathrm{d}\tau}\,\mathrm{d}\tau,\end{aligned} \tag{9.5}$$

where $u^a \equiv \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$ is the tangent 4-velocity vector.

9.1.2 Integration by Parts and the Boundary Term

The variation of the action becomes:

$$\delta S = \frac{m}{c}\int \left[\frac{1}{2}\left(\partial_i g_{ab}\right) u^a u^b \delta x^i + g_{ab} u^a \frac{\mathrm{d}(\delta x^b)}{\mathrm{d}\tau}\right] c\,\mathrm{d}\tau. \tag{9.6}$$

Integrating the second term by parts with respect to proper time τ:

$$\begin{aligned}\int g_{ab} u^a \frac{\mathrm{d}(\delta x^b)}{\mathrm{d}\tau}\,\mathrm{d}\tau &= \left[g_{ab} u^a \delta x^b\right]_1^2 - \int \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ab} u^a\right)\delta x^b\,\mathrm{d}\tau \\ &= \left[g_{ia} u^a \delta x^i\right]_1^2 - \int \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ia} u^a\right)\delta x^i\,\mathrm{d}\tau. \end{aligned}\tag{9.7}$$

Combining terms gives the complete functional variation of the action:

$$\delta S = m\int_{\tau_1}^{\tau_2}\left[\frac{1}{2}\left(\partial_i g_{ab}\right) u^a u^b - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ia} u^a\right)\right]\delta x^i\,\mathrm{d}\tau + \left[m g_{ia} u^a \delta x^i\right]_1^2. \tag{9.8}$$

Physical Insight & Motivation: Canonical 4-Momentum in Curved Spacetime

Evaluating the boundary term at the upper endpoint gives the on-shell covariant **canonical 4-momentum**:

$$P_i \equiv \left.\frac{\partial S}{\partial x^i}\right|_{\text{on-shell}} = m g_{ia} u^a = m u_i. \tag{9.9}$$

The mass-shell constraint in curved spacetime is then identically:

$$g^{ab} P_a P_b = m^2 g^{ab} u_a u_b = m^2(-c^2) = -m^2 c^2, \tag{9.10}$$

yielding the relativistic Hamilton–Jacobi equation:

$$g^{ab}(x)\frac{\partial S}{\partial x^a}\frac{\partial S}{\partial x^b} = -m^2 c^2. \tag{9.11}$$

9.1.3 Euler–Lagrange Equations and Symmetrization

Demanding stationarity $\delta S = 0$ for all arbitrary variations with fixed endpoints ($\delta x^i(1) = \delta x^i(2) = 0$):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ik} u^k\right) - \frac{1}{2}\left(\partial_i g_{ab}\right) u^a u^b = 0. \tag{9.12}$$

Expanding the total proper time derivative using the chain rule $\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}x^m}{\mathrm{d}\tau}\partial_m = u^m \partial_m$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ik} u^k\right) &= g_{ik}\frac{\mathrm{d}u^k}{\mathrm{d}\tau} + \frac{\mathrm{d}g_{ik}}{\mathrm{d}\tau}u^k \\ &= g_{ik}\frac{\mathrm{d}u^k}{\mathrm{d}\tau} + \left(\partial_m g_{ik}\right) u^m u^k. \end{aligned}\tag{9.13}$$

Substituting into Equation (9.12):

$$g_{ik}\frac{\mathrm{d}u^k}{\mathrm{d}\tau} = \frac{1}{2}\left(\partial_i g_{ab}\right) u^a u^b - \left(\partial_m g_{ik}\right) u^m u^k. \tag{9.14}$$

Because the product $u^a u^b$ is strictly symmetric under index interchange ($u^a u^b = u^b u^a$), we can rewrite the second term by renaming dummy indices $m \to a$, $k \to b$ and symmetrizing:

$$\left(\partial_a g_{ib}\right) u^a u^b = \frac{1}{2}\left(\partial_a g_{ib} + \partial_b g_{ia}\right) u^a u^b. \tag{9.15}$$

Combining both terms on the right-hand side:

$$g_{ik}\frac{\mathrm{d}u^k}{\mathrm{d}\tau} = -\frac{1}{2}\left(\partial_a g_{ib} + \partial_b g_{ia} - \partial_i g_{ab}\right) u^a u^b. \tag{9.16}$$

9.1.4 The Christoffel Symbols of the Second Kind

Multiplying Equation (9.16) by the inverse metric g^{im} and using $g^{im}g_{ik} = \delta^m{}_k$:

$$\delta^m{}_k \frac{\mathrm{d}u^k}{\mathrm{d}\tau} = \frac{\mathrm{d}u^m}{\mathrm{d}\tau} = -\frac{1}{2} g^{im} \left(\partial_a g_{ib} + \partial_b g_{ia} - \partial_i g_{ab}\right) u^a u^b. \tag{9.17}$$

Definition: Christoffel Symbols (Affine Connection)

The **Christoffel symbols of the first kind** are defined by:

$$\Gamma_{\lambda\mu\nu} \equiv \frac{1}{2} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right), \tag{9.18}$$

and the **Christoffel symbols of the second kind** are obtained by raising the first index with the inverse metric $g^{m\lambda}$:

$$\Gamma^m_{\mu\nu} \equiv g^{m\lambda}\Gamma_{\lambda\mu\nu} = \frac{1}{2} g^{m\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right). \tag{9.19}$$

By inspection of the definition, the connection is strictly symmetric in its lower indices:

$$\Gamma^m_{\mu\nu} = \Gamma^m_{\nu\mu} \quad \text{(Torsion-Free Condition)}. \tag{9.20}$$

Key Concept: Inferring the Connection from Ambient Higher-Dimensional Embedding

One can always embed the geometry in a higher dimension where a Cartesian coordinate system exists. In such a coordinate basis, perform a coordinate transformation such that our embedded manifold becomes a coordinate surface, allowing us to express the induced coordinates in terms of the embedding space coordinates to calculate how the basis changes from point to point. Since:

$$\nabla_\mu \mathbf{e}_\nu = \Gamma^\alpha_{\mu\nu} \mathbf{e}_\alpha, \tag{9.21}$$

this allows us to infer the connection terms directly from the existence of such a coordinate system itself.

Example: 2-Sphere S^2 Embedded in $\mathbb{R}^3$. Consider a two-dimensional sphere of radius R embedded in three-dimensional Euclidean space. Let (θ, ϕ) be the usual spherical coordinates on the sphere, so that the embedding into $\mathbb{R}^3$ is:

$$x = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi, \qquad z = R\cos\theta. \tag{9.22}$$

Here θ is the polar angle and ϕ is the azimuthal angle.
Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ denote the constant Cartesian basis vectors of the ambient $\mathbb{R}^3$. The coordinate basis vectors on the sphere are obtained by differentiating the embedding:

$$\mathbf{e}_\theta = \frac{\partial}{\partial\theta} = R\cos\theta\cos\phi\,\mathbf{e}_x + R\cos\theta\sin\phi\,\mathbf{e}_y - R\sin\theta\,\mathbf{e}_z, \tag{9.23}$$

$$\mathbf{e}_\phi = \frac{\partial}{\partial\phi} = -R\sin\theta\sin\phi\,\mathbf{e}_x + R\sin\theta\cos\phi\,\mathbf{e}_y. \tag{9.24}$$

It is also useful to introduce the outward-pointing unit normal:

$$\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z. \tag{9.25}$$

Notice that $\mathbf{e}_r$ is normal to the sphere, whereas $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are tangent to it.

Since the Cartesian basis is constant ($\partial_\mu \mathbf{e}_x = \partial_\mu \mathbf{e}_y = \partial_\mu \mathbf{e}_z = 0$), when differentiating the coordinate basis vectors, we simply differentiate their components.
First,

$$\partial_\theta \mathbf{e}_\theta = -R\sin\theta\cos\phi\,\mathbf{e}_x - R\sin\theta\sin\phi\,\mathbf{e}_y - R\cos\theta\,\mathbf{e}_z = -R\mathbf{e}_r. \tag{9.26}$$

Similarly,

$$\partial_\theta \mathbf{e}_\phi = -R\cos\theta\sin\phi\,\mathbf{e}_x + R\cos\theta\cos\phi\,\mathbf{e}_y = \frac{\cos\theta}{\sin\theta}\mathbf{e}_\phi = \cot\theta\,\mathbf{e}_\phi. \tag{9.27}$$

For the derivatives with respect to ϕ, we find:

$$\partial_\phi \mathbf{e}_\theta = -R\cos\theta\sin\phi\,\mathbf{e}_x + R\cos\theta\cos\phi\,\mathbf{e}_y = \cot\theta\,\mathbf{e}_\phi, \tag{9.28}$$

while

$$\partial_\phi \mathbf{e}_\phi = -R\sin\theta\cos\phi\,\mathbf{e}_x - R\sin\theta\sin\phi\,\mathbf{e}_y. \tag{9.29}$$

To express this in the spherical basis, we use:

$$\partial_\phi \mathbf{e}_\phi = -R\sin^2\theta\,\mathbf{e}_r - \sin\theta\cos\theta\,\mathbf{e}_\theta. \tag{9.30}$$

Thus, the derivatives of the coordinate basis vectors are:

$$\partial_\theta \mathbf{e}_\theta = -R\mathbf{e}_r, \qquad \partial_\theta \mathbf{e}_\phi = \cot\theta\,\mathbf{e}_\phi, \tag{9.31}$$

$$\partial_\phi \mathbf{e}_\theta = \cot\theta\,\mathbf{e}_\phi, \qquad \partial_\phi \mathbf{e}_\phi = -R\sin^2\theta\,\mathbf{e}_r - \sin\theta\cos\theta\,\mathbf{e}_\theta. \tag{9.32}$$

The terms proportional to $\mathbf{e}_r$ are normal to the sphere and therefore describe the extrinsic curvature of the embedded surface. For the intrinsic connection on the sphere, we retain only the components tangent to the sphere. Thus:

$$\nabla_\theta \mathbf{e}_\theta = 0, \qquad \nabla_\theta \mathbf{e}_\phi = \cot\theta\,\mathbf{e}_\phi, \tag{9.33}$$

$$\nabla_\phi \mathbf{e}_\theta = \cot\theta\,\mathbf{e}_\phi, \qquad \nabla_\phi \mathbf{e}_\phi = -\sin\theta\cos\theta\,\mathbf{e}_\theta. \tag{9.34}$$

Comparing these results with $\nabla_\mu \mathbf{e}_\nu = \Gamma^\alpha_{\mu\nu}\mathbf{e}_\alpha$, we obtain:

$$\boxed{\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta}, \qquad \boxed{\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta}, \tag{9.35}$$

with all other Christoffel symbols vanishing.
Indeed, the induced metric on the sphere is $\mathrm{d}s^2 = R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\,\mathrm{d}\phi^2$, and the Christoffel symbols computed from the metric formula (9.19) match these identically!

Postulate / Principle: The Geodesic Equation (Affine Parameterization)

The trajectory of a freely falling particle in curved spacetime satisfies the **Geodesic Equation**:

$$\frac{\mathrm{d}u^m}{\mathrm{d}\tau} + \Gamma^m_{ab}u^a u^b = 0 \iff \frac{\mathrm{d}^2x^m}{\mathrm{d}\tau^2} + \Gamma^m_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau} = 0. \tag{9.36}$$

9.2 Covariant Derivative and Non-Tensorial Transformation Laws

9.2.1 Why Partial Derivatives Fail on Curved Manifolds

In flat Cartesian spacetime, the directional acceleration $\frac{\mathrm{d}u^m}{\mathrm{d}\tau} = u^i \partial_i u^m$ is a valid Lorentz vector. However, under a general non-linear coordinate transformation $x^a \to x'^a(x)$, the 4-velocity transforms as:

$$u'^m = \frac{\partial x'^m}{\partial x^k} u^k. \tag{9.37}$$

Taking the partial derivative with respect to x'^i:

$$\begin{aligned} \partial'_i u'^m &= \frac{\partial x^j}{\partial x'^i} \partial_j \left(\frac{\partial x'^m}{\partial x^k} u^k \right) \\ &= \frac{\partial x^j}{\partial x'^i} \frac{\partial x'^m}{\partial x^k} \partial_j u^k + \frac{\partial x^j}{\partial x'^i} \frac{\partial^2 x'^m}{\partial x^j \partial x^k} u^k. \end{aligned} \tag{9.38}$$

The appearance of the inhomogeneous second derivative $\frac{\partial^2 x'^m}{\partial x^j \partial x^k}$ proves that $\partial_i u^m$ is **not a tensor**!

9.2.2 Transformation Law of the Christoffel Symbols

Under a coordinate transformation $x^a \to x'^a$, direct differentiation of the metric in Equation (9.19) yields the non-tensorial transformation law:

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^i} \frac{\partial x^j}{\partial x'^b} \frac{\partial x^k}{\partial x'^c} \Gamma^i_{jk} + \frac{\partial x'^a}{\partial x^m} \frac{\partial^2 x^m}{\partial x'^b \partial x'^c}. \tag{9.39}$$

Using the identity $\frac{\partial x'^a}{\partial x^m} \frac{\partial x^m}{\partial x'^c} = \delta^a{}_c \implies \frac{\partial x'^a}{\partial x^m} \frac{\partial^2 x^m}{\partial x'^b \partial x'^c} = -\frac{\partial x^j}{\partial x'^b} \frac{\partial x^k}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^j \partial x^k}$, this can be written as:

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^i} \frac{\partial x^j}{\partial x'^b} \frac{\partial x^k}{\partial x'^c} \Gamma^i_{jk} - \frac{\partial x^j}{\partial x'^b} \frac{\partial x^k}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^j \partial x^k}. \tag{9.40}$$

Key Concept: Christoffel Symbols Are Not Tensors!

- The first term in Equation (9.39) is the standard tensorial transformation.
- The second term is an **inhomogeneous piece** proportional to the second derivative of the coordinate map.
- Because of this inhomogeneous term, Γ^a_{bc} is **not a tensor**. In particular, Γ^a_{bc} can be transformed to zero at any chosen event P by adopting a Local Inertial Frame, even though true curvature is non-zero.

9.2.3 Definition of the Covariant Derivative

Remarkably, the inhomogeneous term in $\partial'_i u'^m$ (9.38) and the inhomogeneous term in Γ'^m_{ia} (9.40) have **equal magnitude and opposite signs**! When added together:

$$\nabla_i u^m \equiv \partial_i u^m + \Gamma^m_{ia} u^a \tag{9.41}$$

the inhomogeneous pieces cancel identically, producing a genuine **rank-(1,1) mixed tensor**:

$$\nabla'_i u'^m = \frac{\partial x^j}{\partial x'^i} \frac{\partial x'^m}{\partial x^k} \nabla_j u^k. \tag{9.42}$$

9.2.4 Proper 4-Acceleration and Orthogonality

The directional covariant derivative along the particle's trajectory defines the **proper 4-acceleration**:

$$a^m \equiv \frac{\mathrm{D}u^m}{\mathrm{d}\tau} \equiv u^i \nabla_i u^m = \frac{\mathrm{d}u^m}{\mathrm{d}\tau} + \Gamma^m_{ab} u^a u^b. \tag{9.43}$$

Hence:

$$\text{Geodesic Motion} \iff a^m = 0 \quad \text{(Free Fall = Zero Proper Acceleration)}. \tag{9.44}$$

Foundational Question: Can the Relativistic Hamiltonian Always Be Interpreted as Physical Energy?

In classical non-relativistic mechanics, the Hamiltonian $H = p\dot{q} - L$ is identified with the total mechanical energy $E = T+V$. When deriving the relativistic Lagrangian and canonical momentum from the covariant action $S = -mc\int \mathrm{d}\tau$, does the associated Hamiltonian always represent physical energy?

- **Inertial Minkowski Spacetime:** The Hamilton–Jacobi relation $P_0 = -\frac{E}{c} = \frac{\partial S}{\partial x^0}$ directly connects the temporal momentum component to the observer's conserved energy $E = \gamma mc^2$.
- **Curved Spacetime with a Timelike Killing Vector $\xi = \partial_t$:** If the metric is stationary ($\partial_t g_{ab} = 0$), the conserved quantity $E \equiv -\xi^a P_a = -P_0$ is the genuine conserved energy measured by asymptotic static observers.
- **General Time-Dependent Spacetimes (e.g. Expanding FLRW Universe):** There is no global timelike Killing vector ($\partial_t g_{ab} \neq 0$). Consequently, there is **no global conservation of energy**, and $P_0(t)$ is not a constant of motion (leading to cosmological photon redshift). Energy is only well-defined locally in a Local Inertial Frame via $E_{\text{local}} = -u^a_{\text{obs}} P_a$.

Proof of Orthogonality $a^m u_m = 0$ in Curved Spacetime. Because the Levi-Civita connection is metric-compatible ($\nabla_i g_{ab} = 0$):

$$a^m u_m = (u^i \nabla_i u^m) u_m = \frac{1}{2} u^i \nabla_i \left(g_{mn} u^m u^n\right) = \frac{1}{2} u^i \nabla_i \left(-c^2\right) = 0. \tag{9.45}$$

The 4-acceleration is always orthogonal to the 4-velocity in any curved metric. □

9.3 Local Inertial Frames and Normal Coordinates

9.3.1 Riemann Normal Coordinates at Event P

Let x^a be an arbitrary coordinate system with event P located at $x^a_P = x^a(P)$. We define new local coordinates x'^a via:

$$x'^a = (x^a - x^a_P) + \frac{1}{2}\Gamma^a_{bc}(P)(x^b - x^b_P)(x^c - x^c_P) + \mathcal{O}\left((x - x_P)^3\right). \tag{9.46}$$

Evaluating the Christoffel symbols in the primed frame at P:

$$\Gamma'^a_{bc}(P) = 0, \qquad g'_{ab}(P) = \eta_{ab}, \qquad \partial'_c g'_{ab}(P) = 0. \tag{9.47}$$

However, the derivatives of the Christoffel symbols do **not** vanish:

$$\partial'_d \Gamma'^a_{bc}(P) \neq 0. \tag{9.48}$$

These non-vanishing derivatives encode the gauge-invariant **Riemann Curvature Tensor** $R^a{}_{bcd}(P)$. To compare the connection associated with two different locally inertial coordinate systems, it is useful to begin by distinguishing the three coordinate systems involved.

Let x'^a and x''^a be two local coordinate systems, with x'^a chosen to be locally inertial at an event P, and x''^a chosen to be locally inertial at another event Q. We introduce a third coordinate system x^a, defined on a region that overlaps both coordinate systems. Thus, on the relevant overlap regions,

$$x'^a = x'^a(x), \qquad x^a = x^a(x'), \tag{9.49}$$

and

$$x''^a = x''^a(x), \qquad x^a = x^a(x''). \tag{9.50}$$

Since x^a overlaps both x'^a and x''^a, the two local coordinate systems can also be related directly by composition:

$$x'^a = x'^a(x(x'')), \qquad x''^a = x''^a(x(x')). \tag{9.51}$$

Thus x^a serves as a common coordinate system through which the two local inertial frames can be compared.

The connection coefficients are not themselves tensor components, so their values depend on the chosen coordinates. We first recall how the connection arises from the geodesic equation in a locally inertial coordinate system.

Consider a freely falling particle with affine parameter s. Since x'^a is locally inertial at P, the geodesic equation at P takes the form

$$\frac{d^2x'^a}{ds^2} = 0 \qquad \text{at } P. \tag{9.52}$$

Using the transformation

$$x'^a = x'^a(x), \tag{9.53}$$

the first derivative is

$$\frac{dx'^a}{ds} = \frac{\partial x'^a}{\partial x^b}\frac{dx^b}{ds}, \tag{9.54}$$

and hence

$$\frac{d^2x'^a}{ds^2} = \frac{\partial x'^a}{\partial x^d}\frac{d^2x^d}{ds^2} + \frac{\partial^2 x'^a}{\partial x^b \partial x^c}\frac{dx^b}{ds}\frac{dx^c}{ds}. \tag{9.55}$$

Using Eq. (9.52),

$$\frac{\partial x'^a}{\partial x^d}\frac{d^2x^d}{ds^2} + \frac{\partial^2 x'^a}{\partial x^b \partial x^c}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0. \tag{9.56}$$

Multiplying by the inverse Jacobian $\partial x^d/\partial x'^a$, we obtain

$$\frac{d^2x^d}{ds^2} + \frac{\partial x^d}{\partial x'^a}\frac{\partial^2 x'^a}{\partial x^b \partial x^c}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0. \tag{9.57}$$

This motivates the identification

$$\boxed{\Gamma^d_{bc} = \frac{\partial x^d}{\partial x'^a}\frac{\partial^2 x'^a}{\partial x^b \partial x^c}}. \tag{9.58}$$

The geodesic equation in the arbitrary coordinates x^a therefore takes the familiar form

$$\boxed{\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0.} \tag{9.59}$$

We now distinguish the roles of the two local inertial systems.

Since x'^a is chosen to be locally inertial at P,

$$\boxed{\Gamma'^a_{bc}(P) = 0.} \tag{9.60}$$

On the other hand, the coordinates x''^a are chosen to be locally inertial at the different event Q, and therefore

$$\boxed{\Gamma''^a_{bc}(Q) = 0.} \tag{9.61}$$

These two equations refer to different coordinate systems and, importantly, to different events.

To determine what the second local inertial system x''^a assigns to the connection at P, we first use the common coordinate system x^a. Starting from the geodesic equation in the x^a coordinates,

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{de}\frac{dx^d}{ds}\frac{dx^e}{ds} = 0, \tag{9.62}$$

and transforming from x^a to x''^a, we have

$$\frac{dx''^r}{ds} = \frac{\partial x''^r}{\partial x^a}\frac{dx^a}{ds}, \tag{9.63}$$

and therefore

$$\frac{d^2x''^r}{ds^2} = \frac{\partial x''^r}{\partial x^a}\frac{d^2x^a}{ds^2} + \frac{\partial^2 x''^r}{\partial x^a \partial x^d}\frac{dx^a}{ds}\frac{dx^d}{ds}. \tag{9.64}$$

Substituting the geodesic equation in the x^a coordinates and rewriting the velocities in terms of dx''^b/ds gives

$$\frac{d^2x''^r}{ds^2} + \Gamma''^r_{bc}\frac{dx''^b}{ds}\frac{dx''^c}{ds} = 0, \tag{9.65}$$

where

$$\boxed{\Gamma''^r_{bc} = \frac{\partial x''^r}{\partial x^d}\frac{\partial x^e}{\partial x''^b}\frac{\partial x^f}{\partial x''^c}\Gamma^d_{ef} + \frac{\partial x''^r}{\partial x^d}\frac{\partial^2 x^d}{\partial x''^b \partial x''^c}}. \tag{9.66}$$

Thus, at the event P,

$$\boxed{\Gamma''^r_{bc}(P) = \left[\frac{\partial x''^r}{\partial x^d}\frac{\partial x^e}{\partial x''^b}\frac{\partial x^f}{\partial x''^c}\Gamma^d_{ef} + \frac{\partial x''^r}{\partial x^d}\frac{\partial^2 x^d}{\partial x''^b \partial x''^c}\right]_P}. \tag{9.67}$$

At this stage, the role of the common coordinate system x^a is clear: it provides a coordinate description that overlaps both local inertial systems and allows us to discuss the connection coefficients of x'^a and x''^a at the same event.

We can now relate the two local inertial systems directly. Since x'^a and x''^a are connected through the common coordinate system x^a, we may regard x'^a as a function of x''^a:

$$x'^a = x'^a(x''), \qquad x''^a = x''^a(x'). \tag{9.68}$$

Starting with the geodesic equation in the x'^a coordinates,

$$\frac{d^2x'^a}{ds^2} + \Gamma'^a_{ef}\frac{dx'^e}{ds}\frac{dx'^f}{ds} = 0, \tag{9.69}$$

we use

$$\frac{dx'^a}{ds} = \frac{\partial x'^a}{\partial x''^b}\frac{dx''^b}{ds}, \tag{9.70}$$

and therefore

$$\frac{d^2x'^a}{ds^2} = \frac{\partial x'^a}{\partial x''^d}\frac{d^2x''^d}{ds^2} + \frac{\partial^2 x'^a}{\partial x''^b \partial x''^c}\frac{dx''^b}{ds}\frac{dx''^c}{ds}. \tag{9.71}$$

Substituting this into the geodesic equation gives

$$0 = \frac{\partial x'^a}{\partial x''^d}\frac{d^2x''^d}{ds^2}$$

$$+\left(\frac{\partial^2 x'^a}{\partial x''^b \partial x''^c}+\Gamma'^a_{ef}\frac{\partial x'^e}{\partial x''^b}\frac{\partial x'^f}{\partial x''^c}\right)\frac{dx''^b}{ds}\frac{dx''^c}{ds}. \tag{9.72}$$

Multiplying by $\partial x''^r/\partial x'^a$ and comparing with

$$\frac{d^2x''^r}{ds^2}+\Gamma''^r_{bc}\frac{dx''^b}{ds}\frac{dx''^c}{ds}=0, \tag{9.73}$$

we obtain the direct transformation law

$$\boxed{\Gamma''^r_{bc}=\frac{\partial x''^r}{\partial x'^a}\frac{\partial x'^e}{\partial x''^b}\frac{\partial x'^f}{\partial x''^c}\Gamma'^a_{ef}+\frac{\partial x''^r}{\partial x'^a}\frac{\partial^2 x'^a}{\partial x''^b\partial x''^c}}. \tag{9.74}$$

This equation makes the relation between the two local inertial coordinate systems explicit. Although the connection coefficients in x'^a vanish at P,

$$\Gamma'^a_{bc}(P)=0, \tag{9.75}$$

the transformed connection coefficients in the x''^a coordinates at the same event are

$$\boxed{\Gamma''^r_{bc}(P)=\left.\frac{\partial x''^r}{\partial x'^a}\frac{\partial^2 x'^a}{\partial x''^b\partial x''^c}\right|_P}. \tag{9.76}$$

Thus x''^a, despite being locally inertial at Q, is in general not locally inertial at P, and consequently

$$\Gamma''^r_{bc}(P)\neq 0 \tag{9.77}$$

in general.

Conversely, because x''^a is locally inertial at Q,

$$\Gamma''^r_{bc}(Q)=0, \tag{9.78}$$

and the same transformation law gives

$$\boxed{0=\left.\frac{\partial x''^r}{\partial x'^a}\frac{\partial x'^e}{\partial x''^b}\frac{\partial x'^f}{\partial x''^c}\Gamma'^a_{ef}\right|_Q+\left.\frac{\partial x''^r}{\partial x'^a}\frac{\partial^2 x'^a}{\partial x''^b\partial x''^c}\right|_Q}. \tag{9.79}$$

Therefore the two statements

$$\Gamma'^a_{bc}(P)=0, \qquad \Gamma''^a_{bc}(Q)=0, \tag{9.80}$$

express the local inertial character of the two coordinate systems at their respective events. Away from those events, the connection coefficients need not vanish. The connection at P in the x''^a coordinates is obtained from the transformation (9.74), and its nonzero value is determined by the second derivatives of the coordinate transformation between the two local inertial systems.

Foundational Question: Where Does a Local Coordinate Chart End?

How far from event P does a Riemann Normal Coordinate (or Local Inertial Frame) patch remain valid?

- **Curvature Scale Limit:** The approximation $g_{ab}\approx\eta_{ab}+\mathcal{O}(x^2)$ is accurate only within a radius $r\ll L_{\text{curvature}}\sim 1/\sqrt{|R_{\mu\nu\rho\sigma}|}$.
- **Geodesic Crossing / Focal Points (Cut Locus):** A Riemann Normal chart is

constructed by shooting geodesics radially outward from P. The chart terminates abruptly where neighboring geodesics intersect (focal/caustic points, conjugate points) or when the exponential map fails to be a diffeomorphism.

- **Coordinate vs. Physical Singularities:** In black hole physics (e.g. Schwarzschild $r = 2GM/c^2$), the apparent boundary is merely a chart artifact, removable by transitioning to Eddington–Finkelstein or Kruskal–Szekeres coordinates, whereas a true physical singularity (e.g. $r = 0$) terminates all geodesics and cannot be removed by any chart.

9.4 Reparameterization Invariance and Non-Affine Geodesics

9.4.1 Arbitrary Parameter Transformations

The square-root action (9.1) is explicitly **reparameterization invariant**: under any smooth monotonic reparameterization $\tau \to \lambda(\tau)$, the integral $\int \mathrm{d}\tau = \int \frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\mathrm{d}\lambda$ is numerically unchanged.

How does the geodesic equation transform when expressed in terms of an arbitrary non-affine parameter λ? Using the chain rule:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}, \tag{9.81}$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\right) = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} + \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}. \tag{9.82}$$

Substituting into the affine geodesic equation $\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0$:

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2\left[\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\right] + \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = 0. \tag{9.83}$$

Dividing by $\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2$:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = -\frac{\mathrm{d}^2\lambda/\mathrm{d}\tau^2}{(\mathrm{d}\lambda/\mathrm{d}\tau)^2}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \equiv f(\lambda)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}. \tag{9.84}$$

Definition: Affine vs. Non-Affine Parameters

- If $f(\lambda) = 0$, the parameter λ is an **Affine Parameter**. The acceleration vector is strictly zero ($u^i\nabla_i u^a = 0$).
- Any linear transformation preserves affinity:

$$\lambda = a\tau + b \implies \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2} = 0 \implies f(\lambda) = 0. \tag{9.85}$$

- If $f(\lambda) \neq 0$, the parameter λ is **Non-Affine**. The directional acceleration is parallel to the tangent vector ($u^i\nabla_i u^a = f(\lambda)u^a$). The particle still traces the exact same geometric curve in spacetime, but with a non-uniform velocity parameterization.

9.4.2 Constructing the Affine Parameter from a Non-Affine Parameter

Suppose we are given a geodesic described with respect to a non-affine parameter λ satisfying:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = f(\lambda)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}. \tag{9.86}$$

Can we always construct an affine parameter $\mu(\lambda)$ such that the geodesic equation takes standard form?

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\mu^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\mu}\frac{\mathrm{d}x^k}{\mathrm{d}\mu} = 0. \tag{9.87}$$

Derivation of the Affine Parameter $\mu(\lambda)$. Transforming derivatives from λ to μ:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} = \left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}^2x^i}{\mathrm{d}\mu^2} + \frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^i}{\mathrm{d}\mu}. \tag{9.88}$$

Substituting into the non-affine geodesic equation:

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^2\left[\frac{\mathrm{d}^2x^i}{\mathrm{d}\mu^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\mu}\frac{\mathrm{d}x^k}{\mathrm{d}\mu}\right] + \frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^i}{\mathrm{d}\mu} = f(\lambda)\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}x^i}{\mathrm{d}\mu}. \tag{9.89}$$

Demanding the bracketed term vanish yields the ordinary differential equation for $\mu(\lambda)$:

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2} = f(\lambda)\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \iff \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\ln\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right) = f(\lambda). \tag{9.90}$$

Integrating both sides:

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = C\exp\left(\int f(\lambda)\,\mathrm{d}\lambda\right) \implies \mu(\lambda) = C\int\exp\left(\int f(\lambda')\,\mathrm{d}\lambda'\right)\mathrm{d}\lambda + D. \tag{9.91}$$

□

Physical Insight & Motivation: Using Cosmological Redshift z and Scale Factor $a(t)$ as Non-Affine Parameters

In observational cosmology (FLRW spacetime), light rays traveling from distant galaxies to our telescopes are frequently parameterized not by affine parameter λ, but by observable monotonic quantities such as:

- **Cosmic Scale Factor $a(t)$:** Parameterizing null geodesics by a leads to a non-vanishing acceleration term $f(a) = -\frac{\ddot{a}}{\dot{a}^2} = \frac{1+q}{a}$, where q is the deceleration parameter.
- **Cosmological Redshift $z = \frac{a_0}{a} - 1$:** Differentiating photon paths with respect to z yields non-affine geodesic equations that directly integrate cosmological distance measures (luminosity distance $d_L(z)$, angular diameter distance $d_A(z)$). Equation (9.91) provides the exact bridge to restore the fundamental affine parameter $\lambda(z)$.

Foundational Question: Can a Parameterization-Dependent Action Be Converted into a Reparameterization-Invariant Action?

Yes! This is achieved precisely through the **einbein (gauge field) formulation**:

- The quadratic action $S_{\text{quad}} = \frac{1}{2}\int g_{ab}\dot{x}^a\dot{x}^b\mathrm{d}\lambda$ depends explicitly on the choice of parameter λ (scaling $\lambda \to c\lambda$ changes the numerical value of S_{quad}).

- By introducing an independent 1D metric $e(\lambda)$ (the einbein) on the worldline, the action becomes:

$$S[x,e] = \frac{1}{2}\int \left[\frac{1}{e(\lambda)} g_{ab}\dot{x}^a\dot{x}^b - e(\lambda)m^2c^2\right] \mathrm{d}\lambda. \tag{9.92}$$

  Under a worldline diffeomorphism $\lambda \to \lambda'(\lambda)$, $e(\lambda)$ transforms as a 1D density $e'(\lambda') = e(\lambda)\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}$, rendering the entire action **strictly reparameterization-invariant**!

9.5 The Quadratic Kinetic Action and Null Geodesics

9.5.1 Failure of the Square-Root Action for Light Rays

For massless particles (photons, gravitons), the spacetime interval is null:

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\mathrm{d}x^b = 0 \implies \mathrm{d}\tau = 0. \tag{9.93}$$

Consequently, the square-root action $S = -mc\int \mathrm{d}s \equiv 0$ vanishes identically and cannot be used to derive photon trajectories.

9.5.2 The Quadratic (Energy) Action

To resolve this limitation and provide a unified variational principle for both massive and massless particles, we define the **quadratic action**:

$$S_{\text{quad}}[x^a(\lambda)] \equiv \frac{1}{2}\int_{\lambda_1}^{\lambda_2} g_{ab}(x)\,\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_{\lambda_1}^{\lambda_2} \mathcal{L}_{\text{quad}}\,(x,\dot{x})\,\mathrm{d}\lambda, \tag{9.94}$$

with Lagrangian:

$$\mathcal{L}_{\text{quad}}(x,\dot{x}) = \frac{1}{2} g_{ab}(x)\,\dot{x}^a\dot{x}^b. \tag{9.95}$$

9.5.3 Euler–Lagrange Derivation of the Affine Geodesic

Applying the Euler–Lagrange equations $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^m}\right) - \frac{\partial\mathcal{L}}{\partial x^m} = 0$:

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^m} = \frac{1}{2} g_{ab}\left(\delta^a_m\dot{x}^b + \dot{x}^a\delta^b_m\right) = g_{mb}\dot{x}^b, \tag{9.96}$$

$$\frac{\partial\mathcal{L}}{\partial x^m} = \frac{1}{2}\left(\partial_m g_{ab}\right)\dot{x}^a\dot{x}^b. \tag{9.97}$$

Hence:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{mb}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right) - \frac{1}{2}\left(\partial_m g_{ab}\right)\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0. \tag{9.98}$$

Expanding the derivative and multiplying by g^{im} yields directly:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = 0. \tag{9.99}$$

Key Concept: Virtues of the Quadratic Action

1. **Automatic Affine Parameterization:** The Euler–Lagrange equations of S_{quad} **automatically select an affine parameter** λ (no square-root singular reparameterization ambiguity).

2. **Universal Applicability:** It is equally valid for timelike ($g_{ab}\dot{x}^a\dot{x}^b = -c^2$), spacelike ($g_{ab}\dot{x}^a\dot{x}^b > 0$), and null ($g_{ab}\dot{x}^a\dot{x}^b = 0$) paths.

3. **Conservation of the Metric Norm (First Integral of Motion):** Along any geodesic trajectory satisfying the affine equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right) = \dot{x}^c\nabla_c\left(g_{ab}\dot{x}^a\dot{x}^b\right) = 2g_{ab}\dot{x}^a\left(\dot{x}^c\nabla_c\dot{x}^b\right) = 0. \tag{9.100}$$

   Therefore, the quantity $g_{ab}\dot{x}^a\dot{x}^b = \text{constant}$ is a strict constant of motion:

   A timelike geodesic remains timelike, and a null geodesic remains null for all λ! (9.101)

Chapter 10

Null Geodesics, Conformal Invariance, Fermat's Principle, and Tensor Calculus

10.1 Null Geodesics and the Massless Particle Limit

10.1.1 Affine vs. Non-Affine Tangent Vectors on Light Cones

For a massless particle (such as a photon or graviton), the tangent vector $k^a \equiv \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ to the worldline $x^a(\lambda)$ is everywhere null:

$$k_a k^a = g_{ab}(x) k^a k^b = 0. \tag{10.1}$$

Because the proper time interval vanishes identically along lightlike trajectories ($\mathrm{d}\tau = 0$), proper time cannot serve as the parameter.

In general, with respect to an arbitrary parameter λ, the geodesic equation for a null ray takes the non-affine form:

$$k^a \nabla_a k^b = f(\lambda) k^b. \tag{10.2}$$

10.1.2 Affine Rescaling of Null Tangent Vectors

Can we always rescale the tangent vector k^a by a smooth scalar function $\mu(x)$ to obtain an affinely parameterized null tangent vector $\ell^a \equiv \mu k^a$?

$$\begin{aligned}
\ell^a \nabla_a \ell^b &= (\mu k^a)\, \nabla_a \left(\mu k^b\right) \\
&= \mu k^a \left(\mu \nabla_a k^b + k^b \nabla_a \mu\right) \\
&= \mu^2 \left(k^a \nabla_a k^b\right) + \mu k^b \left(k^a \nabla_a \mu\right) \\
&= \mu^2 f(\lambda) k^b + \mu k^b \left(k^a \partial_a \mu\right) \\
&= \left[\mu f(\lambda) + k^a \partial_a \mu\right] \ell^b.
\end{aligned} \tag{10.3}$$

Demanding that ℓ^a satisfy the **affinely parameterized null geodesic equation** $\ell^a \nabla_a \ell^b = 0$:

$$k^a \partial_a \mu + f(\lambda)\mu = 0 \iff \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} + f(\lambda)\mu = 0. \tag{10.4}$$

Integrating yields:

$$\mu(\lambda) = \exp\left(-\int f(\lambda)\, \mathrm{d}\lambda\right). \tag{10.5}$$

The parameter $v(\lambda)$ associated with $\ell^a = \frac{\mathrm{d}x^a}{\mathrm{d}v}$ satisfies $\frac{\mathrm{d}v}{\mathrm{d}\lambda} = \frac{1}{\mu(\lambda)}$, yielding the standard affine equation:

$$\ell^a \nabla_a \ell^b = 0 \iff \frac{\mathrm{d}^2 x^b}{\mathrm{d}v^2} + \Gamma^b_{cd} \frac{\mathrm{d}x^c}{\mathrm{d}v} \frac{\mathrm{d}x^d}{\mathrm{d}v} = 0. \tag{10.6}$$

10.1.3 4-Momentum and Observer Energy for Massless Particles

The 4-momentum of a photon is defined as an affinely parameterized null tangent vector:

$$P^a = \hbar k^a_{\text{wave}} = \ell^a, \qquad P_a P^a = 0. \tag{10.7}$$

For an observer moving with timelike 4-velocity u^a_{obs} ($u_a u^a = -c^2$), the physically measured photon energy is the invariant scalar projection:

$$E = -P_a u^a_{\text{obs}} = -\ell_a u^a_{\text{obs}} = \hbar \omega_{\text{obs}}. \tag{10.8}$$

Physical Insight & Motivation: The Meaning of Affine Freedom for Photons

While the overall magnitude of the 4-velocity for massive particles is fixed by $u_a u^a = -c^2$, for photons $\ell_a \ell^a = 0$ leaves the constant affine scaling $\ell^a \to \alpha \ell^a$ unconstrained by metric geometry. This freedom corresponds physically to specifying the **photon frequency/energy** at the emission event!

10.1.4 The Einbein (Polyakov) Action for Smooth Massless Limits

To avoid the singular $m \to 0$ behavior of the square-root action, we introduce an auxiliary worldline metric field (an **einbein** $e(\lambda) > 0$):

$$S[x^a, e] = \frac{1}{2} \int \left[\frac{1}{e(\lambda)} g_{ab}(x) \dot{x}^a \dot{x}^b - e(\lambda) m^2 c^2 \right] \mathrm{d}\lambda. \tag{10.9}$$

Varying with respect to the einbein $e(\lambda)$:

$$\frac{\delta S}{\delta e} = -\frac{1}{2e^2} g_{ab} \dot{x}^a \dot{x}^b - \frac{1}{2} m^2 c^2 \stackrel{!}{=} 0 \implies e(\lambda) = \frac{\sqrt{-g_{ab} \dot{x}^a \dot{x}^b}}{mc}. \tag{10.10}$$

Substituting this algebraic equation of motion back into Equation (10.9) reproduces the square-root action for $m \neq 0$:

$$S_{\text{on-shell}} = -mc \int \sqrt{-g_{ab} \dot{x}^a \dot{x}^b}\, \mathrm{d}\lambda. \tag{10.11}$$

Crucially, the einbein action (10.9) is **strictly non-singular in the limit** $m \to 0$:

$$S_{m=0}[x^a, e] = \frac{1}{2} \int \frac{1}{e(\lambda)} g_{ab}(x) \dot{x}^a \dot{x}^b \, \mathrm{d}\lambda. \tag{10.12}$$

Varying with respect to $e(\lambda)$ enforces the null constraint $g_{ab} \dot{x}^a \dot{x}^b = 0$, while varying with respect to $x^a(\lambda)$ gives the geodesic equation in the gauge $e(\lambda) = \text{const}$.

Foundational Question: Does General Relativity Automatically Predict Particle Trajectories from the EFE?

In Newtonian gravity or Maxwellian electrodynamics, the field equations ($\nabla^2 \Phi = 4\pi G \rho$, $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$) do not determine the equations of motion of test charges; the Lorentz force law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ must be postulated independently.

Does General Relativity require an independent postulate for geodesic motion?

- **The Geodesic Hypothesis as an Integrability Condition:** In General Relativity, the non-linear Einstein Field Equations $G_{ab} = \frac{8\pi G}{c^4} T_{ab}$ automatically demand the local conservation of energy-momentum via the contracted Bianchi identities:

$$\nabla^a G_{ab} \equiv 0 \implies \nabla^a T_{ab} = 0. \tag{10.13}$$

- **The Einstein–Infeld–Hoffmann (EIH) Theorem:** Modeling a test particle as a localized, non-singular stress-energy distribution T_{ab}, the conservation equation $\nabla^a T_{ab} = 0$ rigorously forces the center-of-mass worldline of the particle to follow a **geodesic** of spacetime!

- **Conclusion:** Free fall along geodesics is **not an independent axiom** of General Relativity, but a direct mathematical consequence of the non-linear geometric field equations themselves.

10.2 Geometry and Induced Metric of Null Hypersurfaces

In Chapter 8, we developed the differential geometry of non-null (spacelike and timelike) hypersurfaces Σ. For those surfaces, a unit normal n^a with non-zero norm ($n_a n^a = \epsilon = \pm 1$) allowed us to project spacetime tensors onto Σ via the standard orthogonal projector $h^a{}_b = \delta^a_b - \epsilon n^a n_b$, endowing the hypersurface with a non-degenerate 3-metric h_{ab}.

However, in General Relativity, some of the most essential physical boundaries are **null hypersurfaces** $\mathcal{N}$—including black hole event horizons, Cauchy horizons, and expanding light wavefronts. On these surfaces, the standard geometric machinery breaks down completely, requiring a fundamentally different, 2-dimensional perspective.

10.2.1 The Geometric Dilemma: When a Normal is Also a Tangent

Consider a 3-dimensional hypersurface $\mathcal{N}$ defined as a level surface of a smooth scalar function $\Phi(x) = \Phi_0$. The gradient 1-form is $\ell_a \equiv A(x)\partial_a \Phi$. On a **null hypersurface**, the raised normal vector $\ell^a = g^{ab}\ell_b$ has zero norm:

$$\ell_a \ell^a = g_{ab}\ell^a \ell^b = 0. \tag{10.14}$$

Key Concept: The Geometric Pathology of Null Hypersurfaces

The vanishing of $\ell_a \ell^a$ introduces two remarkable geometric features:

1. **The Normal is Tangent to the Surface:** By definition, the 1-form ℓ_a is orthogonal to any vector v^a tangent to $\mathcal{N}$ ($\ell_a v^a = 0$). But because $\ell_a \ell^a = 0$, the vector $\boldsymbol{\ell}$ is orthogonal to *itself*! Consequently, the normal vector $\boldsymbol{\ell}$ is simultaneously **normal to $\mathcal{N}$ and tangent to $\mathcal{N}$**:

$$\boldsymbol{\ell} \in T_p\mathcal{N} \quad \text{and} \quad \boldsymbol{\ell} \perp T_p\mathcal{N}. \tag{10.15}$$

   Physical Intuition: A null hypersurface is not a stationary wall in space; it is a light wavefront propagating at the speed of light. Its direction of motion (tangent vector) is identical to its spatial wavefront normal!

2. **Failure of Naive Projectors:** If we attempt the standard projector $N_{ab} \equiv g_{ab} - \epsilon \ell_a \ell_b$,

contracting with the null normal gives:

$$N_{ab}\ell^b = g_{ab}\ell^b - \epsilon\ell_a(\ell_b\ell^b) = \ell_a - 0 = \ell_a \neq 0! \tag{10.16}$$

Because $\ell_b\ell^b = 0$, subtracting $\ell_a\ell_b$ cannot remove the longitudinal component along $\boldsymbol{\ell}$.

Furthermore, the integral curves of ℓ^a are affinely parameterized null geodesics ($\ell^b\nabla_b\ell^a = 0$), known as the **null generators** of $\mathcal{N}$. The entire null hypersurface is ruled by these light rays.

10.2.2 The Auxiliary Null Vector and the Projector $\Pi^a{}_b$

To define an orthogonal projection and measure geometry on a null hypersurface, we must introduce a second direction that points *out* of the surface. We choose an **auxiliary null vector** k^a transverse to $\mathcal{N}$, satisfying the cross-normalization:

$$\ell_a k^a = -1, \qquad k_a k^a = 0. \tag{10.17}$$

Physical Meaning: If $\boldsymbol{\ell}$ represents outgoing light rays generating the surface (e.g., along the horizon), $\boldsymbol{k}$ represents an ingoing family of light rays crossing the surface transversely.

We now construct an asymmetric projector $\Pi^a{}_b$ onto the tangent space $T_p\mathcal{N}$:

$$\Pi^a{}_b \equiv \delta^a_b + k^a\ell_b. \tag{10.18}$$

Let us check its properties step by step:

- **Eliminates the normal direction:** Contracting with ℓ_a:

$$\Pi^a{}_b\ell_a = \delta^a_b\ell_a + k^a\ell_b\ell_a = \ell_b + (\ell_a k^a)\ell_b = \ell_b - \ell_b = 0. \tag{10.19}$$

  Therefore, for any arbitrary 4-vector V^a, the projected vector $V^b_{\parallel} \equiv \Pi^b{}_a V^a$ satisfies $\ell_b V^b_{\parallel} = 0$, placing it squarely inside the tangent space $T_p\mathcal{N}$.

- **Idempotency ($\Pi^2 = \Pi$):** Applying the operator twice:

$$\begin{aligned}\Pi^c{}_b\Pi^b{}_a &= (\delta^c_b + k^c\ell_b)\left(\delta^b_a + k^b\ell_a\right)\\ &= \delta^c_a + k^c\ell_a + k^c\ell_a + k^c\left(\ell_b k^b\right)\ell_a\\ &= \delta^c_a + 2k^c\ell_a - k^c\ell_a = \delta^c_a + k^c\ell_a = \Pi^c{}_a.\end{aligned} \tag{10.20}$$

10.2.3 The Symmetric 2-Metric q_{ab} on the Transverse Screen Space

While $\Pi^a{}_b$ is an algebraic projection operator, the lowered tensor $\Pi_{ab} = g_{ab} + k_a\ell_b$ is not symmetric under $a \leftrightarrow b$. To measure physical distances and spatial areas, we construct the symmetric candidate:

$$q_{ab} \equiv g_{ab} + \ell_a k_b + k_a\ell_b = \Pi_{ab} + \ell_a k_b. \tag{10.21}$$

Raising the first index yields the mixed operator:

$$q^a{}_b = \delta^a_b + \ell^a k_b + k^a\ell_b. \tag{10.22}$$

Let us analyze what happens when $q^a{}_b$ acts on the two null vectors $\boldsymbol{\ell}$ and $\boldsymbol{k}$:

$$q^a{}_b\ell^b = \ell^a + \ell^a\left(k_b\ell^b\right) + k^a\left(\ell_b\ell^b\right) = \ell^a - \ell^a + 0 = 0, \tag{10.23}$$

$$q^a{}_b k^b = k^a + \ell^a\left(k_b k^b\right) + k^a\left(\ell_b k^b\right) = k^a + 0 - k^a = 0. \tag{10.24}$$

Now let us compute the square of $q^a{}_b$ to verify idempotency ($q^2 = q$):

$$q^a{}_b q^b{}_c = q^a{}_c + \ell^a k_b q^b{}_c. \tag{10.25}$$

For $q^a{}_b$ to act as a true projector, the extra term must vanish:

$$k_b q^b{}_c = k_b \left(\delta^b_c + \ell^b k_c + k^b \ell_c\right) = k_c - k_c + \left(k_b k^b\right) \ell_c = \left(k_b k^b\right) \ell_c \overset{!}{=} 0. \tag{10.26}$$

Since $\ell_c \neq 0$, this confirms our earlier requirement: **the auxiliary vector k^a must be strictly null ($k_a k^a = 0$)!**

Definition: The 2D Transverse Screen Metric q_{ab}

For any null hypersurface with normal ℓ^a and auxiliary null vector k^a ($\ell_a k^a = -1, k_a k^a = 0$), the symmetric tensor:

$$q_{ab} = g_{ab} + \ell_a k_b + k_a \ell_b \tag{10.27}$$

acts as an orthogonal projector onto the **2-dimensional Riemannian screen space** $\mathcal{S}_2 \subset T_p\mathcal{M}$ orthogonal to both $\boldsymbol{\ell}$ and $\boldsymbol{k}$. The dimension of this subspace is confirmed by its trace:

$$\mathrm{Tr}(q) = q^a{}_a = \delta^a_a + \ell^a k_a + k^a \ell_a = 4 - 1 - 1 = 2. \tag{10.28}$$

10.2.4 Canonical Null Basis and Metric Completeness

To describe tensors on the null surface in components, we introduce a **canonical null basis** (a Newman–Penrose-style null tetrad) at each point:

$$\mathbf{v}_{(a)} = (\boldsymbol{\ell}, \boldsymbol{k}, \mathbf{e}_1, \mathbf{e}_2)\,, \tag{10.29}$$

where $\mathbf{e}_A$ ($A = 1, 2$) are two spacelike vectors spanning the 2D transverse screen space $\mathcal{S}_2$:

$$\ell_a e^a_A = 0, \qquad k_a e^a_A = 0, \qquad g_{ab} e^a_A e^b_B = q_{ab} e^a_A e^b_B \equiv q_{AB}. \tag{10.30}$$

Denoting the matrix inverse of the 2D metric q_{AB} by q^{AB} ($q^{AC} q_{CB} = \delta^A_B$), the dual basis of one-forms is:

$$\mathbf{v}^{*(a)} = \left(-\underline{\boldsymbol{k}}, -\underline{\boldsymbol{\ell}}, \underline{\mathbf{e}}^1, \underline{\mathbf{e}}^2\right), \qquad \text{where} \quad e^A_a \equiv q^{AB} g_{ab} e^b_B, \quad e^A_a e^a_B = \delta^A_B. \tag{10.31}$$

This gives the complete decomposition of the 4D identity tensor and spacetime metric:

$$\delta^a_b = -\ell^a k_b - k^a \ell_b + q^{AB} e^a_A (e_B)_b, \tag{10.32}$$

$$g_{ab} = -\ell_a k_b - k_a \ell_b + q_{AB} e^A_a e^B_b, \tag{10.33}$$

$$g^{ab} = -\ell^a k^b - k^a \ell^b + q^{AB} e^a_A e^b_B. \tag{10.34}$$

The transverse metric components transform seamlessly between 4D and 2D:

$$q_{ab} = q_{AB} e^A_a e^B_b, \qquad q^{ab} = q^{AB} e^a_A e^b_B. \tag{10.35}$$

10.2.5 Erecting Adapted Coordinates and Degeneracy of the 3-Metric

To work directly on the null hypersurface $\mathcal{N}$, we set up an adapted coordinate system $y^\alpha = (\lambda, z^1, z^2)$:

- λ is an affine parameter along the null geodesic rays, so that displacements along the rays are $\mathrm{d}x^a = \ell^a \mathrm{d}\lambda \implies e^a_\lambda \equiv \frac{\partial x^a}{\partial \lambda} = \ell^a$.

- $z^A = (z^1, z^2)$ are two transverse coordinates that remain constant along each null generator, uniquely labeling the individual light rays: $e^a_A \equiv \frac{\partial x^a}{\partial z^A}$.

Evaluating the induced 3-metric $h_{\alpha\beta} = g_{ab} e^a_\alpha e^b_\beta$ in the coordinate basis (λ, z^1, z^2):

$$\begin{aligned} h_{\lambda\lambda} &= g_{ab}\ell^a\ell^b = 0 \quad \text{(since } \boldsymbol{\ell} \text{ is null)}, \\ h_{\lambda A} &= g_{ab}\ell^a e^b_A = 0 \quad \text{(since } \boldsymbol{\ell} \text{ is normal to all vectors tangent to } \mathcal{N}), \\ h_{AB} &= g_{ab} e^a_A e^b_B = q_{AB}. \end{aligned} \tag{10.36}$$

Thus, the matrix form of the induced 3-metric is manifestly block-diagonal:

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_{11} & q_{12} \\ 0 & q_{12} & q_{22} \end{pmatrix}. \tag{10.37}$$

Notice that the determinant of $h_{\alpha\beta}$ is **identically zero**:

$$\det\left(h_{\alpha\beta}\right) \equiv 0. \tag{10.38}$$

The Physical Consequence: The intrinsic line element on a null hypersurface is **strictly 2-dimensional**:

$$\mathrm{d}s^2_{\mathcal{N}} = h_{\alpha\beta}\,\mathrm{d}y^\alpha \mathrm{d}y^\beta = 0 \cdot \mathrm{d}\lambda^2 + q_{AB}\,\mathrm{d}z^A \mathrm{d}z^B = q_{AB}\,\mathrm{d}z^A \mathrm{d}z^B. \tag{10.39}$$

Displacements along the null generators $\mathrm{d}\lambda$ contribute zero metric distance. All non-trivial geometry on a null hypersurface lives entirely within the 2D spacelike cross-sections q_{AB}!

10.2.6 Invariant Directed Surface Element on Null Hypersurfaces

For any coordinate system $y^\alpha = (y^1, y^2, y^3)$ on $\mathcal{N}$, the invariant directed surface 3-element is given by:

$$\mathrm{d}\Sigma_a = \epsilon_{abcd} e^b_1 e^c_2 e^d_3\,\mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3. \tag{10.40}$$

Because contracting $\mathrm{d}\Sigma_a$ with any tangent vector e^a_α vanishes by the antisymmetry of ϵ_{abcd}, $\mathrm{d}\Sigma_a$ is strictly directed along the normal ℓ_a:

$$\epsilon_{abcd} e^b_1 e^c_2 e^d_3 = f\ell_a \implies f = -\epsilon_{abcd} k^a e^b_1 e^c_2 e^d_3. \tag{10.41}$$

In the adapted null coordinates (λ, z^1, z^2), this simplifies to the universal invariant formula:

$$\mathrm{d}\Sigma_a = \ell_a \sqrt{q}\,\mathrm{d}\lambda\,\mathrm{d}z^1 \mathrm{d}z^2, \tag{10.42}$$

where $q \equiv \det(q_{AB}) = q_{11}q_{22} - q_{12}^2$ is the determinant of the 2D transverse metric.

10.2.7 Working Physical Examples

Physical Insight & Motivation: Example 1: The Schwarzschild Event Horizon at $r = r_h = 2GM$

Consider the Schwarzschild metric in standard coordinates (t, r, θ, ϕ):

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{10.43}$$

The event horizon $\mathcal{H}^+$ is the hypersurface $r = r_h \equiv 2GM$, where the metric coefficient $f(r) = 1 - 2GM/r = 0$.

1. **Restricting to the Horizon ($\mathrm{d}r = 0$):** Freezing $r = r_h$, the line element becomes:

$$\mathrm{d}s^2\Big|_{r=r_h} = -\left(1 - \frac{2GM}{r_h}\right)\mathrm{d}t^2 + r_h^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) = 0\cdot\mathrm{d}t^2 + r_h^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{10.44}$$

   The time coordinate t has zero metric weight ($g_{tt}(r_h) = 0$). The tangent vector $\ell = \partial_t$ is the null generator of the horizon.

2. **The Horizon as a 2D Sphere:** The induced 3-metric in coordinates (t, θ, ϕ) is degenerate:

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_h^2 & 0 \\ 0 & 0 & r_h^2\sin^2\theta \end{pmatrix}, \qquad \det h = 0. \tag{10.45}$$

   The intrinsic geometry of the event horizon is strictly a **2-dimensional Riemannian sphere of radius** $r_h = 2GM$, with transverse 2-metric:

$$q_{AB} = \begin{pmatrix} r_h^2 & 0 \\ 0 & r_h^2\sin^2\theta \end{pmatrix}, \qquad \sqrt{q} = r_h^2\sin\theta, \qquad \mathrm{Area}(\mathcal{H}^+) = \int\sqrt{q}\,\mathrm{d}\theta\mathrm{d}\phi = 4\pi r_h^2 = 16\pi G^2M^2. \tag{10.46}$$

3. **Why We Need the Transverse Null Direction k^a:** Since $\ell = \partial_t$ lies tangent to the horizon, it only travels along the horizon surface. To describe matter and observers crossing *into* the black hole, we must introduce an ingoing auxiliary null vector k^a pointing across $\mathcal{H}^+$ into the interior, with $\ell_a k^a = -1$ and $k_a k^a = 0$.

Physical Insight & Motivation: Example 2: Ingoing Eddington–Finkelstein and Double Null Coordinates

In **ingoing Eddington–Finkelstein coordinates** (v, r, θ, ϕ) where $v = t + r_*$ is advanced time ($r_* = r + 2GM\ln|r/2GM - 1|$):

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\mathrm{d}\Omega^2. \tag{10.47}$$

On the horizon $r = r_h$ ($\mathrm{d}r = 0$), the line element directly yields $\mathrm{d}s^2|_{r=r_h} = r_h^2\mathrm{d}\Omega^2$. The two complementary null vectors spanning the (v, r) plane are:

$$\begin{aligned} &\ell^a\partial_a = \partial_v \quad \text{(along horizon generators)}, \\ &k^a\partial_a = -\partial_r + \frac{1}{2}\left(1 - \frac{2GM}{r}\right)\partial_v \xrightarrow{r\to r_h} -\partial_r \quad \text{(transverse null vector into hole)}, \end{aligned} \tag{10.48}$$

which precisely satisfy $\ell_a\ell^a = 0$, $k_a k^a = 0$, and $\ell_a k^a = -1$.

The Manifest Picture in Double Null Coordinates (u, v, θ, ϕ):
Using retarded time $u = t - r_*$ and advanced time $v = t + r_*$, the 4D line element takes the double null form:

$$\mathrm{d}s^2 = -F(u,v)\,\mathrm{d}u\,\mathrm{d}v + r^2(u,v)\,\mathrm{d}\Omega^2, \qquad \text{where} \quad F(u,v) = 1 - \frac{2GM}{r}. \tag{10.49}$$

The future horizon is a surface of constant $u = \infty$ (or $U = 0$ in Kruskal null coordinates $\mathrm{d}s^2 = -F(U,V)\,\mathrm{d}U\,\mathrm{d}V + r^2\mathrm{d}\Omega^2$). Setting $\mathrm{d}U = 0$ on the horizon:

$$\mathrm{d}s^2\Big|_{U=0} = -F(0,V)\cdot(0)\cdot\mathrm{d}V + r_h^2\mathrm{d}\Omega^2 = r_h^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{10.50}$$

In double null coordinates, the two null directions $\boldsymbol{\ell} \propto \partial_V$ and $\boldsymbol{k} \propto \partial_U$ are manifestly decoupled from the angular 2-sphere cross-sections, demonstrating that any null hypersurface is universally characterized by its 2D transverse metric q_{AB}.

10.3 Conformal Transformations and Invariance of Null Geodesics

10.3.1 Definition of a Conformal Transformation

A **conformal transformation** is a local, position-dependent rescaling of the spacetime metric tensor that preserves angles and null cones while altering distances and durations:

$$\tilde{g}_{ab}(x) = \Omega^2(x)g_{ab}(x), \qquad \tilde{g}^{ab}(x) = \Omega^{-2}(x)g^{ab}(x), \tag{10.51}$$

where $\Omega(x) > 0$ is a smooth, strictly positive conformal factor.

10.3.2 Transformation of the Christoffel Symbols

We calculate the Christoffel symbols $\tilde{\Gamma}^a_{bc}$ of the conformally transformed metric $\tilde{g}_{ab}$:

$$\tilde{\Gamma}^a_{bc} = \frac{1}{2}\tilde{g}^{ad}\left(\partial_b\tilde{g}_{cd} + \partial_c\tilde{g}_{bd} - \partial_d\tilde{g}_{bc}\right). \tag{10.52}$$

Using $\partial_b\tilde{g}_{cd} = \partial_b\left(\Omega^2 g_{cd}\right) = \Omega^2\partial_b g_{cd} + 2\Omega(\partial_b\Omega)g_{cd}$:

$$\begin{aligned}\tilde{\Gamma}^a_{bc} &= \frac{1}{2}\Omega^{-2}g^{ad}\left[\Omega^2\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right) + 2\Omega\left(g_{cd}\partial_b\Omega + g_{bd}\partial_c\Omega - g_{bc}\partial_d\Omega\right)\right] \\ &= \Gamma^a_{bc} + \frac{1}{\Omega}g^{ad}\left(g_{cd}\partial_b\Omega + g_{bd}\partial_c\Omega - g_{bc}\partial_d\Omega\right).\end{aligned} \tag{10.53}$$

Using $\frac{\partial_b\Omega}{\Omega} = \partial_b\ln\Omega$:

Postulate / Principle: Conformal Transformation of the Connection

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta^a_b\partial_c\ln\Omega + \delta^a_c\partial_b\ln\Omega - g_{bc}g^{ad}\partial_d\ln\Omega. \tag{10.54}$$

10.3.3 Transformation of Geodesic Trajectories

Let $x^a(\lambda)$ be an affinely parameterized geodesic in metric g_{ab} ($\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$). Evaluating its acceleration with respect to the conformal connection $\tilde{\Gamma}^a_{bc}$:

$$\begin{aligned}\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \tilde{\Gamma}^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda} &= \left(\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \Gamma^a_{bc}\dot{x}^b\dot{x}^c\right) + \left(\delta^a_b\partial_c\ln\Omega + \delta^a_c\partial_b\ln\Omega - g_{bc}g^{ad}\partial_d\ln\Omega\right)\dot{x}^b\dot{x}^c \\ &= 0 + 2\left(\dot{x}^c\partial_c\ln\Omega\right)\dot{x}^a - \left(g_{bc}\dot{x}^b\dot{x}^c\right)g^{ad}\partial_d\ln\Omega.\end{aligned} \tag{10.55}$$

Key Concept: Why Null Geodesics Are Conformally Invariant

1. **Timelike Geodesics ($g_{bc}\dot{x}^b\dot{x}^c = -c^2 \neq 0$):** The second term in Equation (10.55) is $+c^2 g^{ad}\partial_d \ln\Omega \neq 0$. This gradient vector is orthogonal to $\dot{x}^a$, bending the trajectory off its original path. **Timelike geodesics are not conformally invariant.**

2. **Null Geodesics ($g_{bc}\dot{x}^b\dot{x}^c = 0$):** The second term **vanishes identically**! Equation (10.55) reduces to:

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \tilde{\Gamma}^a_{bc}\dot{x}^b\dot{x}^c = 2\left(\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\lambda}\right)\dot{x}^a. \tag{10.56}$$

Because the right-hand side is strictly proportional to the tangent vector $\dot{x}^a$, the worldline $x^a(\lambda)$ remains an unaccelerated **null geodesic** in $\tilde{g}_{ab}$ (under a non-affine parameter λ).

Physical Insight & Motivation: Foundation of Penrose Conformal Diagrams

Because conformal transformations preserve the geometric trajectories of null geodesics and the light-cone structure of spacetime everywhere, one can bring spatial and null infinity ($\mathscr{I}^{\pm}, i^0, i^{\pm}$) to a finite coordinate distance via a conformal factor $\Omega(x) \to 0$ at infinity without distorting causal relationships.

10.4 Fermat's Principle in Static Spacetimes

10.4.1 The Optical (Fermat) Metric

A spacetime $(\mathcal{M}, g_{ab})$ is **static** if it possesses a hypersurface-orthogonal timelike Killing vector field $\xi = \partial_t$. In adapted coordinates (ct, x^α) $(\alpha = 1, 2, 3)$, the metric coefficients satisfy:

$$g_{0\alpha} = 0, \qquad \partial_0 g_{\mu\nu} = 0. \tag{10.57}$$

The spacetime line element is:

$$\mathrm{d}s^2 = g_{00}(\vec{x})c^2\mathrm{d}t^2 + g_{\alpha\beta}(\vec{x})\,\mathrm{d}x^\alpha\mathrm{d}x^\beta. \tag{10.58}$$

Along a null geodesic, $\mathrm{d}s^2 = 0$:

$$g_{00}(\vec{x})c^2\mathrm{d}t^2 + g_{\alpha\beta}(\vec{x})\,\mathrm{d}x^\alpha\mathrm{d}x^\beta = 0 \implies c\,\mathrm{d}t = \sqrt{-\frac{g_{\alpha\beta}(\vec{x})}{g_{00}(\vec{x})}\,\mathrm{d}x^\alpha\mathrm{d}x^\beta}. \tag{10.59}$$

Definition: The Optical (Fermat) Metric $H_{\alpha\beta}$

The 3-dimensional Riemannian **optical metric** (or Fermat metric) on the spatial slice Σ_3 is defined by:

$$H_{\alpha\beta}(\vec{x}) \equiv -\frac{g_{\alpha\beta}(\vec{x})}{g_{00}(\vec{x})}. \tag{10.60}$$

Because $g_{00} < 0$ and $g_{\alpha\beta}$ is positive-definite, $H_{\alpha\beta}$ is a strictly positive-definite Riemannian metric on space.

10.4.2 Variational Formulation of Fermat's Principle

The total coordinate arrival time t_{AB} for a light signal traveling between spatial points A and B along a spatial path $\vec{x}(\sigma)$ parameterized by $\sigma \in [\sigma_1, \sigma_2]$ is:

$$c\, t_{AB}[\vec{x}(\sigma)] = \int_{\sigma_1}^{\sigma_2} \sqrt{H_{\alpha\beta}(\vec{x}) \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}}\, \mathrm{d}\sigma. \tag{10.61}$$

Postulate / Principle: Fermat's Principle of Stationary Travel Time in General Relativity

In a static gravitational field, the spatial path traversed by a light ray between two fixed spatial events A and B renders the coordinate arrival time **stationary**:

$$\delta \int_A^B \mathrm{d}t = 0. \tag{10.62}$$

Equivalently: **Light rays in static curved spacetimes follow spatial geodesics of the 3D Riemannian optical metric $H_{\alpha\beta}(\vec{x})$.**

10.4.3 Effective Gravitational Refractive Index and Light Bending

In isotropic coordinates for a weak, static Newtonian gravitational field $\Phi(\vec{x})$:

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right), \qquad g_{\alpha\beta} = \left(1 - \frac{2\Phi}{c^2}\right)\delta_{\alpha\beta}. \tag{10.63}$$

The optical line element becomes:

$$\mathrm{d}\ell^2_{\mathrm{opt}} = H_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\beta = \frac{1 - \frac{2\Phi}{c^2}}{1 + \frac{2\Phi}{c^2}} |\mathrm{d}\vec{x}|^2 \approx \left(1 - \frac{4\Phi(\vec{x})}{c^2}\right) |\mathrm{d}\vec{x}|^2. \tag{10.64}$$

Taking the square root defines the **effective refractive index** $n(\vec{x})$ of curved space:

$$n(\vec{x}) \equiv \sqrt{H(\vec{x})} \approx 1 - \frac{2\Phi(\vec{x})}{c^2} > 1 \quad (\text{since } \Phi < 0). \tag{10.65}$$

Key Concept: Origin of the Factor of 2 in Gravitational Light Deflection

- A naive Newtonian particle model of light experiences only the temporal potential perturbation g_{00}, giving an effective index $n_{\mathrm{Newton}} \approx 1 - \frac{\Phi}{c^2}$ and deflection angle $\theta = \frac{2GM}{c^2 b}$.
- In General Relativity, both the temporal metric g_{00} (time dilation) and the spatial metric $g_{\alpha\beta}$ (spatial curvature) contribute equal amounts to the optical metric $H_{\alpha\beta} = -g_{\alpha\beta}/g_{00}$.
- This produces the famous double deflection angle observed by Eddington:

$$\theta_{\mathrm{Einstein}} = \frac{4GM}{c^2 b}. \tag{10.66}$$

10.5 Metric Compatibility and Tensor Differential Calculus

10.5.1 The Metric Compatibility Condition

The Levi-Civita covariant derivative ∇_a is uniquely determined by two geometric axioms:

1. **Torsion-Free Condition:** $\Gamma^a_{bc} = \Gamma^a_{cb}$.
2. **Metric Compatibility (Ricci's Lemma):** The metric is covariantly constant:

$$\nabla_c g_{ab} = 0. \tag{10.67}$$

Verification of Metric Compatibility. Writing out the covariant derivative of the rank-(0,2) metric tensor:

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad} = \partial_c g_{ab} - \Gamma_{bca} - \Gamma_{acb}. \tag{10.68}$$

Using the definition of the Christoffel symbols of the first kind $\Gamma_{cab} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab})$:

$$\begin{aligned} \Gamma_{bca} + \Gamma_{acb} &= \frac{1}{2}\left(\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca}\right) + \frac{1}{2}\left(\partial_c g_{ba} + \partial_b g_{ac} - \partial_a g_{cb}\right) \\ &= \frac{1}{2}\left(2\partial_c g_{ab}\right) = \partial_c g_{ab}. \end{aligned} \tag{10.69}$$

Hence:

$$\nabla_c g_{ab} = \partial_c g_{ab} - \partial_c g_{ab} = 0. \tag{10.70}$$

□

10.5.2 Covariant Derivative of Arbitrary Rank Tensors

For a general rank-(p, q) tensor $T^{a_1 \ldots a_p}_{b_1 \ldots b_q}$, each upper contravariant index receives a positive connection term, and each lower covariant index receives a negative connection term:

$$\nabla_c T^{a_1 \ldots a_p}_{b_1 \ldots b_q} = \partial_c T^{a_1 \ldots a_p}_{b_1 \ldots b_q} + \sum_{i=1}^{p} \Gamma^{a_i}_{cd} T^{a_1 \ldots d \ldots a_p}_{b_1 \ldots b_q} - \sum_{j=1}^{q} \Gamma^{d}_{cb_j} T^{a_1 \ldots a_p}_{b_1 \ldots d \ldots b_q}. \tag{10.71}$$

Because $\nabla_c g_{ab} = 0$ and $\nabla_c g^{ab} = 0$, the covariant derivative commutes with index raising and lowering:

$$\nabla_c \left(g_{ab} V^b\right) = g_{ab} \nabla_c V^b \iff \nabla_c V_a = (\nabla_c V)_a. \tag{10.72}$$

10.5.3 Fundamental Connection Contraction Identities

Contraction over Mixed Indices Γ^a_{ba}

Using the definition of the Christoffel symbol:

$$\Gamma^a_{ba} = \frac{1}{2} g^{ac} \left(\partial_b g_{ca} + \partial_a g_{bc} - \partial_c g_{ab}\right). \tag{10.73}$$

The last two terms cancel because $g^{ac}\partial_a g_{bc} = g^{ca}\partial_c g_{ba} = g^{ac}\partial_c g_{ab}$:

$$\Gamma^a_{ba} = \frac{1}{2} g^{ac} \partial_b g_{ac}. \tag{10.74}$$

Using Jacobi's formula $\partial_b \sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{ac}\partial_b g_{ac}$ derived in Chapter 8:

Postulate / Principle: Trace of the Christoffel Symbol

$$\Gamma^a_{ba} = \frac{1}{\sqrt{-g}}\partial_b\sqrt{-g} = \partial_b\left(\ln\sqrt{-g}\right). \tag{10.75}$$

Contraction over Metric Indices $g^{bc}\Gamma^a_{bc}$

Multiplying the Christoffel symbol by the inverse metric:

$$g^{bc}\Gamma^a_{bc} = \frac{1}{2}g^{bc}g^{am}\left(\partial_b g_{mc} + \partial_c g_{bm} - \partial_m g_{bc}\right) = g^{bc}g^{am}\partial_b g_{mc} - \frac{1}{2}g^{am}\left(g^{bc}\partial_m g_{bc}\right). \tag{10.76}$$

Using $\partial_b\left(g^{am}g_{mc}\right) = \partial_b(\delta^a_c) = 0 \implies g_{mc}\partial_b g^{am} = -g^{am}\partial_b g_{mc}$:

$$g^{bc}g^{am}\partial_b g_{mc} = -g^{bc}g_{mc}\partial_b g^{am} = -\delta^b_m\partial_b g^{am} = -\partial_m g^{am}. \tag{10.77}$$

Using $\frac{1}{2}g^{bc}\partial_m g_{bc} = \frac{\partial_m\sqrt{-g}}{\sqrt{-g}}$:

$$g^{bc}\Gamma^a_{bc} = -\partial_m g^{am} - g^{am}\frac{\partial_m\sqrt{-g}}{\sqrt{-g}} = -\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,\partial_m g^{am} + g^{am}\partial_m\sqrt{-g}\right). \tag{10.78}$$

Applying the product rule:

Postulate / Principle: Contracted Inverse Metric Connection Identity

$$g^{bc}\Gamma^a_{bc} = -\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}\,g^{am}\right). \tag{10.79}$$

10.5.4 Covariant Divergence and the Laplace–Beltrami Operator

Definition: Covariant Divergence of a Vector Field

Using the trace formula (10.75), the covariant divergence of any vector field V^a is:

$$\begin{aligned}\nabla_a V^a &= \partial_a V^a + \Gamma^a_{ab}V^b = \partial_a V^a + V^b\partial_b\left(\ln\sqrt{-g}\right)\\ &= \partial_a V^a + \frac{V^a}{\sqrt{-g}}\partial_a\sqrt{-g} = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,V^a\right).\end{aligned} \tag{10.80}$$

Definition: The Invariant Laplace–Beltrami Operator $\Box$

For an arbitrary scalar field $\phi(x)$, the covariant d'Alembertian (Laplace–Beltrami operator) is the divergence of its gradient:

$$\Box\phi \equiv \nabla_a\nabla^a\phi = \nabla_a\left(g^{ab}\partial_b\phi\right) = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,g^{ab}\partial_b\phi\right). \tag{10.81}$$

Equation (10.81) provides the universal tool for writing wave equations and scalar field dynamics in any curved or curvilinear coordinate system!

Chapter 11

Curved Spacetime Divergences, Coordinate Charts, Gauss's Theorem, and Frobenius Hypersurface Orthogonality

11.1 Tensor Divergences and Conservation Laws in Curved Geometry

11.1.1 Covariant Divergence of Antisymmetric Rank-2 Tensors

In flat spacetime, the divergence of a rank-2 contravariant tensor field is simply $\partial_a Q^{ab}$. In curved spacetime, minimal coupling replaces ordinary partial derivatives with covariant derivatives:

$$\nabla_a Q^{ab} = \partial_a Q^{ab} + \Gamma^a_{ac} Q^{cb} + \Gamma^b_{ac} Q^{ac}. \tag{11.1}$$

When the tensor field Q^{ab} is **strictly antisymmetric** ($Q^{ab} = -Q^{ba}$), a dramatic algebraic simplification occurs:

1. The last term involves the contraction of the Christoffel connection Γ^b_{ac} with Q^{ac}. Because the Levi-Civita connection is torsion-free, it is symmetric in its lower indices ($\Gamma^b_{ac} = \Gamma^b_{ca}$), whereas Q^{ac} is antisymmetric ($Q^{ac} = -Q^{ca}$). Consequently:

$$\Gamma^b_{ac} Q^{ac} = \Gamma^b_{ca}(-Q^{ca}) = -\Gamma^b_{ac} Q^{ac} \implies \Gamma^b_{ac} Q^{ac} \equiv 0. \tag{11.2}$$

2. The middle term involves the contracted Christoffel symbol Γ^a_{ac}, which we established in Chapter 10 equals the logarithmic derivative of the metric volume determinant factor:

$$\Gamma^a_{ac} = \partial_c \ln \sqrt{-g} = \frac{1}{\sqrt{-g}} \partial_c \sqrt{-g}. \tag{11.3}$$

Substituting these two results back into Eq. (11.1), we obtain:

$$\begin{aligned} \nabla_a Q^{ab} &= \partial_a Q^{ab} + \left(\frac{1}{\sqrt{-g}} \partial_c \sqrt{-g} \right) Q^{cb} \\ &= \partial_a Q^{ab} + \left(\frac{1}{\sqrt{-g}} \partial_a \sqrt{-g} \right) Q^{ab} \\ &= \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, \partial_a Q^{ab} + Q^{ab}\, \partial_a \sqrt{-g} \right). \end{aligned} \tag{11.4}$$

Recognizing the product rule in the numerator, we arrive at the fundamental identity:

$$\boxed{\nabla_a Q^{ab} = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,Q^{ab}\right)} \qquad (\text{for } Q^{ab} = -Q^{ba}). \tag{11.5}$$

Key Concept: Curved-Spacetime Maxwell Equations and Gauge Curvature

Applying this identity directly to the electromagnetic field strength tensor $F^{ab} = -F^{ba}$, the inhomogeneous curved-space Maxwell equations $\nabla_a F^{ab} = \mu_0 J^b$ take the remarkably elegant coordinate form:

$$\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,F^{ab}\right) = \mu_0 J^b, \tag{11.6}$$

where $F^{ab} = g^{ac}g^{bd}F_{cd}$.
Furthermore, the electromagnetic field strength tensor is defined in terms of the gauge 4-potential A_a by the covariant exterior derivative:

$$F_{ab} \equiv \nabla_a A_b - \nabla_b A_a. \tag{11.7}$$

Expanding the covariant derivatives in terms of Christoffel symbols:

$$F_{ab} = (\partial_a A_b - \Gamma^c_{ab}A_c) - (\partial_b A_a - \Gamma^c_{ba}A_c) = \partial_a A_b - \partial_b A_a. \tag{11.8}$$

Because the connection is torsion-free ($\Gamma^c_{ab} = \Gamma^c_{ba}$), the Christoffel symbols cancel identically. Hence, **the definition of the gauge field strength $F_{ab} = \partial_a A_b - \partial_b A_a$ is invariant under minimal coupling** and identical in curved and flat spacetimes.

11.1.2 Covariant Divergence of Symmetric Rank-2 Tensors

Next, consider a **symmetric rank-2 tensor** $T^{ki} = T^{ik}$, such as the energy-momentum-stress tensor of matter. In mixed component form $T^k{}_i = g_{im}T^{km}$, the covariant divergence is given by:

$$\nabla_k T^k{}_i = \partial_k T^k{}_i + \Gamma^k_{km}T^m{}_i - \Gamma^m_{ik}T^k{}_m. \tag{11.9}$$

Using the determinant identity $\Gamma^k_{km} = \frac{1}{\sqrt{-g}}\partial_m\sqrt{-g}$, the first two terms combine into a total derivative under the volume element:

$$\partial_k T^k{}_i + \Gamma^k_{km}T^m{}_i = \partial_k T^k{}_i + \left(\frac{1}{\sqrt{-g}}\partial_k\sqrt{-g}\right)T^k{}_i = \frac{1}{\sqrt{-g}}\partial_k\left(\sqrt{-g}\,T^k{}_i\right). \tag{11.10}$$

Now, let us examine the remaining connection term $-\Gamma^m_{ik}T^k{}_m$. Substituting the explicit expression for the Christoffel connection in terms of the metric:

$$\begin{aligned}\Gamma^m_{ik}T^k{}_m &= \left[\frac{1}{2}g^{mn}\left(\partial_i g_{nk} + \partial_k g_{in} - \partial_n g_{ik}\right)\right]T^k{}_m \\ &= \frac{1}{2}\left(\partial_i g_{nk} + \partial_k g_{in} - \partial_n g_{ik}\right)\left(g^{mn}T^k{}_m\right) \\ &= \frac{1}{2}\left(\partial_i g_{nk} + \partial_k g_{in} - \partial_n g_{ik}\right)T^{kn}.\end{aligned} \tag{11.11}$$

Because T^{kn} is symmetric ($T^{kn} = T^{nk}$), let us evaluate the contraction with the last two terms in the bracket:

$$\left(\partial_k g_{in} - \partial_n g_{ik}\right)T^{kn} = \partial_k g_{in}T^{kn} - \partial_n g_{ik}T^{kn}. \tag{11.12}$$

Relabeling the dummy indices $k \leftrightarrow n$ in the second term:

$$\partial_n g_{ik} T^{kn} = \partial_k g_{in} T^{nk} = \partial_k g_{in} T^{kn}. \tag{11.13}$$

Therefore, these two terms cancel identically:

$$\left(\partial_k g_{in} - \partial_n g_{ik}\right) T^{kn} = 0. \tag{11.14}$$

Only the first term $\partial_i g_{nk}$ survives! Thus:

$$\Gamma^m_{ik} T^k{}_m = \frac{1}{2} T^{kn} \partial_i g_{nk}. \tag{11.15}$$

Substituting this back into Eq. (11.9) yields the master identity for symmetric tensors:

$$\boxed{\nabla_k T^k{}_i = \frac{1}{\sqrt{-g}} \partial_k \left(\sqrt{-g}\, T^k{}_i\right) - \frac{1}{2} T^{kn} \partial_i g_{nk}} \qquad (\text{for } T^{kn} = T^{nk}). \tag{11.16}$$

Physical Insight & Motivation: Physical Interpretation: Gravitational Exchange of Energy-Momentum

In flat spacetime ($\partial_i g_{nk} = 0$, $\sqrt{-g} = 1$), local energy-momentum conservation takes the familiar form $\partial_k T^k{}_i = 0$, leading to strictly conserved total energy and momentum.
In curved spacetime, general covariance demands $\nabla_k T^k{}_i = 0$. By Eq. (11.16), this implies:

$$\partial_k \left(\sqrt{-g}\, T^k{}_i\right) = \frac{1}{2} \sqrt{-g}\, T^{kn} \partial_i g_{nk}. \tag{11.17}$$

The right-hand side represents the **gravitational force density** acting on matter. Matter alone does not possess a conserved energy-momentum current because it exchanges energy and momentum with the dynamical geometry of spacetime! Only when spacetime admits a continuous symmetry (a Killing vector field ξ^a, so that $\mathcal{L}_\xi g_{ab} = 0$) does a true conserved current $\mathcal{J}^a = T^a{}_b \xi^b$ exist.

11.1.3 Integral Conservation with Antisymmetric Tensors

When the covariant divergence of an antisymmetric tensor field vanishes ($\nabla_a Q^{ab} = 0$), integrating over an arbitrary 4-volume $\mathcal{M}$ weighted by the invariant volume measure yields:

$$\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \nabla_a Q^{ab} = \int_{\mathcal{M}} \mathrm{d}^4x\, \partial_a \left(\sqrt{-g}\, Q^{ab}\right) = \oint_{\partial\mathcal{M}} \mathrm{d}^3x \left(\sqrt{-g}\, Q^{ab}\right) n_a = 0. \tag{11.18}$$

This indicates that the flux of the tensor density $\sqrt{-g}\, Q^{ab}$ across any closed 3D boundary $\partial\mathcal{M}$ is identically zero, establishing that $\sqrt{-g}\, Q^{ab}$ defines conserved topological surface charges on foliated boundaries.

11.2 Foundations of Coordinate Charts, Boundaries, and Horizon Singularities

11.2.1 Local Inertial Coordinates vs. Global Coordinate Systems

In General Relativity, coordinates x^μ are purely arbitrary smooth labeling maps $\phi : U \subset \mathcal{M} \to \mathbb{R}^4$ defined on open subsets U of the spacetime manifold $\mathcal{M}$. They have no standalone physical meaning until metric intervals $\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ are evaluated.

Foundational Question: Pedagogical Inquiry: Why Are Local Inertial Frames Strictly Local?

Question: *In GR, we have Local Inertial Coordinates (LIC) and global coordinates, like the Schwarzschild coordinate system. Both coordinate choices have no inherent physical interpretation on their own. For local inertial coordinates, physical Cartesian-like interpretations exist only in a small region close to the base point P. Why is it that LIC is valid only in a small neighborhood, whereas Schwarzschild coordinates are able to cover a vastly larger patch of spacetime?*

Resolution and Geometric Analysis:

1. **The Rigid Constraint of Local Inertial Coordinates (LIC):** A Local Inertial Coordinate system (or Riemann Normal Coordinate system, RNC) centered at an event P is constructed by demanding that at P:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}, \qquad \partial_\lambda g_{\mu\nu}(P) = 0 \iff \Gamma^\lambda_{\mu\nu}(P) = 0. \tag{11.19}$$

   RNC coordinates are assigned by shooting geodesics radially outward in all spacetime directions from P with initial tangent vectors $v^a \in T_P\mathcal{M}$. Events along each geodesic are assigned coordinates $x^\mu = v^\mu \tau$.

   However, spacetime possesses non-vanishing curvature ($R^\rho{}_{\sigma\mu\nu} \neq 0$). By the Jacobi geodesic deviation equation ($\frac{D^2\xi^a}{\mathrm{d}\tau^2} = R^a{}_{bcd}u^b u^c \xi^d$), neighboring geodesics inevitably accelerate toward or away from each other due to tidal gravitational fields. As one moves away from P, the metric components deviate quadratically:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}(P)x^\alpha x^\beta + \mathcal{O}(x^3). \tag{11.20}$$

   Beyond a characteristic curvature scale $r_{\text{curv}} \sim 1/\sqrt{|R|}$, the outward-shooting geodesics begin to cross each other (caustics and conjugate points). At any point where two geodesics intersect, two distinct tangent vectors $v_1^\mu \neq v_2^\mu$ map to the exact same physical event. At that point, the exponential map $\exp_P : T_P\mathcal{M} \to \mathcal{M}$ ceases to be bijective, and the coordinate system breaks down entirely!

2. **The Global Adaptability of Symmetry-Adapted Coordinates (e.g., Schwarzschild):** Schwarzschild coordinates (t, r, θ, ϕ) are **not** constructed by shooting straight geodesic rays from a single point. Instead, they are constructed by adapting the coordinate grid directly to the **global isometries (Killing vector fields)** of the spacetime manifold:

   - The coordinate t is chosen along the orbits of the timelike Killing vector field $\xi = \partial_t$ (stationarity).
   - The angular coordinates (θ, ϕ) are chosen along the orbits of the spatial $SO(3)$ isometry group (spherical symmetry).
   - The radial coordinate r is defined geometrically such that the area of a sphere of constant (t, r) is identically $4\pi r^2$ ($r = \sqrt{A/4\pi}$).

   Because these coordinates follow the intrinsic symmetries of the spacetime geometry rather than an artificial straight-line congruence radiating from a single origin, they remain perfectly non-singular and well-behaved across the entire exterior manifold $2M < r < \infty$.

11.2.2 Gravitational Lensing as Physical Proof of Chart Breakdown

Foundational Question: Pedagogical Inquiry: Gravitational Lensing and the Breakdown of Riemann Normal Coordinates

Question: *In Misner, Thorne, and Wheeler (MTW), for Riemann Normal Coordinates, one sends out many geodesics assuming they never intersect to label events using selector parameters and affine parameters. Is the observed physical phenomenon of gravitational lensing a definitive proof that such coordinate systems must inevitably become ill-defined globally?*

Resolution and Geometric Proof: Yes, absolutely. Gravitational lensing is the direct observational manifestation of conjugate points and caustics in null geodesic congruences. Consider an observer at event P. If the observer attempts to construct a global coordinate chart by shooting geodesics radially outward:

- When a massive deflector (such as a galaxy or black hole) lies between a distant source (e.g., a quasar) and the observer, multiple null geodesics leaving the source converge at the observer's telescope.
- By time-reversal symmetry of null geodesics, rays shot outward by the observer with different initial directional angles $\theta_1 \neq \theta_2$ will bend around opposite sides of the lensing mass and intersect at the source event S.
- At the intersection event S, the coordinate map $\phi : \mathcal{M} \to \mathbb{R}^4$ assigns **two distinct coordinate tuples** to a single physical event in spacetime.
- Mathematically, the differential of the exponential map $d(\exp_P)$ loses maximal rank, the Jacobian determinant vanishes:

$$J = \det\left(\frac{\partial x^{\mu}_{\text{RNC}}}{\partial x^{\alpha}_{\text{manifold}}}\right) = 0, \tag{11.21}$$

  and the metric determinant in RNC develops a coordinate singularity.

Thus, astronomical observations of multiple lensed images are an undeniable empirical proof that local inertial / Riemann normal coordinate charts cannot cover the universe globally.

11.2.3 Detecting Coordinate Boundaries and Horizon Singularities

Foundational Question: Pedagogical Inquiry: How Do We Identify Chart Boundaries and Event Horizons?

Question: *How do I know that an event I am trying to study does not lie within my chosen coordinate system? How do we locate the boundaries of a coordinate chart? For instance, inside the black hole event horizon, standard Schwarzschild coordinates fail, and we must transition to Eddington–Finkelstein or Kruskal coordinates. Similarly, in Rindler spacetime, $x \to \infty$ in certain charts only covers the right Rindler wedge.*

Diagnostic Framework for Chart Boundaries: In differential geometry, a manifold is covered by an atlas of overlapping coordinate patches $\{(U_\alpha, \phi_\alpha)\}$. A single chart covers only the open subset U_α. To determine whether a boundary is reached and whether spacetime continues beyond it:

1. **Metric Degeneracy and Coordinate Infinities:** Check if the metric components $g_{\mu\nu}$ or inverse components $g^{\mu\nu}$ blow up or if the determinant vanishes ($\det g = 0$ or $\det g = \infty$).
   - In Schwarzschild coordinates, at $r = 2M$, $g_{00} = -(1 - 2M/r) \to 0$ and $g_{rr} = (1 - 2M/r)^{-1} \to \infty$.
   - In Rindler coordinates, at $\bar{x} = -c^2/a$, $g_{00} = -(1 + a\bar{x}/c^2)^2 \to 0$.
2. **Geodesic Incompleteness Test:** Calculate the trajectory of a freely falling particle or photon $x^\mu(\lambda)$ using the geodesic equation. If the trajectory reaches the boundary of the coordinate chart ($r \to 2M$ or $\bar{x} \to -c^2/a$) at a **finite value of the affine parameter / proper time** ($\tau_{\text{horizon}} < \infty$), while the coordinate time diverges ($t \to \infty$), the boundary is an artificial barrier of the coordinate chart, not the end of physical spacetime.
3. **Curvature Invariant Computation (Kretschmann Scalar):** Evaluate coordinate-independent scalar invariants constructed from the Riemann curvature tensor:
$$K \equiv R_{abcd}R^{abcd}, \qquad R \equiv g^{ab}R_{ab}, \qquad C^2 \equiv C_{abcd}C^{abcd}. \tag{11.22}$$
   - If K is **finite and smooth** at the boundary, the singularity is purely a **coordinate singularity**. The manifold can and must be extended into a larger chart (e.g., Eddington–Finkelstein, Painlevé–Gullstrand, or Kruskal–Szekeres coordinates for Schwarzschild, and global Minkowski coordinates (T, X) for Rindler).
   - In Schwarzschild:
$$K = \frac{48G^2M^2}{c^4r^6} \implies K|_{r=2M} = \frac{3c^4}{4G^2M^4} < \infty. \tag{11.23}$$
     The horizon $r = 2M$ is completely non-singular.
   - If $K \to \infty$ (e.g., at $r = 0$, where $K \to \infty$), it is a **true physical curvature singularity** (spacetime curvature is infinite, tidal forces become unbounded, and geodesics terminate).

11.3 Curvilinear Tensor Calculus and the Laplace–Beltrami Operator

11.3.1 Covariant Divergence in Curvilinear 3D Coordinates

In general 3-dimensional curvilinear coordinates (such as cylindrical, spherical polar, or toroidal coordinates), the metric is Riemannian ($g = \det(g_{ij}) > 0$). The covariant divergence of a contravariant vector field A^i is:
$$\vec{\nabla} \cdot \vec{A} = \nabla_i A^i = \frac{1}{\sqrt{g}}\partial_i \left(\sqrt{g}\, A^i\right). \tag{11.24}$$

11.3.2 The Laplace–Beltrami Operator

The invariant Laplacian acting on a smooth scalar field $\Phi(x)$ on any (pseudo-)Riemannian manifold $(\mathcal{M}, g_{ij})$ is the **Laplace–Beltrami operator**:
$$\Box\Phi \equiv \nabla_i\nabla^i\Phi = \nabla_i \left(g^{ij}\nabla_j\Phi\right) = \nabla_i \left(g^{ij}\partial_j\Phi\right). \tag{11.25}$$

Applying the divergence identity Eq. (11.24) to the contravariant vector $A^i = g^{ij}\partial_j\Phi$:

$$\boxed{\Box\Phi = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}\, g^{ij}\partial_j\Phi\right)}. \tag{11.26}$$

11.3.3 Coordinate Basis vs. Orthonormal (Physical) Basis: Lamé Scale Factors

In practical physics applications, vector components are often specified with respect to a normalized, dimensionless **orthonormal basis** $\{\hat{e}_i\}$ rather than the natural **coordinate basis** $\{\vec{e}_i = \partial_i\}$.

For an orthogonal metric where $g_{ij} = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$, the **Lamé scale factors** are:

$$h_i \equiv \sqrt{g_{ii}} \quad (\text{no sum over } i), \qquad |\vec{e}_i| = \sqrt{g_{ii}} = h_i. \tag{11.27}$$

The unit orthonormal basis vectors are defined by:

$$\hat{e}_i \equiv \frac{\vec{e}_i}{h_i} \iff \vec{e}_i = h_i\hat{e}_i. \tag{11.28}$$

Expanding an arbitrary physical vector field $\vec{V}$ in both bases:

$$\vec{V} = A^i\vec{e}_i = A^i\,(h_i\hat{e}_i) = \left(A^i h_i\right)\hat{e}_i \equiv V_{(i)}\hat{e}_i, \tag{11.29}$$

where $V_{(i)}$ denotes the **physical / orthonormal components** of $\vec{V}$. Thus, the dictionary between coordinate components A^i and physical components $V_{(i)}$ is:

$$\boxed{A^i = \frac{V_{(i)}}{h_i} = \frac{V_{(i)}}{\sqrt{g_{ii}}}}. \tag{11.30}$$

11.3.4 Detailed Worked Example: Spherical Polar Coordinates

Consider standard 3D Euclidean space parameterized by spherical polar coordinates (r, θ, ϕ):

$$\mathrm{d}s^2_{\text{space}} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{11.31}$$

The metric tensor and its inverse are:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{11.32}$$

The metric determinant is $g = 1 \cdot r^2 \cdot (r^2\sin^2\theta) = r^4\sin^2\theta$, giving the invariant volume element:

$$\sqrt{g} = r^2\sin\theta. \tag{11.33}$$

The Lamé scale factors are:

$$h_r = 1, \qquad h_\theta = r, \qquad h_\phi = r\sin\theta. \tag{11.34}$$

1. Contravariant Coordinate Divergence

Using Eq. (11.24):

$$\begin{aligned}\nabla_i A^i &= \frac{1}{r^2\sin\theta}\left[\partial_r\left(r^2\sin\theta\, A^r\right)+\partial_\theta\left(r^2\sin\theta\, A^\theta\right)+\partial_\phi\left(r^2\sin\theta\, A^\phi\right)\right]\\ &= \frac{1}{r^2}\partial_r\left(r^2A^r\right)+\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\, A^\theta\right)+\partial_\phi A^\phi.\end{aligned} \tag{11.35}$$

2. Orthonormal Vector Calculus Divergence

Transforming to physical components using Eq. (11.30):

$$A^r = V_{(r)}, \qquad A^\theta = \frac{V_{(\theta)}}{r}, \qquad A^\phi = \frac{V_{(\phi)}}{r\sin\theta}. \tag{11.36}$$

Substituting these into Eq. (11.35):

$$\begin{aligned}\vec{\nabla}\cdot\vec{V} &= \frac{1}{r^2}\partial_r\left(r^2V_{(r)}\right)+\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\frac{V_{(\theta)}}{r}\right)+\partial_\phi\left(\frac{V_{(\phi)}}{r\sin\theta}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2V_{(r)}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, V_{(\theta)}\right)+\frac{1}{r\sin\theta}\frac{\partial V_{(\phi)}}{\partial\phi}.\end{aligned} \tag{11.37}$$

This exactly reproduces the familiar divergence formula of undergraduate vector calculus!

3. Spherical Polar Laplacian

Evaluating the Laplace–Beltrami operator on a scalar function $\Phi(r,\theta,\phi)$ via Eq. (11.26):

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2\sin\theta}\partial_i\left(r^2\sin\theta\, g^{ik}\partial_k\Phi\right)\\ &= \frac{1}{r^2\sin\theta}\left[\partial_r\left(r^2\sin\theta\, g^{rr}\partial_r\Phi\right)+\partial_\theta\left(r^2\sin\theta\, g^{\theta\theta}\partial_\theta\Phi\right)+\partial_\phi\left(r^2\sin\theta\, g^{\phi\phi}\partial_\phi\Phi\right)\right]\\ &= \frac{1}{r^2\sin\theta}\partial_r\left(r^2\sin\theta\,\partial_r\Phi\right)+\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\Phi\right)+\frac{1}{r^2\sin\theta}\partial_\phi\left(\frac{1}{\sin\theta}\,\partial_\phi\Phi\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}.\end{aligned} \tag{11.38}$$

11.4 Commutator of Covariant Derivatives on Scalar Fields

11.4.1 Proof of Torsion-Free Commutation $[\nabla_a,\nabla_b]\Phi = 0$

Let $\Phi(x)$ be an arbitrary smooth scalar field on $\mathcal{M}$. The first covariant derivative is simply the partial derivative:

$$\nabla_b\Phi = \partial_b\Phi. \tag{11.39}$$

The second covariant derivative treats $\nabla_b\Phi = \partial_b\Phi$ as a rank-(0,1) covariant tensor:

$$\nabla_a\nabla_b\Phi = \partial_a\left(\partial_b\Phi\right)-\Gamma^c_{ab}\left(\partial_c\Phi\right). \tag{11.40}$$

Reversing the order of differentiation:

$$\nabla_b\nabla_a\Phi = \partial_b\left(\partial_a\Phi\right)-\Gamma^c_{ba}\left(\partial_c\Phi\right). \tag{11.41}$$

Subtracting Eq. (11.41) from Eq. (11.40):

$$[\nabla_a,\nabla_b]\Phi = \nabla_a\nabla_b\Phi-\nabla_b\nabla_a\Phi$$

$$= (\partial_a \partial_b \Phi - \partial_b \partial_a \Phi) - (\Gamma^c_{ab} - \Gamma^c_{ba})\, \partial_c \Phi. \tag{11.42}$$

By Schwarz's theorem on mixed partial derivatives, $\partial_a \partial_b \Phi = \partial_b \partial_a \Phi$. Furthermore, for the Levi-Civita connection, the torsion tensor ${T^c}_{ab} \equiv \Gamma^c_{ab} - \Gamma^c_{ba}$ vanishes identically:

$$\boxed{[\nabla_a, \nabla_b]\Phi = 0}. \tag{11.43}$$

Physical Insight & Motivation: Contrast with Vector Fields: The Riemann Curvature Tensor

While covariant derivatives commute when acting on scalars ($[\nabla_a, \nabla_b]\Phi = 0$), they **do not commute** when acting on vector fields V^c:

$$[\nabla_a, \nabla_b] V^c = {R^c}_{dab} V^d. \tag{11.44}$$

The non-commutativity of covariant derivatives on vectors is the fundamental definition of the **Riemann curvature tensor** ${R^c}_{dab}$!

11.5 Covariant Gauss's Theorem and Spacetime Conservation Laws

11.5.1 Derivation of the 4D Covariant Divergence Theorem

Let $\mathcal{M}$ be a 4-dimensional spacetime manifold with boundary $\partial\mathcal{M}$, endowed with metric g_{ab}. Let A^i be a smooth contravariant vector field. Integrating the covariant divergence over $\mathcal{M}$:

$$\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \nabla_i A^i = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g}\, A^i \right) \right] = \int_{\mathcal{M}} \mathrm{d}^4x\, \partial_i \left(\sqrt{-g}\, A^i \right). \tag{11.45}$$

Applying the classical multi-variable Gauss's divergence theorem in $\mathbb{R}^4$:

$$\int_{\mathcal{M}} \mathrm{d}^4x\, \partial_i \left(\sqrt{-g}\, A^i \right) = \oint_{\partial\mathcal{M}} \mathrm{d}^3x \left(\sqrt{-g}\, A^i \right) \hat{n}_i = \oint_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|}\, A^i n_i \epsilon, \tag{11.46}$$

where:

- $h_{\mu\nu}$ is the induced metric on the 3D boundary hypersurface $\partial\mathcal{M}$ with determinant $h \equiv \det(h_{\mu\nu})$,
- n_i is the unit normal 1-form pointing outward from the boundary,
- $\epsilon \equiv n_i n^i = g^{ij} n_i n_j = \pm 1$ is the causal indicator ($\epsilon = -1$ for a timelike normal / spacelike boundary, and $\epsilon = +1$ for a spacelike normal / timelike boundary).

11.5.2 Foliation of Spacetime and Boundary Decompositions

Consider a spacetime volume bounded between two constant-time Cauchy hypersurfaces $\Sigma(t_1)$ and $\Sigma(t_2)$, and enclosed laterally by a spatial cylinder $\mathcal{B}_\infty$ at spatial infinity ($r \to \infty$), as illustrated in Figure 11.1:

$$\partial\mathcal{M} = \Sigma(t_2) \cup \Sigma(t_1) \cup \mathcal{B}_\infty. \tag{11.47}$$

Decomposing the boundary integral into temporal (spacelike hypersurfaces) and spatial components:

$$\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \nabla_i A^i = \left[\int_{\Sigma(t_2)} \mathrm{d}^3x \sqrt{h}\, A^0 n_0 - \int_{\Sigma(t_1)} \mathrm{d}^3x \sqrt{h}\, A^0 n_0 \right] + \int_{\mathcal{B}_\infty} \mathrm{d}^3x \sqrt{|h|}\, A^\alpha n_\alpha. \tag{11.48}$$

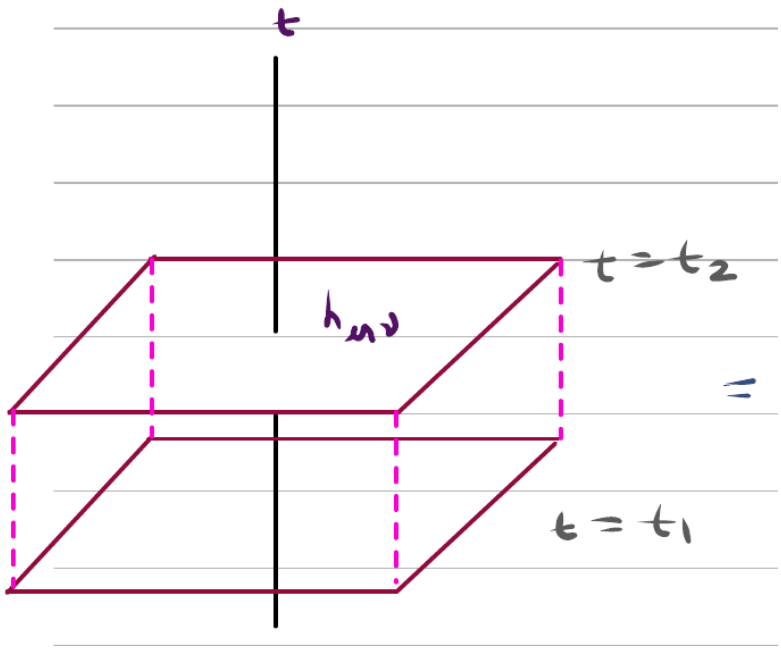


Figure 11.1: Spacetime integration volume $\mathcal{M}$ bounded by two spacelike Cauchy surfaces $\Sigma(t_1)$ and $\Sigma(t_2)$ (with induced 3-metric $h_{\mu\nu}$ and timelike normals n_0) and the spatial boundary cylinder $\mathcal{B}_\infty$ at spatial infinity ($r \to \infty$).

Important Requirement: The Crucial Role of Spatial Boundary Conditions at Infinity

In physics, we frequently drop the spatial boundary term at spatial infinity:

$$\int_{\mathcal{B}_\infty} \mathrm{d}^3x \sqrt{|h|}\, A^\alpha n_\alpha = 0. \tag{11.49}$$

This step is only legitimate if the spatial field components fall off sufficiently rapidly as $|\vec{x}| \to \infty$:

$$A^\alpha(\vec{x}, t) \xrightarrow{|\vec{x}|\to\infty} 0 \quad \text{such that } |A^\alpha| = \wr\left(\frac{1}{r^2}\right). \tag{11.50}$$

When this decay condition holds, all boundary flux escapes purely through the temporal surfaces $\Sigma(t)$, ensuring that:

$$\int_{\Sigma(t_2)} \mathrm{d}^3x \sqrt{h}\, A^0 n_0 - \int_{\Sigma(t_1)} \mathrm{d}^3x \sqrt{h}\, A^0 n_0 = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \nabla_i A^i. \tag{11.51}$$

Physical Insight & Motivation: Connection to Open vs. Closed Strings and On-Shell Momentum

In variational field theory, discarding the spatial boundary term at infinity ensures that the canonical momentum density conjugate to a field ϕ:

$$P_i = \left.\frac{\partial \mathcal{L}}{\partial(\partial^0\phi)}\right|_{\text{on-shell}} = \dot{\phi}, \qquad S = \frac{1}{2}\int \mathrm{d}^4x \sqrt{-g}\, \partial_a\phi\partial^a\phi, \tag{11.52}$$

leads to conserved global canonical generators $P_\mu = \int_\Sigma \mathrm{d}^3x\, T^0{}_\mu$.
In **String Theory**, this boundary decomposition dictates the fundamental distinction between:

- **Closed Strings:** The worldsheet manifold has no spatial boundaries ($\partial\Sigma_{\text{string}} = \emptyset$). Boundary terms vanish automatically due to periodic boundary conditions.
- **Open Strings:** The worldsheet has spatial endpoints ($\sigma = 0, \pi$). Non-vanishing boundary terms require either Neumann ($\partial_\sigma X^\mu = 0$) or Dirichlet ($X^\mu = \text{const}$) boundary conditions, which physically force open string endpoints to remain anchored to **D-branes**!

11.5.3 Current Conservation and Invariant Global Charge

Let $J^i = (\rho, \vec{J}) = (\rho, \rho\vec{v})$ be a conserved 4-current density satisfying the local covariant continuity equation:

$$\nabla_i J^i = 0 \iff \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}\, J^i\right) = 0. \tag{11.53}$$

In flat spacetime (or in coordinates where $\sqrt{-g} = 1$), this reduces to the classical continuity equation:

$$\partial_0 J^0 + \partial_\alpha J^\alpha = 0 \iff \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0. \tag{11.54}$$

Integrating $\nabla_i J^i = 0$ over the spacetime manifold $\mathcal{M}$ using Gauss's Theorem:

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\nabla_i J^i = 0 \implies \oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\, J^i n_i = 0. \tag{11.55}$$

Decomposing into Cauchy surfaces $\Sigma(t_1)$, $\Sigma(t_2)$ and the spatial boundary $\mathcal{B}_\infty$:

$$\left[\int_{\Sigma(t_2)} \mathrm{d}^3x\sqrt{h}\, J^0 n_0 - \int_{\Sigma(t_1)} \mathrm{d}^3x\sqrt{h}\, J^0 n_0\right] + \int_{\mathcal{B}_\infty} \mathrm{d}^3x(\rho\vec{v}\cdot\hat{n}) = 0. \tag{11.56}$$

Imposing isolated boundary conditions ($\vec{J}\cdot\hat{n} \to 0$ as $r \to \infty$), the spatial boundary integral vanishes:

$$\int_{\Sigma(t_1)} \mathrm{d}^3x\sqrt{h}\, J^0 = \int_{\Sigma(t_2)} \mathrm{d}^3x\sqrt{h}\, J^0. \tag{11.57}$$

We define the **total invariant electric / scalar charge** on a Cauchy hypersurface $\Sigma(t)$ by:

$$\boxed{Q(t) \equiv \int_{\Sigma(t)} \mathrm{d}^3x\sqrt{h}\, J^0 = \text{constant in time}}. \tag{11.58}$$

11.5.4 Flux of Antisymmetric 2-Forms and Gauss's Law

For an antisymmetric 2-form $Q^{ik} = -Q^{ki}$ whose covariant divergence vanishes or sources a current ($\nabla_i Q^{ik} = 0$), the total flux across a closed 3-boundary is:

$$\mathcal{Q}^k \equiv \oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\, Q^{ik} n_i = 0. \tag{11.59}$$

Applying this to the electromagnetic field tensor F^{ik} on a spatial 3-ball V bounded by a 2-sphere $\partial V = S^2$:

$$Q_{\text{electric}} = \oint_{S^2} \mathrm{d}^2x\sqrt{\sigma}\, F^{0r} = \frac{q}{\varepsilon_0}. \tag{11.60}$$

11.6 Frobenius' Theorem and Hypersurface Orthogonality

11.6.1 The Integrability Condition for Hypersurface Orthogonality

A non-vanishing vector field n_a (or 1-form $n = n_a \mathrm{d}x^a$) is said to be **hypersurface-orthogonal** if there exists a foliation of spacetime by a family of 3-dimensional hypersurfaces $\Sigma_c : \Phi(x) = c = \text{const}$ such that n_a is everywhere normal to Σ_c.

Theorem: Frobenius' Theorem for 1-Forms

A nowhere-vanishing smooth 1-form n_a is hypersurface-orthogonal if and only if it satisfies the **Frobenius integrability condition**:

$$\boxed{n_{[a}\nabla_b n_{c]} = 0 \iff n \wedge \mathrm{d}n = 0}. \tag{11.61}$$

Proof. Let us expand the fully antisymmetrized 3-index tensor $n_{[a}\nabla_b n_{c]}$:

$$n_{[a}\nabla_b n_{c]} = \frac{1}{3!}\left[n_a\nabla_b n_c + n_b\nabla_c n_a + n_c\nabla_a n_b - n_b\nabla_a n_c - n_c\nabla_b n_a - n_a\nabla_c n_b\right]. \tag{11.62}$$

First, write all cyclic permutations of the indices (a, b, c):

$$+n_a\nabla_b n_c + n_b\nabla_c n_a + n_c\nabla_a n_b, \tag{11.63}$$

and then subtract the odd permutations obtained by swapping any two indices:

$$-n_b\nabla_a n_c - n_c\nabla_b n_a - n_a\nabla_c n_b. \tag{11.64}$$

Now, let us expand the covariant derivatives in terms of partial derivatives and Christoffel symbols: $\nabla_b n_c = \partial_b n_c - \Gamma^d_{bc} n_d$. The connection terms contribute:

$$\begin{aligned} -\frac{1}{6}\Big[&n_a\Gamma^d_{bc}n_d + n_b\Gamma^d_{ca}n_d + n_c\Gamma^d_{ab}n_d \\ &- n_b\Gamma^d_{ac}n_d - n_c\Gamma^d_{ba}n_d - n_a\Gamma^d_{cb}n_d\Big]. \end{aligned} \tag{11.65}$$

Because the connection is symmetric ($\Gamma^d_{bc} = \Gamma^d_{cb}$), every positive term cancels its transposed counterpart identically:

$$n_a\Gamma^d_{bc}n_d - n_a\Gamma^d_{cb}n_d = 0, \quad n_b\Gamma^d_{ca}n_d - n_b\Gamma^d_{ac}n_d = 0, \quad n_c\Gamma^d_{ab}n_d - n_c\Gamma^d_{ba}n_d = 0. \tag{11.66}$$

Thus, the connection drops out completely, and the covariant Frobenius condition is strictly equivalent to the partial-derivative (exterior calculus) condition:

$$n_{[a}\nabla_b n_{c]} = n_{[a}\partial_b n_{c]} = 0. \tag{11.67}$$

By Frobenius' integrability theorem, condition (11.61) is satisfied if and only if there exists a smooth non-zero scalar **integrating factor** $\mu(x)$ and a scalar field $\Phi(x)$ such that:

$$\boxed{n_a = -\mu(x)\,\partial_a\Phi(x) = -\mu(x)\,\nabla_a\Phi(x)}. \tag{11.68}$$

The level surfaces $\Phi(x) = \text{constant}$ form the orthogonal hypersurfaces. □

11.6.2 Worked Example: Normal Vectors to Cylindrical Hypersurfaces in Static Spacetimes

Consider a general static, spherically symmetric spacetime metric in standard curvature coordinates (t, r, θ, ϕ):

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{11.69}$$

We wish to determine the unit normal 1-form n_a and vector n^a orthogonal to the constant-radius cylindrical hypersurface:

$$\Sigma_R : \Phi(x) \equiv r = R = \text{constant}. \tag{11.70}$$

1. Constructing the Normal 1-Form

By Frobenius' Theorem, the normal 1-form is proportional to the gradient of Φ:

$$n_a = -\mu(r)\,\partial_a r = -\mu(r)\begin{pmatrix}\partial_t r\\ \partial_r r\\ \partial_\theta r\\ \partial_\phi r\end{pmatrix}^T = -\mu(r)(0,1,0,0) = (0, -\mu(r), 0, 0)\,. \tag{11.71}$$

2. Enforcing Unit Normalization

The hypersurface Σ_R is a timelike cylinder (t is a coordinate on the surface), so its normal must be **spacelike**, satisfying $n_a n^a = +1$:

$$n_a n^a = g^{ab} n_a n_b = g^{rr}(n_r)^2 = 1. \tag{11.72}$$

From the inverse metric $g^{rr} = f(r)$, we have:

$$f(r)\cdot(-\mu)^2 = 1 \implies \mu^2 f(r) = 1 \implies \mu(r) = \frac{1}{\sqrt{f(r)}}. \tag{11.73}$$

3. Covariant and Contravariant Components

Thus, the unit normal 1-form n_a is:

$$n_a = \left(0, -\frac{1}{\sqrt{f(r)}}, 0, 0\right). \tag{11.74}$$

The contravariant unit normal vector $n^a = g^{ab} n_b$ is obtained by raising the index with $g^{rr} = f(r)$:

$$n^a = \left(0, -f(r)\cdot\frac{1}{\sqrt{f(r)}}, 0, 0\right) = \left(0, -\sqrt{f(r)}, 0, 0\right). \tag{11.75}$$

Checking the invariant norm:

$$n_a n^a = n_r n^r = \left(-\frac{1}{\sqrt{f(r)}}\right)\left(-\sqrt{f(r)}\right) = +1, \tag{11.76}$$

confirming exact unit spacelike normalization.

Chapter 12

Parallel Transport, Holonomy, and Riemann Curvature

12.1 Variational Mechanics, Boundary Physics, and the Einbein Action

12.1.1 Finite-Distance Boundaries vs. Asymptotic Spatial Boundaries

In the variational calculus of field theory, integrating by parts over a spacetime volume $\mathcal{M}$ generates a bulk Euler–Lagrange equation alongside a surface integral over the boundary $\partial\mathcal{M}$:

$$\delta S = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_a\frac{\partial\mathcal{L}}{\partial(\nabla_a\phi)}\right)\delta\phi + \oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,n_a\left(\frac{\partial\mathcal{L}}{\partial(\nabla_a\phi)}\right)\delta\phi. \tag{12.1}$$

Foundational Question: Why Boundary Terms Matter at Finite Distances vs. Asymptotic Infinity

A common question in variational calculus is why boundary terms can be freely discarded at spatial infinity ($r \to \infty$), but cannot be dropped when the boundary is located at a finite physical distance:

1. **Asymptotic Boundaries at Spatial Infinity ($r \to \infty$):** For isolated physical systems, the energy density and field configurations are localized in a compact region. As $|\vec{x}| = r \to \infty$, physical fields and their variations fall off rapidly:

$$\phi(r) = \mathcal{O}(r^{-1}), \quad \nabla_a\phi = \mathcal{O}(r^{-2}), \quad \delta\phi \xrightarrow{r\to\infty} 0. \tag{12.2}$$

   Because the surface area of a 2-sphere grows as r^2, the boundary integrand decays as $\mathcal{O}(r^{-2}) \times \mathcal{O}(r^{-1}) \times r^2 = \mathcal{O}(r^{-1}) \to 0$. Hence, the boundary flux through spatial infinity vanishes unconditionally.

2. **Boundaries at Finite Physical Distance (Horizons, Cavity Walls, Causal Diamonds):** When a boundary $\partial\mathcal{M}$ is located at a finite distance (such as the event horizon of a black hole, the acceleration horizon of a Rindler observer, or the conducting walls of a cavity):

   - The field values and normal derivatives $\partial_n\phi|_{\partial\mathcal{M}}$ are **strictly non-zero**.
   - Fixing Dirichlet boundary conditions ($\delta\phi|_{\partial\mathcal{M}} = 0$) eliminates $\delta\phi$, but fixing Neumann or metric boundary conditions requires adding explicit boundary counter-terms to the action.

- In General Relativity, the Einstein–Hilbert action $S_{\rm EH} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,R$ contains second derivatives of the metric ($\partial^2 g$). Varying $S_{\rm EH}$ on a manifold with a finite boundary leaves a non-vanishing normal derivative variation $\delta(\partial_n g_{\mu\nu}) \neq 0$. To obtain a well-posed variational problem, one **must** augment the bulk action with the **Gibbons–Hawking–York (GHY) boundary term**:

$$S_{\rm total} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,R + \frac{1}{8\pi G}\oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,\epsilon K, \tag{12.3}$$

  where $K = h^{ab}\nabla_a n_b$ is the extrinsic curvature trace of the boundary hypersurface.
- Dropping these boundary terms arbitrarily at finite boundaries throws away physical quasi-local energy-momentum (the **Brown–York stress tensor** $T^{\mu\nu}_{\rm BY} = \frac{2}{\sqrt{|h|}}\frac{\delta S}{\delta h_{\mu\nu}}$) and violates black hole thermodynamic entropy relations ($S_{\rm BH} = A/4G$).

12.1.2 First-Principles Derivation of the Einbein / Polyakov Action

The standard action for a free relativistic massive point particle is proportional to the invariant proper time along its worldline:

$$S = -m\int \mathrm{d}\tau\sqrt{-\dot{x}^2} = -m\int \mathrm{d}\tau\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}, \tag{12.4}$$

where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ is the four-velocity with respect to an arbitrary worldline parameter τ, and we adopt the mostly-plus metric signature $(-,+,+,+)$.

First-Principles Derivation via Dirac Constraint Theory. To transition to the Hamiltonian formalism, we perform a Legendre transformation. We first compute the canonical momenta conjugate to the coordinates x^μ:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \implies p^2 = p^\mu p_\mu = \frac{m^2\dot{x}_\mu\dot{x}^\mu}{-\dot{x}^2} = -m^2.$$

In a standard non-constrained system, one inverts this relation to express the velocities $\dot{x}^\mu$ as functions of the coordinates and momenta. However, this Legendre transformation is **non-invertible** because the Hessian matrix of the Lagrangian with respect to the velocities is singular:

$$\det\left(\frac{\partial^2 L}{\partial\dot{x}^\mu\partial\dot{x}^\nu}\right) = 0.$$

Instead of an invertible relationship, squaring the canonical momentum yields an algebraic identity relating the momenta themselves: $p^\mu p_\mu + m^2 = 0$. In Dirac's theory of constrained Hamiltonian systems, this is known as a **primary constraint**.

Next, we calculate the canonical Hamiltonian H_c:

$$\dot{x}^\mu p_\mu = \dot{x}^\mu\left(\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\right) = -m\sqrt{-\dot{x}^2} = L \implies H_c \equiv \dot{x}^\mu p_\mu - L = 0.$$

The canonical Hamiltonian vanishes identically ($H_c = 0$), which is a universal hallmark of fully reparametrization-invariant theories.

Because the Legendre transformation is non-invertible and the canonical Hamiltonian vanishes, we incorporate the primary constraint into the Lagrangian formalism via a Lagrange multiplier $\eta(\tau)$ to restore non-trivial dynamics:

$$L = \dot{x}^\mu p_\mu - \frac{\eta}{2}(p^2 + m^2) - H_c \quad (\text{where } H_c = 0).$$

From the Euler–Lagrange equation of motion for the auxiliary canonical momentum p_μ:

$$\left(\frac{\partial L}{\partial p_\mu}\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{p}_\mu}\right) = 0 \implies \dot{x}^\mu - \eta p^\mu = 0 \implies p^\mu = \frac{1}{\eta}\dot{x}^\mu.$$

Eliminating p^μ from the Lagrangian gives the second-order coordinate velocity form:

$$L = \dot{x}^\mu\left(\frac{\dot{x}_\mu}{\eta}\right) - \frac{\eta}{2}\left(\left(\frac{\dot{x}}{\eta}\right)^2 + m^2\right) = \frac{\dot{x}^2}{\eta} - \frac{\dot{x}^2}{2\eta} - \frac{\eta}{2}m^2 = \frac{1}{2}\left(\frac{\dot{x}^2}{\eta} - \eta m^2\right).$$

This is the **Polyakov action** (or **einbein action**) for the point particle. □

Equivalence of Auxiliary Multiplier Parameterizations: Whether one introduces the auxiliary field as a Lagrange multiplier $\eta(\tau)$ enforcing the Dirac mass-shell constraint $p^2+m^2 = 0$, or as the 1D worldline lapse/einbein $e(\tau)$, or as $\lambda(\tau)/2$, the resulting action and equations of motion are **identical**:

$$S = \frac{1}{2}\int \mathrm{d}\tau\left(\frac{\dot{x}^2}{\eta} - \eta m^2\right) \equiv \frac{1}{2}\int \mathrm{d}\tau\left(\frac{\dot{x}^2}{e} - e m^2\right) \equiv \int \mathrm{d}\tau\left(\frac{\dot{x}^2}{\lambda} - \frac{\lambda}{4}m^2\right). \tag{12.5}$$

By introducing the Lagrange multiplier to systematically manage the non-invertible Legendre transformation, we obtain a formulation that completely avoids the non-analytic square root and seamlessly accommodates massless particles ($m = 0$).

In particular, choosing the gauge parameter $\eta = 1/m^2$ yields the much simpler, non-singular quadratic action:

$$S = \frac{1}{2}\int \mathrm{d}\tau\left(\frac{\dot{x}^2}{1/m^2} - \frac{1}{m^2}m^2\right) = \frac{1}{2}\int \mathrm{d}\tau\left(m^2\dot{x}^2 - 1\right) = \frac{m^2}{2}\int \mathrm{d}\tau\, g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{1}{2}\int \mathrm{d}\tau. \tag{12.6}$$

Because the worldline parameter interval $[\tau_1, \tau_2]$ has fixed boundary endpoints, the variation of the constant term vanishes identically ($\delta\int \mathrm{d}\tau = 0$), so the $-\frac{1}{2}$ drops out of the Euler–Lagrange equations of motion. Dividing out the constant mass factor (or rescaling the worldline parameter), this leaves the standard quadratic kinetic action universally employed in General Relativity:

$$S_{\text{quad}} = \frac{1}{2}\int \mathrm{d}\tau\, g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu. \tag{12.7}$$

Why is S_{quad} the preferred action, and how does standard General Relativity routinely "forget" the Lagrange multiplier?

1. **Analytical Simplicity (No Square Roots):** Unlike the square-root proper-time action $S = -m\int \mathrm{d}\tau\sqrt{-\dot{x}^2}$, the Lagrangian $\mathcal{L}_{\text{quad}} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ is purely polynomial in velocities. Its Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}_{\text{quad}}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}_{\text{quad}}}{\partial x^\alpha} = 0 \implies \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0, \tag{12.8}$$

directly yield the affine geodesic equation without singular square-root denominators, smoothly applying to both massive ($m > 0$) and massless ($m = 0$) particles.

2. **The Variational Consequence of Gauge Fixing:** Setting $\eta = 1/m^2$ directly inside the action *prior* to variation fixes the gauge, thereby eliminating η from the variational calculus. As a result, we discard the algebraic Euler–Lagrange equation for the multiplier:

$$\frac{\partial L}{\partial \eta} = -\frac{\dot{x}^2}{2\eta^2} - \frac{m^2}{2} = 0 \implies \dot{x}^2 = -\eta^2 m^2 = -\frac{1}{m^2} \iff g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\frac{1}{m^2}, \tag{12.9}$$

which is precisely the physical mass-shell constraint. The gauge-fixed action S_{quad} varies solely with respect to $x^\mu(\tau)$ and has no dynamical awareness of the particle's mass.

3. **Why the Multiplier Can Be Safely Forgotten:** Along any solution of the geodesic equation, metric compatibility ($\nabla_\sigma g_{\mu\nu} = 0$) guarantees that the norm of the velocity vector is an exact **first integral of motion**:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) = 2g_{\mu\nu}\dot{x}^\mu \left(\frac{\mathrm{D}\dot{x}^\nu}{\mathrm{d}\tau}\right) = 0 \implies g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \text{constant}. \tag{12.10}$$

Because the norm is strictly conserved along geodesics, the geodesic equations ensure it remains constant, but cannot determine what constant it equals. The standard GR trick of forgetting the Lagrange multiplier succeeds because one can simply impose the missing constraint ($g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$ for timelike particles, or 0 for light rays) **by hand as an initial condition** at $\tau = 0$, which is then conserved for all τ.

12.2 Geometric Foundations of Parallel Transport and Holonomy

12.2.1 Global Flat-Space Parallelism vs. Curved-Space Holonomy

In flat Euclidean space (or flat Minkowski spacetime), parallel transport is an unambiguous, path-independent global operation. In Cartesian coordinates (x, y, z), the natural basis vectors $\{\hat{x}, \hat{y}, \hat{z}\}$ are everywhere constant throughout the manifold ($\partial_\mu \hat{e}_\nu = 0$). A vector field $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ is parallel transported along an arbitrary curve $x^\mu(t)$ simply by demanding that its components remain constant:

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = 0 \iff \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\partial_\mu A_i = 0. \tag{12.11}$$

In a curved Riemannian or Lorentzian manifold $(\mathcal{M}, g_{ab})$, there exists no global Cartesian coordinate system. Tangent spaces $T_P\mathcal{M}$ and $T_Q\mathcal{M}$ at distinct events $P \neq Q$ are distinct vector spaces:

- Two vectors living at different points cannot be added or subtracted directly.
- To compare a vector $V^a(P) \in T_P\mathcal{M}$ with a vector at Q, one must specify a smooth curve γ connecting P to Q and **parallel transport** V^a along γ.
- As illustrated in Figure 12.2b, parallel transporting a vector around a closed circuit on a curved surface (such as a 2-sphere) results in a net rotation angle $\Delta\theta = \Omega_{\text{solid}} = \iint \mathcal{K}\,\mathrm{d}A \neq 0$, where $\mathcal{K}$ is the Gaussian curvature. This phenomenon is called **geometric holonomy**.

Foundational Question: Can We Borrow a Global Vector Space by Embedding?

To add or subtract vectors that reside at different points on a manifold, and thus live in distinct tangent spaces, one requires a map to transport vectors from one space to another. Because diffeomorphism symmetry prevents any unique, canonical identification between different tangent spaces, no such map exists globally by default.

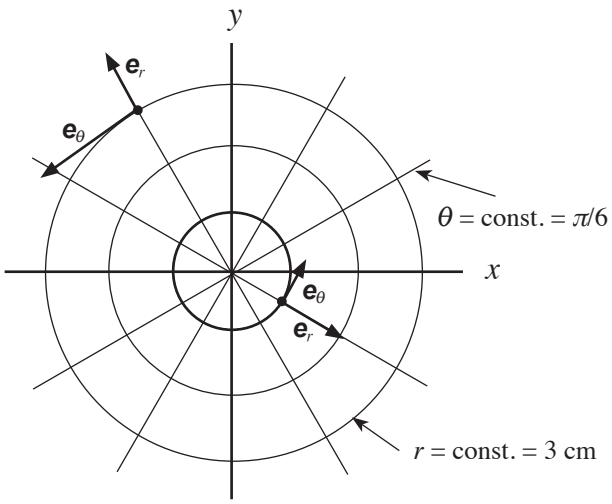
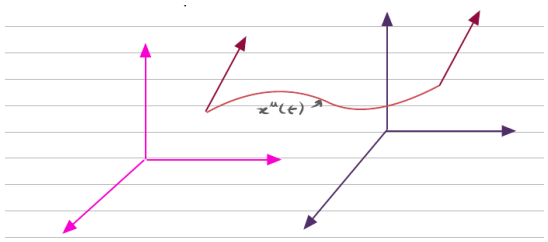
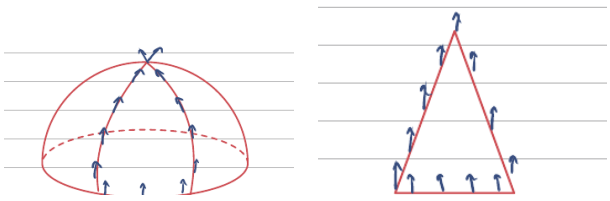


Figure 12.1: The polar-coordinate basis vectors change direction from point to point. Thus, as we move through the coordinate system, the local basis at one point is rotated relative to the basis at another. A vector that is purely radial with respect to one local basis therefore need not remain purely radial when compared with the basis at a neighboring point. The connection coefficients encode precisely this change of the local basis from one point to another, and hence appear in the parallel-transport equation.



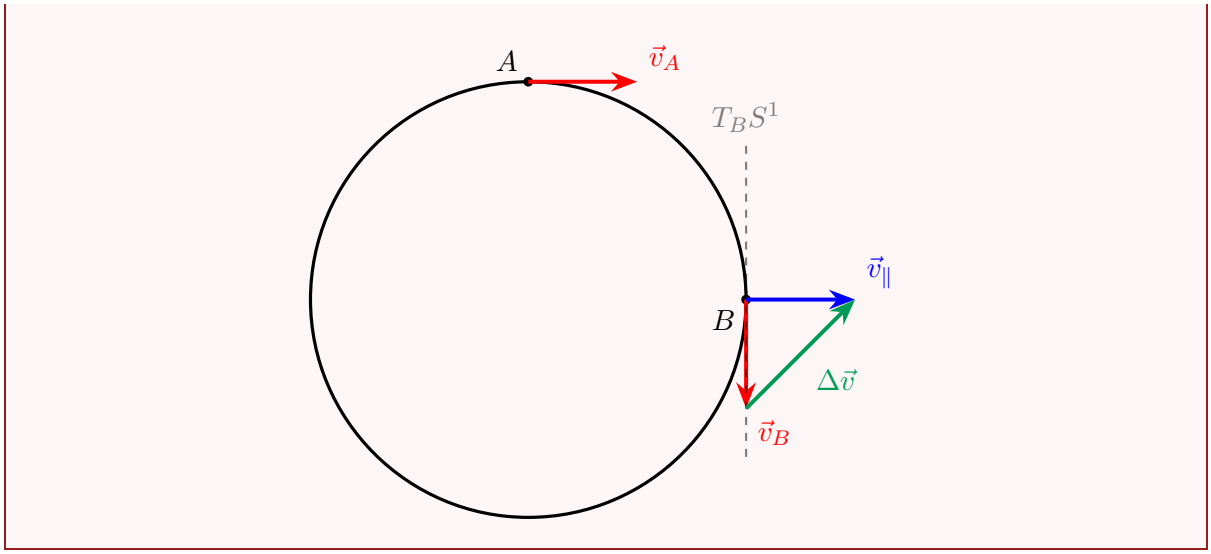
(a) Flat Cartesian coordinates: basis vectors do not vary with position.



(b) Closed-loop transport: 2-sphere anholonomy $\Delta\theta \neq 0$ (left) vs. flat plane closure $\Delta\theta = 0$ (right).

Figure 12.2: Geometric distinction between parallel transport in flat vs. curved manifolds.

However, specifying a path or introducing an additional field (such as a metric or a vector field) provides a prescription for constructing a transport map. This path- and structure-dependent choice is precisely why multiple notions of differentiation (e.g., the covariant derivative, the Lie derivative) exist on curved spaces.
The most naive attempt to bypass this intrinsic structure would be to embed the manifold S^1 into flat Euclidean space $\mathbb{R}^2$ and borrow the global vector space structure of the ambient space. Yet, this approach immediately fails: even though $\vec{v}_A \in T_A S^1$ and $\vec{v}_B \in T_B S^1$ can both be viewed as vectors in $\mathbb{R}^2$, their difference computed in $\mathbb{R}^2$ is not a tangent vector to S^1.



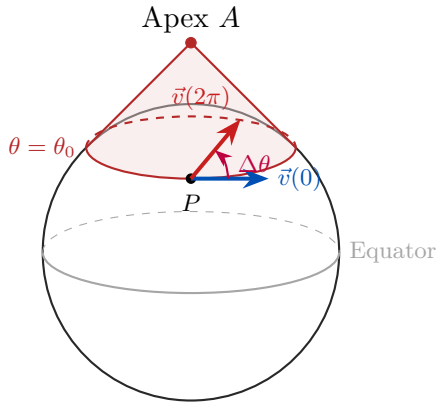
12.2.2 Intrinsic Parallel Transport via Coordinate Charts and Flattening Maps

Key Concept: The Foundational Role of Coordinate Charts: Flattening, Operating, and Pulling Back

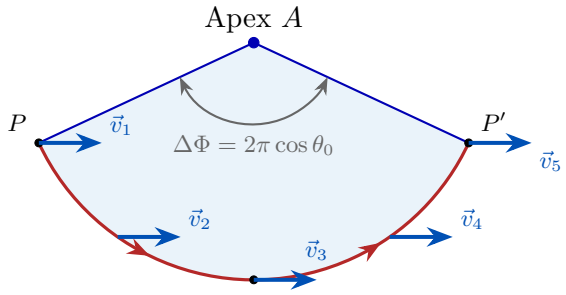
The profound geometric significance of a **coordinate chart** (U, ϕ) becomes transparent here. A curved manifold $\mathcal{M}$ possesses no global Cartesian coordinate frame and no canonical vector addition between disparate tangent spaces. A coordinate chart $\phi : U \subset \mathcal{M} \to \mathbb{R}^n$ is precisely the mathematical tool that allows us to map curves and lines from an arbitrary curved surface onto a flat Euclidean space $\mathbb{R}^n$.
In this flat space, we can execute standard geometric operations—such as Euclidean parallel transport ($\frac{\mathrm{d}\vec{v}}{\mathrm{d}\lambda} = 0$), vector comparison, and differentiation—while the metric $g_{\mu\nu}$ and connection coefficients $\Gamma^\lambda_{\mu\nu}$ systematically account for the coordinate distortion. Once the operation is performed in the flat representation, the inverse chart map $\phi^{-1} : \mathbb{R}^n \to U$ pulls the result back onto the arbitrary curved surface.
The two physical constructions below—the tangent cone method and the polyhedral discretization method—are direct geometric realizations of this fundamental chart principle.

To construct an intrinsic parallel transport law on a curved 2D surface (such as a 2-sphere S^2) using Euclidean intuition, we examine two explicit flattening constructions:

1. The Tangent Cone Method: Along any latitude circle $\theta = \theta_0$ on S^2, construct the tangent cone that touches the sphere along that parallel. The cone has zero intrinsic Gaussian curvature ($\mathcal{K}_{\text{cone}} = 0$). We can cut the cone open and unroll it flat onto the Euclidean plane without stretching or distorting intrinsic distances. Parallel transport on the unrolled sector is ordinary Euclidean parallelism:



(a) Tangent Cone on S^2



(b) Unrolled Flat Sector

Derivation of the Holonomy Angle via the Tangent Cone: Consider a 2-sphere of radius R and a loop along a latitude circle at colatitude $\theta_0 \in (0, \pi/2)$.

1. **Contact Circle Perimeter:** The radius of the latitude circle is $r_0 = R\sin\theta_0$, giving a perimeter $C = 2\pi R\sin\theta_0$.

2. **Cone Geometry:** The cone tangent to the sphere along $\theta = \theta_0$ has its apex on the symmetry axis at $z_A = R/\cos\theta_0$. The slant height L from apex A to the contact circle is:

$$L = \sqrt{r_0^2 + (z_A - z_0)^2} = \sqrt{R^2\sin^2\theta_0 + \left(\frac{R}{\cos\theta_0} - R\cos\theta_0\right)^2} = R\tan\theta_0. \tag{12.12}$$

3. **Unrolling into $\mathbb{R}^2$:** Cutting the cone open along a generator yields a planar sector in $\mathbb{R}^2$ of radius L and boundary arc length C. The opening angle $\Delta\Phi$ of the sector is:

$$\Delta\Phi = \frac{C}{L} = \frac{2\pi R\sin\theta_0}{R\tan\theta_0} = 2\pi\cos\theta_0. \tag{12.13}$$

4. **Parallel Transport on the Flat Sheet:** Because the cone has zero intrinsic Gaussian curvature ($\mathcal{K}_{\text{cone}} = 0$), Christoffel symbols vanish in planar Cartesian coordinates on the unrolled sheet ($\Gamma^i_{jk} = 0$). Parallel transport along the circular arc is strictly constant Euclidean translation:

$$\frac{\mathrm{d}V^i}{\mathrm{d}\lambda} = 0 \implies V^i(\lambda) = \text{constant Cartesian components.} \tag{12.14}$$

5. **Re-gluing and Angular Deficit:** Gluing the two boundary edges of the sector back together identifies the initial and final points. The missing wedge (angular deficit) is:

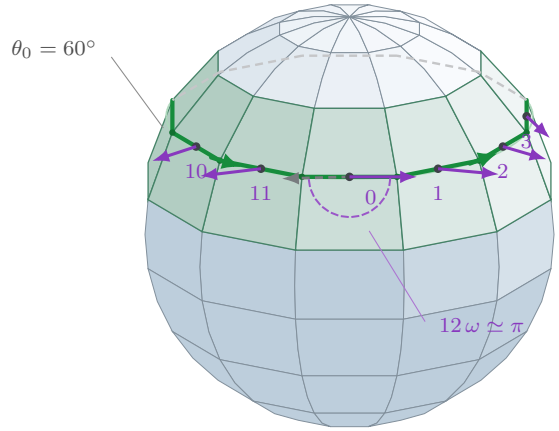
$$\Delta\theta = 2\pi - \Delta\Phi = 2\pi(1 - \cos\theta_0) = \Omega_{\text{enclosed}}, \tag{12.15}$$

which matches identically the solid angle $\Omega = \iint_{\text{cap}} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 2\pi(1 - \cos\theta_0)$ enclosed by the loop on S^2, providing a direct physical demonstration of the Gauss–Bonnet theorem!

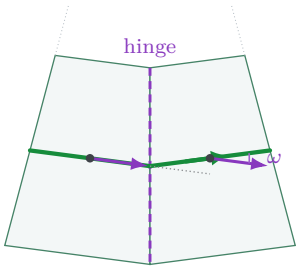
2. The Polyhedral Discretization Method (Regge Calculus Viewpoint): To gain deeper, coordinate-free intuition for how curvature generates holonomy, we consider a piecewise-flat polyhedral approximation of the curved manifold, an approach foundational to discrete geometry and Regge calculus.

We discretize the 2-sphere S^2 into planar quadrilateral facets (hinged faces) bounded by lines of latitude and longitude. Because each individual facet is a flat Euclidean domain ($\mathbb{R}^2$),

the intrinsic Riemann curvature vanishes everywhere in the interior of each patch ($\mathcal{K}_{\text{patch}} = 0$, $\Gamma^k_{ij} = 0$). All intrinsic Gaussian curvature of the smooth manifold is concentrated entirely as conical singularities (deficit angles) at the vertices and along the hinges connecting adjacent facets.



(a) Discretized 12-gon latitude loop.



(b) Transport across one flattened hinge.

Figure 12.3: Polyhedral discretization of parallel transport on S^2: (a) closed 12-gon circuit along a parallel of latitude; (b) local Euclidean translation across an unfolded hinge.

Step-by-Step Mechanism of Discrete Holonomy:

1. **The Walked Path on the Polyhedron:** Consider a closed polygonal circuit formed by $N = 12$ chord segments along the colatitude circle $\theta_0 = 60°$ (Figure 12.3a). This closed 12-gon represents a non-geodesic loop. Inside each flat patch k, parallel transport of a vector $\vec{v}$ is strictly ordinary Euclidean translation ($\frac{\mathrm{d}\vec{v}}{\mathrm{d}\lambda} = 0$). The vector simply glides along the straight chord without turning.

2. **Crossing a Hinge (Local Unfolding):** To transport the vector from patch k to the adjacent patch $k+1$, we flatten the two patches across their shared hinge into a single Euclidean plane $\mathbb{R}^2$ (Figure 12.3b). Across this flattened hinge, the vector maintains its exact Cartesian direction. However, because the spherical band tapers toward the pole, the polygonal path kinks by an exterior hinge angle ω at the boundary. Consequently, upon entering patch $k+1$, the vector is no longer tangent to the path: it lags the direction of travel by precisely ω.

3. **Accumulation of Deficit at Patch Boundaries:** All turning occurs discontinuously at the hinge boundaries, while zero turning occurs in the interior of the flat patches. At each of the 12 hinges, the relative orientation between the vector and the trajectory advances by ω.

Global Holonomy via the Unrolled Sector: When the entire band of $N = 12$ facets is cut along one generator and unrolled flat onto $\mathbb{R}^2$ (Figure 12.4), the facets assemble into a circular sector of a cone rather than a flat annulus.

1. **Total Sector Angle:** The sum of the twelve exterior hinge angles is:

$$\sum_{k=1}^{12} \omega_k = 2\pi \cos\theta_0. \tag{12.16}$$

For $\theta_0 = 60°$, we have $\cos(60°) = 1/2$, so the 12 facets span a half-circle: $\sum \omega_k = \pi$.

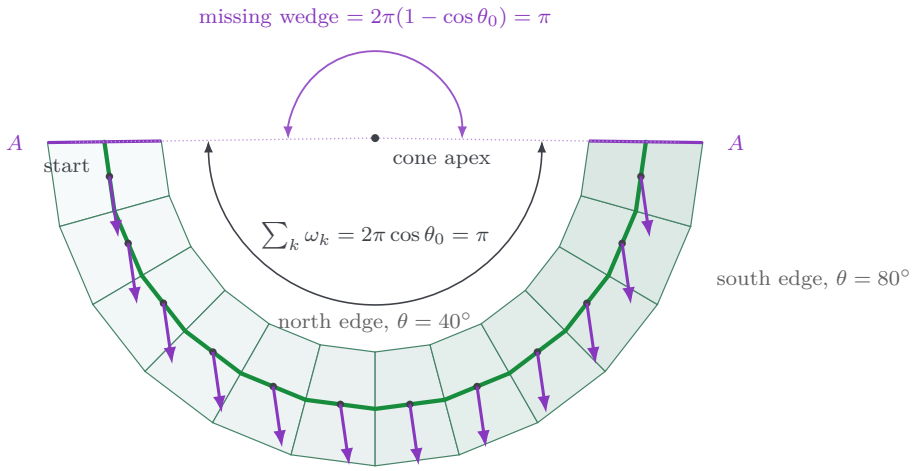


Figure 12.4: Unrolled band of 12 planar facets forming a cone sector: all parallel-transported vectors remain strictly parallel in the Euclidean plane, yielding a net deficit angle $\Delta\theta = \pi$ upon re-gluing the boundary edges A.

2. **Euclidean Parallelism on the Sector:** On the flat unrolled sector, the transported vector field $\vec{v}$ is everywhere parallel (pointing due east in Cartesian coordinates).

3. **Re-identification and Angular Deficit:** To reconstruct the closed polyhedral loop on S^2, we re-glue the two terminal boundary edges marked A. Because the flat sector has an angular deficit (missing wedge) of:

$$\Delta\theta = 2\pi - \sum_{k=1}^{12} \omega_k = 2\pi(1-\cos\theta_0), \tag{12.17}$$

   the vector returns to its starting point rotated by $\Delta\theta$. For $\theta_0 = 60°$, $\Delta\theta = 2\pi(1-1/2) = \pi = 180°$ — exactly reversing the vector's direction upon completing the circuit!

4. **Discrete Gauss–Bonnet Theorem:** In the continuum limit ($N \to \infty$), the polyhedral deficit $\Delta\theta$ converges identically to the enclosed solid angle $\Omega_{\text{enclosed}} = \iint_{\text{cap}} \mathcal{K}\,\mathrm{d}A$, demonstrating that parallel-transport holonomy directly measures the total Gaussian curvature concentrated inside the loop.

12.3 The Covariant Transport Equation and Geodesics

12.3.1 Definition of the Directional Covariant Derivative

Let $\gamma(\lambda)$ be a smooth curve with tangent vector $u^k = \frac{\mathrm{d}x^k}{\mathrm{d}\lambda}$. The **directional covariant derivative** $\frac{D}{\mathrm{d}\lambda}$ along the curve acting on a vector field $V^a(\lambda)$ is defined as:

$$\frac{DV^a}{\mathrm{d}\lambda} \equiv u^k \nabla_k V^a = \frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\left(\partial_k V^a + \Gamma^a_{kb} V^b\right) = \frac{\mathrm{d}V^a}{\mathrm{d}\lambda} + \Gamma^a_{kb}(x(\lambda))\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} V^b. \tag{12.18}$$

Definition: Parallel Transport of a Vector Field

A vector field $V^a(\lambda)$ is **parallel transported** along a curve $\gamma(\lambda)$ if and only if its directional covariant derivative vanishes identically:

$$\frac{DV^a}{\mathrm{d}\lambda} = 0 \iff \frac{\mathrm{d}V^a}{\mathrm{d}\lambda} + \Gamma^a_{kb}(x(\lambda))\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}V^b = 0. \tag{12.19}$$

By the Picard–Lindelöf theorem for linear first-order ODEs, given an initial vector $V^a(\lambda_0) = V^a_0$, there exists a **unique** parallel transported vector field along γ.

12.3.2 Dual Formulations of Geodesics

Setting the transported vector V^a equal to the curve's own tangent vector $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$:

$$\frac{Du^a}{\mathrm{d}\lambda} = u^k\nabla_k u^a = 0 \iff \frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda} = 0. \tag{12.20}$$

Key Concept: Dual Geometric Interpretations of Geodesics

The geodesic equation expresses two complementary physical and geometric principles:

1. **Kinematic Definition:** A geodesic is a curve of zero proper acceleration ($a^a = u^k\nabla_k u^a = 0$), describing the path of a freely falling test particle.
2. **Geometric Definition:** A geodesic is an **autoparallel curve**—a curve that parallel transports its own tangent vector along itself.

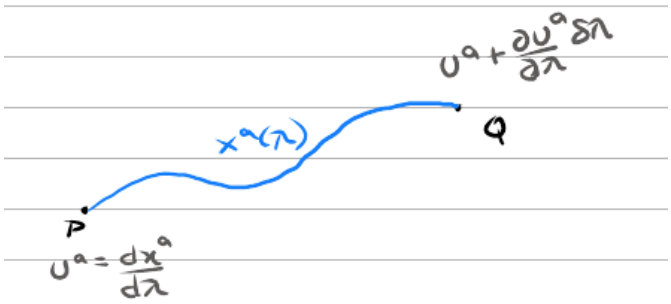


Figure 12.5: Parallel transport of a tangent vector along a curve $x^a(\lambda)$ connecting P to Q.

12.3.3 Non-Affine Transport and Directional Preservation

If we demand only that the tangent vector remains parallel in direction (proportional) without preserving affine scaling:

$$u^a(\lambda + \delta\lambda) = (1 + f(\lambda)\delta\lambda)\left(u^a - \Gamma^a_{bc}u^b u^c\delta\lambda\right), \tag{12.21}$$

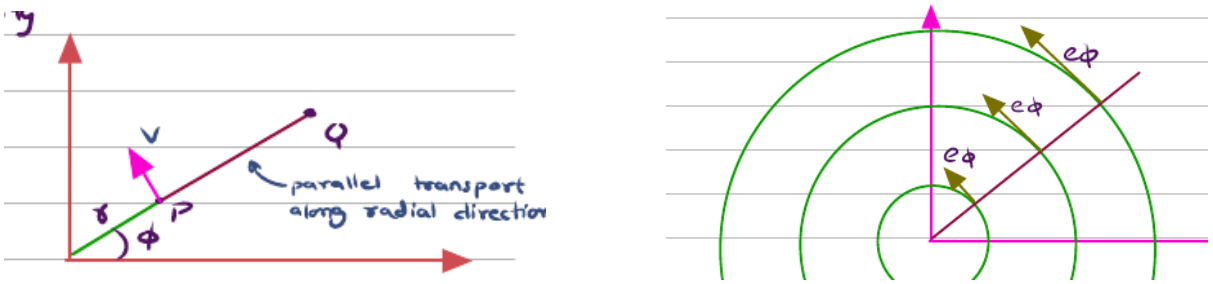
equating first-order terms in $\delta\lambda$ yields the **non-affine geodesic equation**:

$$\frac{\mathrm{d}u^a}{\mathrm{d}\lambda} + \Gamma^a_{bc}u^b u^c = f(\lambda)u^a \iff u^k\nabla_k u^a = f(\lambda)u^a. \tag{12.22}$$

12.4 Worked Example: Parallel Transport in 2D Polar Coordinates

Consider flat Euclidean space $\mathbb{R}^2$ in polar coordinates (r, ϕ):

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2, \qquad \Gamma^r_{\phi\phi} = -r, \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}. \tag{12.23}$$



(a) Radial parallel transport from P to Q.

(b) Coordinate basis vector $\vec{e}_\phi$ growth: $|\vec{e}_\phi| = r$.

Figure 12.6: Parallel transport in polar coordinates: radial transport and coordinate basis scaling.

12.4.1 Transport along a Radial Line

Transporting an initial vector $\vec{V} = V^r\vec{e}_r + V^\phi\vec{e}_\phi$ along a radial ray $\phi = \text{const}$ with tangent $u^a = (1, 0)$:

$$\text{Radial component:} \quad \frac{\mathrm{d}V^r}{\mathrm{d}r} = 0 \implies V^r(r) = V^r_{\text{init}} = \text{constant}. \tag{12.24}$$

$$\text{Angular component:} \quad \frac{\mathrm{d}V^\phi}{\mathrm{d}r} + \frac{1}{r}V^\phi = 0 \implies V^\phi(r) = \frac{A}{r}. \tag{12.25}$$

Physical Resolution: Coordinate vs. Orthonormal Components: The angular coordinate component $V^\phi = A/r$ decays with radius because the coordinate basis vector $\vec{e}_\phi = \partial_\phi$ has squared length $g_{\phi\phi} = r^2$, growing linearly with r.

Expanding in the unit-normalized orthonormal basis $\hat{e}_\phi = \vec{e}_\phi/r$:

$$V_{(\phi)}(r) = rV^\phi(r) = r\left(\frac{A}{r}\right) = A = \text{constant!} \tag{12.26}$$

Thus, the true physical vector has constant Cartesian magnitude and direction.

12.5 Metric Preservation and Closed-Loop Holonomy

12.5.1 Conservation of Scalar Products

For any two vector fields $A^a(\lambda)$ and $B^a(\lambda)$ parallel transported along γ:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ab}A^aB^b\right) = u^k\nabla_k(g_{ab}A^aB^b) = \cancelto{0}{(\nabla_k g_{ab})}A^aB^b + g_{ab}\cancelto{0}{(u^k\nabla_k A^a)}B^b + g_{ab}A^a\cancelto{0}{(u^k\nabla_k B^b)} = 0. \tag{12.27}$$

This establishes two crucial geometric facts:

1. **Norm Conservation:** $|A|^2 = g_{ab}A^aA^b = \text{const}$ along γ. Timelike, spacelike, and null vectors retain their causal classification.
2. **Angle Conservation:** The inner product angle $\cos\theta = \frac{A\cdot B}{|A||B|}$ is strictly constant.

12.5.2 Infinitesimal Loop Holonomy and the Riemann Tensor

Transporting a vector V^a around an infinitesimal closed loop $\mathcal{C}$ bounding surface $\Delta\sigma^{cd}$ starting and ending at P:

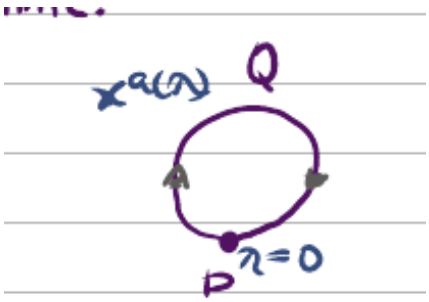


Figure 12.7: Parallel transport around an infinitesimal closed circuit $\mathcal{C}$.

In Riemann Normal Coordinates centered at P ($\Gamma|_P = 0$):

$$\Delta V^a = -\oint_{\mathcal{C}} \Gamma^a_{bc}(x) V^b(x)\, \mathrm{d}x^c = -\left(\partial_d \Gamma^a_{bc}\right)_P V^b(P) \oint_{\mathcal{C}} x^d\, \mathrm{d}x^c + \mathcal{O}(x^3). \tag{12.28}$$

Using Stokes' theorem $\oint_{\mathcal{C}} x^d \mathrm{d}x^c = -\frac{1}{2}\Delta\sigma^{cd}$, this yields the definition of the **Riemann Curvature Tensor**:

$$\Delta V^a = \frac{1}{2} R^a{}_{bcd}\, V^b\, \Delta\sigma^{cd}, \tag{12.29}$$

where:

$$R^a{}_{bcd} \equiv \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}. \tag{12.30}$$

Theorem: Algebraic Symmetries of the Riemann Tensor

The fully covariant Riemann curvature tensor $R_{abcd} \equiv g_{ae} R^e{}_{bcd}$ satisfies:

1. $R_{abcd} = -R_{bacd}$ (antisymmetric in first pair).
2. $R_{abcd} = -R_{abdc}$ (antisymmetric in second pair).
3. $R_{abcd} = +R_{cdab}$ (pairwise interchange symmetry).
4. $R_{a[bcd]} \equiv R_{abcd} + R_{adbc} + R_{acdb} = 0$ (first algebraic Bianchi identity).

In $n = 4$ dimensions, these symmetries reduce the $4^4 = 256$ components to precisely $\frac{4^2(4^2-1)}{12} = 20$ algebraically independent components.

Chapter 13

The Riemann Curvature Tensor, Commutators, and Geodesic Deviation

13.1 The Riemann Curvature Tensor from Commutators

13.1.1 Commutator of Covariant Derivatives on Vector Fields

In Chapter 11, we proved that covariant derivatives commute when acting on scalar fields ($[\nabla_a, \nabla_b]\Phi = 0$). When acting on a contravariant vector field $A^c(x)$, however, covariant derivatives **do not commute**.

Let us calculate the commutator $[\nabla_a, \nabla_b]A^c \equiv \nabla_a \nabla_b A^c - \nabla_b \nabla_a A^c$ in full detail. First, write the tensor ${T_b}^c \equiv \nabla_b A^c = \partial_b A^c + \Gamma^c_{be} A^e$. The second covariant derivative $\nabla_a {T_b}^c$ treats b as a covariant index and c as a contravariant index:

$$\begin{aligned}
\nabla_a \left(\nabla_b A^c\right) &= \partial_a \left(\nabla_b A^c\right) - \Gamma^d_{ab} \left(\nabla_d A^c\right) + \Gamma^c_{ad} \left(\nabla_b A^d\right) \\
&= \partial_a \left(\partial_b A^c + \Gamma^c_{be} A^e\right) - \Gamma^d_{ab} \left(\partial_d A^c + \Gamma^c_{de} A^e\right) + \Gamma^c_{ad} \left(\partial_b A^d + \Gamma^d_{be} A^e\right) \\
&= \partial_a \partial_b A^c + \left(\partial_a \Gamma^c_{be}\right) A^e + \Gamma^c_{be} \partial_a A^e - \Gamma^d_{ab} \partial_d A^c - \Gamma^d_{ab} \Gamma^c_{de} A^e \\
&\quad + \Gamma^c_{ad} \partial_b A^d + \Gamma^c_{ad} \Gamma^d_{be} A^e. \qquad (13.1)
\end{aligned}$$

Now, compute $\nabla_b \left(\nabla_a A^c\right)$ by interchanging the indices $a \leftrightarrow b$:

$$\begin{aligned}
\nabla_b \left(\nabla_a A^c\right) &= \partial_b \partial_a A^c + \left(\partial_b \Gamma^c_{ae}\right) A^e + \Gamma^c_{ae} \partial_b A^e - \Gamma^d_{ba} \partial_d A^c - \Gamma^d_{ba} \Gamma^c_{de} A^e \\
&\quad + \Gamma^c_{bd} \partial_a A^d + \Gamma^c_{bd} \Gamma^d_{ae} A^e. \qquad (13.2)
\end{aligned}$$

Subtracting Eq. (13.2) from Eq. (13.1):

1. The second partial derivatives cancel by Schwarz's theorem: $\partial_a \partial_b A^c - \partial_b \partial_a A^c = 0$.
2. The mixed first-derivative terms cancel: $\Gamma^c_{be} \partial_a A^e - \Gamma^c_{bd} \partial_a A^d = 0$ (relabeling dummy index $e \to d$).
3. The connection terms proportional to the torsion tensor vanish because the Levi-Civita connection is torsion-free:
$$-\left(\Gamma^d_{ab} - \Gamma^d_{ba}\right) \nabla_d A^c = 0. \qquad (13.3)$$

Collecting the remaining terms proportional to A^e:

$$[\nabla_a, \nabla_b]A^c = \left[\partial_a \Gamma^c_{be} - \partial_b \Gamma^c_{ae} + \Gamma^c_{ad} \Gamma^d_{be} - \Gamma^c_{bd} \Gamma^d_{ae}\right] A^e. \qquad (13.4)$$

Definition: The Riemann Curvature Tensor via Vector Commutators

Renaming the indices to standard notation ($a \to b$, $b \to c$, $e \to i$, $c \to a$), the commutator of covariant derivatives acting on an arbitrary contravariant vector field A^a evaluates to:

$$[\nabla_b, \nabla_c]A^a = R^a{}_{ibc}A^i, \tag{13.5}$$

where the **Riemann Curvature Tensor** $R^a{}_{ibc}$ is defined by:

$$R^a{}_{ibc} \equiv \partial_b\Gamma^a_{ci} - \partial_c\Gamma^a_{bi} + \Gamma^a_{bk}\Gamma^k_{ci} - \Gamma^a_{ck}\Gamma^k_{bi}. \tag{13.6}$$

13.1.2 Commutators on Covariant Vectors and General Tensors

By demanding the Leibniz product rule for the scalar invariant $\Phi = A_aB^a$, where $[\nabla_b, \nabla_c](A_aB^a) = 0$:

$$0 = ([\nabla_b, \nabla_c]A_a)\,B^a + A_a\,([\nabla_b, \nabla_c]B^a) = ([\nabla_b, \nabla_c]A_a)\,B^a + A_a\left(R^a{}_{dbc}B^d\right). \tag{13.7}$$

Relabeling indices, we obtain the commutator for a **covariant 1-form**:

$$[\nabla_b, \nabla_c]A_a = -R^d{}_{abc}A_d. \tag{13.8}$$

For an arbitrary rank-(p, q) tensor $T^{a_1\ldots a_p}{}_{b_1\ldots b_q}$, each contravariant index receives a positive Riemann contraction and each covariant index receives a negative Riemann contraction:

$$[\nabla_k, \nabla_n]T^{a_1\ldots a_p}{}_{b_1\ldots b_q} = \sum_{r=1}^{p} R^{a_r}{}_{ikn}T^{a_1\ldots i\ldots a_p}{}_{b_1\ldots b_q} - \sum_{s=1}^{q} R^i{}_{b_skn}T^{a_1\ldots a_p}{}_{b_1\ldots i\ldots b_q}. \tag{13.9}$$

Tensorial Invariance and Spacetime Flatness Criterion: Because A^i is an arbitrary tensor and $[\nabla_b, \nabla_c]A^a$ is a valid rank-$(1, 3)$ tensor, the quotient theorem guarantees that $R^a{}_{ibc}$ **is a true geometric tensor**.

A direct consequence is that if $R^a{}_{bcd} = 0$ at an event in one coordinate chart, it vanishes in **every** coordinate chart. Spacetime is globally flat (isometric to Minkowski spacetime) if and only if $R^a{}_{bcd} \equiv 0$ everywhere.

13.1.3 Integrability of Covariantly Constant Fields

Under what condition does a manifold admit a globally parallel (covariantly constant) vector field $V^a(x)$, satisfying $\nabla_bV^a = 0$ everywhere?

If $\nabla_bV^a = 0$, then $[\nabla_b, \nabla_c]V^a = 0$. By Eq. (13.5):

$$R^a{}_{mbc}V^m = 0. \tag{13.10}$$

For a non-zero vector V^a to exist in all directions, the integrability condition demands $R^a{}_{mbc} \equiv 0$. In a curved spacetime ($R^a{}_{mbc} \neq 0$), parallel transport is fundamentally path-dependent, and globally parallel coordinate frames do not exist.

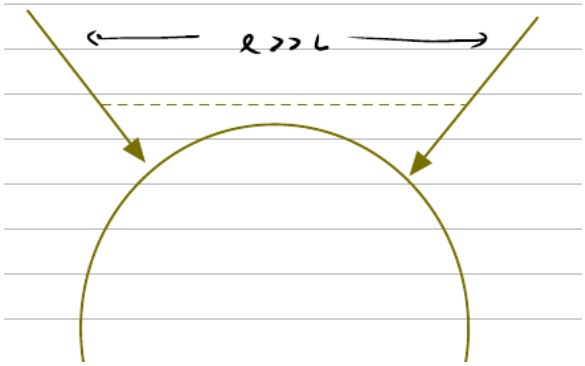
13.2 Physical Equivalence Principle, Inhomogeneous Fields, and Tidal Gravity

13.2.1 The Local Inertial Elevator and Characteristic Curvature Scale

The Equivalence Principle asserts that the local effects of gravity can be eliminated in a freely falling frame (Local Inertial Coordinate system, LIC). However, the elimination of gravity is strictly valid only at a **single event** P, or in a sufficiently small neighborhood of size ℓ.



(a) Freely falling elevator of size $\ell \ll L$: metric is locally Minkowski η_{ab}, $\Gamma = 0$.



(b) Finite size $\ell \sim L$: radial convergence toward Earth reveals irreducible second metric derivatives $\partial^2 g \sim \partial\Gamma \neq 0$.

Figure 13.1: The physical limit of the Equivalence Principle and the onset of tidal gravitational forces.

Let L denote the characteristic spatial length scale over which the gravitational field changes significantly ($L \sim 1/\sqrt{|\partial^2\Phi|/c^2} \sim 1/\sqrt{|R|}$):

- **Microscopic Experiment ($\ell \ll L$):** Inside a compact freely falling elevator of dimension $\ell \ll L$ (Figure 13.1a), the gravitational field is approximately uniform:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}, \qquad \partial_\lambda g_{\mu\nu}(P) = 0 \iff \Gamma^\lambda_{\mu\nu}(P) = 0. \tag{13.11}$$

  All particles accelerate identically at rate $\vec{g}$, rendering the interior indistinguishable from unaccelerated Minkowski space.

- **Macroscopic Separation ($\ell \gtrsim L$):** If two test particles are dropped from different horizontal positions above the Earth (Figure 13.1b), they accelerate radially toward the center of the Earth. As they fall, their lateral separation distance decreases:

$$\frac{\mathrm{d}^2\Delta x}{\mathrm{d}t^2} \neq 0 \implies \partial^2 g_{ab} \sim \partial\Gamma \neq 0. \tag{13.12}$$

  This relative acceleration between neighboring freely falling particles cannot be transformed away by any coordinate transformation. It is the true, invariant physical signature of **spacetime curvature**.

13.3 Newtonian Tidal Acceleration vs. Relativistic Geodesic Deviation

13.3.1 Newtonian Tidal Deviation Equation

In Newtonian gravitation, consider two neighboring particles moving along trajectories $x^\alpha(t)$ and $x^\alpha(t) + \eta^\alpha(t)$ in a gravitational potential $\Phi(\vec{x})$ (Figure 13.2):

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} = -\partial^\alpha\Phi(x). \tag{13.13}$$

Let n be a continuous parameter labeling the family of trajectories, so that the separation vector is $\eta^\alpha \equiv \frac{\partial x^\alpha}{\partial n}$. Differentiating the relative acceleration with respect to n:

$$\frac{\mathrm{d}^2\eta^\alpha}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t^2}\left(\frac{\partial x^\alpha}{\partial n}\right) = \frac{\partial}{\partial n}\left(\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2}\right) = \frac{\partial}{\partial n}\left(-\partial^\alpha\Phi(x)\right)$$

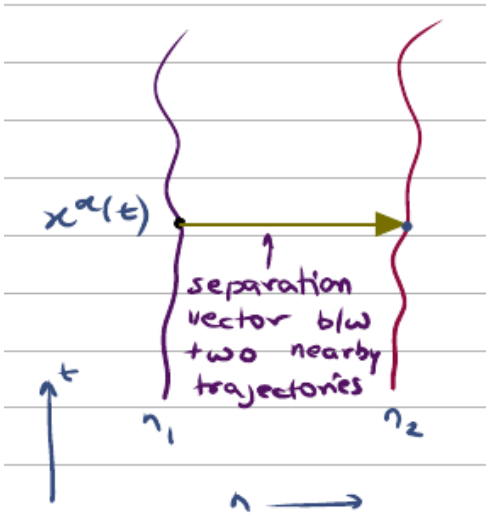


Figure 13.2: Newtonian separation vector $\eta^\alpha(t) = \frac{\partial x^\alpha}{\partial n}$ connecting two neighboring freely falling trajectories.

$$= -\frac{\partial x^\beta}{\partial n}\partial_\beta\partial^\alpha\Phi(x) = -\left(\partial_\beta\partial^\alpha\Phi\right)\eta^\beta. \tag{13.14}$$

We define the **Newtonian tidal tensor** $\mathcal{E}^\alpha{}_\beta \equiv \partial_\beta\partial^\alpha\Phi$. The relative acceleration satisfies:

$$\frac{\mathrm{d}^2\eta^\alpha}{\mathrm{d}t^2} = -\mathcal{E}^\alpha{}_\beta\,\eta^\beta = -\left(\frac{\partial^2\Phi}{\partial x^\beta\partial x_\alpha}\right)\eta^\beta. \tag{13.15}$$

13.3.2 The Relativistic Geodesic Deviation Equation (Jacobi Equation)

Now let us derive the exact general-relativistic counterpart of Eq. (13.15).

Consider a 1-parameter family (congruence) of affinely parameterized geodesics forming a smooth 2-dimensional surface (sheet) in spacetime: $x^i = x^i(\tau, s)$, where τ is the affine parameter along each individual geodesic and s is the continuous selector parameter labeling distinct adjacent geodesics (Figure 13.3).

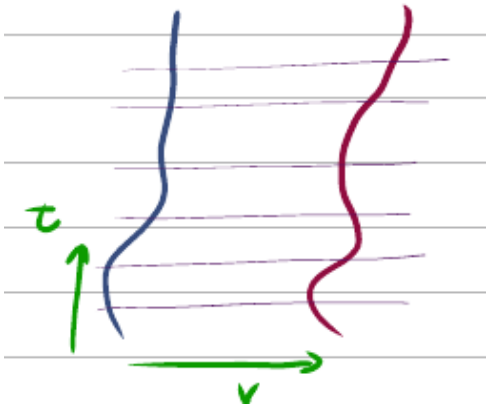


Figure 13.3: A 2-parameter geodesic congruence sheet $x^i(\tau, s)$ with tangent velocity vector $u^i = \frac{\partial x^i}{\partial\tau}$ along the flow and geodesic deviation separation vector $v^i = \frac{\partial x^i}{\partial s}$ across neighboring geodesics.

We define two fundamental vector fields associated with the 2-parameter surface $x^i(\tau, s)$:

- **Tangent 4-Velocity Vector:** The tangent vector along each geodesic curve:

$$u^i \equiv \frac{\partial x^i}{\partial \tau} \qquad \left(\mathbf{u} = \frac{\partial}{\partial \tau}\right). \tag{13.16}$$

- **Geodesic Deviation (Separation) Vector:** The separation vector pointing from one geodesic to an infinitesimally adjacent neighbor at constant affine parameter τ:

$$v^i \equiv \frac{\partial x^i}{\partial s} = \lim_{\Delta s \to 0} \frac{x^i(\tau, s + \Delta s) - x^i(\tau, s)}{\Delta s} \qquad \left(\mathbf{v} = \frac{\partial}{\partial s}\right). \tag{13.17}$$

1. Commutation of Directional Covariant Derivatives ($u^b \nabla_b v^i = v^b \nabla_b u^i$): Let us evaluate the directional covariant derivative of the velocity vector u^i along the deviation vector v^b:

$$\begin{aligned} v^b \nabla_b u^i &= v^b \left(\partial_b u^i + \Gamma^i_{bc} u^c\right) \\ &= \left(\frac{\partial x^b}{\partial s} \frac{\partial}{\partial x^b}\right) u^i + \Gamma^i_{bc} u^c v^b \\ &= \frac{\partial u^i}{\partial s} + \Gamma^i_{bc} u^c v^b. \end{aligned} \tag{13.18}$$

Because τ and s form a smooth coordinate mesh on the 2D surface $x^i(\tau, s)$, the mixed partial derivatives commute:

$$\frac{\partial u^i}{\partial s} = \frac{\partial}{\partial s}\left(\frac{\partial x^i}{\partial \tau}\right) = \frac{\partial^2 x^i}{\partial s \partial \tau} = \frac{\partial^2 x^i}{\partial \tau \partial s} = \frac{\partial}{\partial \tau}\left(\frac{\partial x^i}{\partial s}\right) = \frac{\partial v^i}{\partial \tau}. \tag{13.19}$$

Substituting Eq. (13.19) back into Eq. (13.18):

$$\begin{aligned} v^b \nabla_b u^i &= \frac{\partial v^i}{\partial \tau} + \Gamma^i_{bc} u^c v^b \\ &= \left(\frac{\partial x^c}{\partial \tau} \frac{\partial}{\partial x^c}\right) v^i + \Gamma^i_{cb} v^b u^c \qquad \text{(using torsion-free symmetry } \Gamma^i_{bc} = \Gamma^i_{cb}) \\ &= u^c \partial_c v^i + \Gamma^i_{cb} u^c v^b \\ &= u^c \left(\partial_c v^i + \Gamma^i_{cb} v^b\right) \\ &= u^c \nabla_c v^i = u^b \nabla_b v^i. \end{aligned} \tag{13.20}$$

Thus, we obtain the fundamental commutation relation between directional covariant derivatives:

$$u^b \nabla_b v^i = v^b \nabla_b u^i \iff \frac{Dv^i}{\mathrm{d}\tau} = \frac{Du^i}{\mathrm{d}s}. \tag{13.21}$$

Geometric Interpretation via the Lie Bracket: In the language of differential geometry, $\mathbf{u} = \frac{\partial}{\partial \tau}$ and $\mathbf{v} = \frac{\partial}{\partial s}$ are coordinate basis vector fields on the 2-dimensional embedded sheet. Consequently, their Lie bracket (commutator) vanishes identically:

$$[\mathbf{u}, \mathbf{v}] \equiv \mathcal{L}_u v = 0. \tag{13.22}$$

For any torsion-free connection ($\Gamma^i_{bc} = \Gamma^i_{cb}$), the Lie bracket is expressed in terms of covariant derivatives as:

$$[\mathbf{u}, \mathbf{v}]^i = u^b \nabla_b v^i - v^b \nabla_b u^i = 0, \tag{13.23}$$

which is exactly Eq. (13.21).

2. Second Directional Derivative (Relative Tidal Acceleration): The relative tidal acceleration between neighboring geodesics is given by the second directional covariant derivative $\frac{D^2v^i}{\mathrm{d}\tau^2} \equiv u^a\nabla_a\left(u^b\nabla_b v^i\right)$. Using the commutation property $u^b\nabla_b v^i = v^b\nabla_b u^i$:

$$\begin{aligned} u^a\nabla_a\left(u^b\nabla_b v^i\right) &= u^a\nabla_a\left(v^b\nabla_b u^i\right) \\ &= \left(u^a\nabla_a v^b\right)\nabla_b u^i + u^a v^b\nabla_a\nabla_b u^i \\ &= \left(v^a\nabla_a u^b\right)\nabla_b u^i + u^a v^b\left(\nabla_b\nabla_a u^i + R^i{}_{cab}u^c\right) \\ &= v^a\nabla_a\left(u^b\nabla_b u^i\right) - v^a u^b\nabla_a\nabla_b u^i + u^a v^b\nabla_b\nabla_a u^i + R^i{}_{cab}u^a u^c v^b \\ &= v^a\nabla_a a^i + R^i{}_{cab}u^a u^c v^b, \end{aligned} \tag{13.24}$$

where $a^i \equiv u^b\nabla_b u^i$ is the 4-acceleration of the curves. Since the curves are **geodesics**, the acceleration vanishes identically ($a^i = 0 \implies v^a\nabla_a a^i = 0$).

Using the Riemann tensor antisymmetry $R^i{}_{cab} = -R^i{}_{cba} = R^i{}_{acb}$, we obtain:

Theorem: The Jacobi Geodesic Deviation Equation

The relative acceleration of adjacent freely falling geodesics separated by deviation vector $v^i(\tau)$ with tangent 4-velocity $u^a(\tau)$ is governed by the Riemann curvature tensor:

$$\frac{D^2v^i}{\mathrm{d}\tau^2} = R^i{}_{acb}u^a u^c v^b = -R^i{}_{abc}u^a v^b u^c. \tag{13.25}$$

Newtonian Limit of Geodesic Deviation: In the non-relativistic weak-field limit, $u^a \approx (c, 0, 0, 0)$. Equation (13.25) reduces for spatial components $i = \alpha$ to:

$$\frac{\mathrm{d}^2v^\alpha}{\mathrm{d}t^2} = -c^2R^\alpha{}_{0\beta0}v^\beta. \tag{13.26}$$

Comparing directly with the Newtonian tidal equation (13.15):

$$R^\alpha{}_{0\beta0} \approx \frac{1}{c^2}\frac{\partial^2\Phi}{\partial x^\beta\partial x_\alpha}. \tag{13.27}$$

Taking the trace over spatial indices:

$$R^0{}_0 = R^\alpha{}_{0\alpha0} \approx \frac{1}{c^2}\nabla^2\Phi = \frac{4\pi G}{c^2}\rho_{\text{mass}}. \tag{13.28}$$

This directly links the Riemann curvature tensor to Poisson's equation $\nabla^2\Phi = 4\pi G\rho$! This confirms that the Riemann tensor is the exact curved-spacetime generalization of the Newtonian tidal tensor $\partial_\alpha\partial_\beta\Phi$.

Pedagogical Inquiry: Can Null Particles Move on Non-Geodesic Trajectories? In vacuum General Relativity, minimal coupling and energy-momentum conservation $\nabla_a T^{ab} = 0$ dictate that free, uncharged photons and gravitons **must strictly follow null geodesics**:

$$k^a k_a = 0, \qquad k^a\nabla_a k^b = 0. \tag{13.29}$$

A null trajectory ceases to be a geodesic whenever the massless particle experiences non-gravitational external forces or couplings:

- **Refractive and Plasma Media:** In an astrophysical plasma, photons acquire an effective dispersion relation $\omega^2 = \omega_p^2 + c^2k^2$, deflecting them from vacuum geodesics.

- **Nonlinear Quantum Electrodynamics (QED):** In ultra-strong magnetic fields, vacuum polarization described by the Euler–Heisenberg Lagrangian induces vacuum birefringence.
- **Massless Charged Particles:** A charged particle coupled to F_{ab} obeys the Lorentz force law: $k^a \nabla_a k^b = \frac{q}{\hbar} F^b{}_c k^c \neq 0$.

13.4 Riemann Curvature in Local Inertial Coordinates and Counting

13.4.1 Metric Second Derivatives in Riemann Normal Coordinates

In a Local Inertial Coordinate system centered at an event P:

$$g_{ab}(P) = \eta_{ab}, \qquad \partial_c g_{ab}(P) = 0 \implies \Gamma^a_{bc}|_P = 0. \tag{13.30}$$

At point P, all connection product terms $\Gamma \cdot \Gamma$ in the Riemann tensor drop out:

$$R_{abcd}|_P = g_{ae}\left(\partial_c \Gamma^e_{bd} - \partial_d \Gamma^e_{bc}\right) = \frac{1}{2}\left(\partial_c \partial_b g_{ad} + \partial_d \partial_a g_{bc} - \partial_c \partial_a g_{bd} - \partial_d \partial_b g_{ac}\right). \tag{13.31}$$

Key Concept: Matrix-Based Component Counting for the Riemann Tensor

Another way to view the counting is to regard $R_{\mu\nu\rho\sigma}$ as a matrix whose rows are made by dissecting another anti symmetric matrices and same happens for columns. Since

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \qquad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \tag{13.32}$$

we may take each pair $(\mu\nu)$ and $(\rho\sigma)$ to be ordered, leaving

$$\frac{n(n-1)}{2}$$

independent elements in each row and column if additional symmetry constraints were not imposed. Because of this anti symmetry certain rows will repeat themselves and hence we will only have $\frac{n(n-1)}{2}$ independent rows.
The pair-exchange symmetry $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ then makes sure that even out of those $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ independent elements some are related. The coloured entries below illustrate this symmetry: entries with the same colour are related by the exchange $(\mu\nu) \leftrightarrow (\rho\sigma)$ and therefore are not independent.

	(11)	(12)	(13)	$\cdots$	$(n-1,n)$	(nn)	number of independent component in each row
(11)	R_{1111}	R_{1112}	R_{1113}	$\cdots$	$R_{11\,n-1\,n}$	R_{11nn}	$\frac{n(n-1)}{2}$
(12)	R_{1211}	R_{1212}	R_{1213}	$\cdots$	$R_{12\,n-1\,n}$	R_{12nn}	$\frac{n(n-1)}{2} - 1$
(13)	R_{1311}	R_{1312}	R_{1313}	$\cdots$	$R_{13\,n-1\,n}$	R_{13nn}	$\frac{n(n-1)}{2} - 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$(n-1,n)$	$R_{n-1\,n\,11}$	$R_{n-1\,n\,12}$	$R_{n-1\,n\,13}$	$\cdots$	$R_{n-1\,n\,n-1\,n}$	$R_{n-1\,n\,nn}$	1
(nn)	R_{nn11}	R_{nn12}	R_{nn13}	$\cdots$	$R_{nn\,n-1\,n}$	R_{nnnn}	0

Thus, the first row contains $\frac{n(n-1)}{2}$ independent entries, the second contains $\frac{n(n-1)}{2} - 1$, and so on. Hence,

$$\frac{n(n-1)}{2} + \left(\frac{n(n-1)}{2} - 1\right) + \left(\frac{n(n-1)}{2} - 2\right) + \cdots + 1 + 0 = \frac{1}{2}\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} + 1\right)$$

Then the identity (14.6) reduces the number of independent elements by $\binom{n}{4}$ and hence we get

$$N_{\text{Riem}} = \frac{n^2(n^2-1)}{12}. \tag{13.33}$$

Chapter 14

Bianchi Identities, Ricci Curvature, and the Einstein Tensor

14.1 Algebraic Symmetries and Component Counting

In the preceding chapter, we derived the explicit form of the Riemann curvature tensor at the origin of a Local Inertial Coordinate (LIC) system (Riemann Normal Coordinates, where $\Gamma^{\lambda}_{\mu\nu}\Big|_P = 0$):

$$R_{abcd}|_P = \frac{1}{2}\left(\partial_c\partial_b g_{ad} + \partial_d\partial_a g_{bc} - \partial_c\partial_a g_{bd} - \partial_d\partial_b g_{ac}\right). \tag{14.1}$$

From this representation, the four algebraic symmetry properties of the curvature tensor follow directly:

(1) Antisymmetry in the First Pair of Indices:

$$R_{abcd} = -R_{bacd} \iff R_{(ab)cd} = 0. \tag{14.2}$$

(2) Antisymmetry in the Last Pair of Indices:

$$R_{abcd} = -R_{abdc} \iff R_{ab(cd)} = 0. \tag{14.3}$$

(3) Pairwise Interchange (Block) Symmetry:

$$R_{abcd} = R_{cdab}. \tag{14.4}$$

(4) First (Algebraic) Bianchi Identity (Cyclic Sum):

$$R_{a[bcd]} \equiv \frac{1}{3!}\left(R_{abcd} + R_{adbc} + R_{acdb} - R_{acbd} - R_{abdc} - R_{adcb}\right) = 0, \tag{14.5}$$

which, utilizing the antisymmetry in the last two indices, reduces to the standard cyclic sum:

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{14.6}$$

Verification from Local Inertial Form. We directly verify each property using Eq. (14.1):

1. **Antisymmetry in $[ab]$:** Swapping $a \leftrightarrow b$:

$$\begin{aligned} R_{bacd} &= \frac{1}{2}\left(\partial_c\partial_a g_{bd} + \partial_d\partial_b g_{ac} - \partial_c\partial_b g_{ad} - \partial_d\partial_a g_{bc}\right) \\ &= -\frac{1}{2}\left(\partial_c\partial_b g_{ad} + \partial_d\partial_a g_{bc} - \partial_c\partial_a g_{bd} - \partial_d\partial_b g_{ac}\right) = -R_{abcd}. \end{aligned} \tag{14.7}$$

2. **Antisymmetry in** $[cd]$**:** Swapping $c \leftrightarrow d$:

$$\begin{aligned} R_{abdc} &= \frac{1}{2}\left(\partial_d\partial_b g_{ac} + \partial_c\partial_a g_{bd} - \partial_d\partial_a g_{bc} - \partial_c\partial_b g_{ad}\right) \\ &= -\frac{1}{2}\left(\partial_c\partial_b g_{ad} + \partial_d\partial_a g_{bc} - \partial_c\partial_a g_{bd} - \partial_d\partial_b g_{ac}\right) = -R_{abcd}. \end{aligned} \tag{14.8}$$

3. **Pairwise Interchange** $(ab) \leftrightarrow (cd)$**:**

$$\begin{aligned} R_{cdab} &= \frac{1}{2}\left(\partial_a\partial_d g_{cb} + \partial_b\partial_c g_{da} - \partial_a\partial_c g_{db} - \partial_b\partial_d g_{ca}\right) \\ &= \frac{1}{2}\left(\partial_d\partial_a g_{bc} + \partial_c\partial_b g_{ad} - \partial_c\partial_a g_{bd} - \partial_d\partial_b g_{ac}\right) = R_{abcd}. \end{aligned} \tag{14.9}$$

4. **Cyclic Sum:** Summing cyclic permutations of the last three indices (b, c, d):

$$\begin{aligned} R_{abcd} + R_{acdb} + R_{adbc} = \frac{1}{2}\Big[&(\partial_c\partial_b g_{ad} + \partial_d\partial_a g_{bc} - \partial_c\partial_a g_{bd} - \partial_d\partial_b g_{ac}) \\ &+ (\partial_d\partial_c g_{ab} + \partial_b\partial_a g_{cd} - \partial_d\partial_a g_{cb} - \partial_b\partial_c g_{ad}) \\ &+ (\partial_b\partial_d g_{ac} + \partial_c\partial_a g_{db} - \partial_b\partial_a g_{dc} - \partial_c\partial_d g_{ab})\Big]. \end{aligned} \tag{14.10}$$

   By the commutativity of partial derivatives ($\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$) and metric symmetry ($g_{\mu\nu} = g_{\nu\mu}$), all twelve terms cancel out identically in pairs:

$$\partial_c\partial_b g_{ad} - \partial_b\partial_c g_{ad} = 0, \quad \partial_d\partial_a g_{bc} - \partial_d\partial_a g_{cb} = 0, \quad \text{etc.} \tag{14.11}$$

   Thus, $R_{abcd} + R_{acdb} + R_{adbc} \equiv 0$.

Since these relations are tensorial equations that hold in a Local Inertial Frame at any arbitrary point $P \in \mathcal{M}$, they hold universally in all coordinate charts across the entire manifold. □

14.1.1 Combinatorial Counting of Independent Components

In an n-dimensional spacetime manifold, a generic rank-$(0, 4)$ tensor possesses n^4 independent components. In four spacetime dimensions ($n = 4$), this amounts to $4^4 = 256$ components. The algebraic symmetries constrain this number down to 20:

1. **Antisymmetry in Pairs:** We view $R_{abcd} = R_{[ab][cd]}$ as an $M \times M$ matrix, where each index $A = [ab]$ and $B = [cd]$ labels an antisymmetric pair. The number of independent pairs is:

$$M = \frac{n(n-1)}{2}. \tag{14.12}$$

   For $n = 4$, $M = \frac{4\times 3}{2} = 6$.

2. **Pairwise Interchange Symmetry:** The matrix R_{AB} satisfies $R_{AB} = R_{BA}$, meaning it is symmetric. The number of independent components is:

$$N_{\text{sym}} = \frac{M(M+1)}{2} = \frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} + 1\right)}{2} = \frac{n(n-1)(n^2 - n + 2)}{8}. \tag{14.13}$$

   For $n = 4$, $N_{\text{sym}} = \frac{6\times 7}{2} = 21$.

3. **First Algebraic Bianchi Constraint:** The condition $R_{[abcd]} = 0$ imposes constraints on the totally antisymmetric part. The number of independent totally antisymmetric components in n dimensions is:

$$N_{\text{Bianchi}} = \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}. \tag{14.14}$$

For $n = 4$, $N_{\text{Bianchi}} = \binom{4}{4} = 1$.

Subtracting the Bianchi constraints from the symmetric pair matrix yields the total number of algebraically independent Riemann components:

$$N_{\text{Riem}} = \frac{M(M+1)}{2} - \binom{n}{4} = \frac{n^2(n^2-1)}{12}. \tag{14.15}$$

Dim n	**Generic** n^4	**Antisymmetric** $M = \binom{n}{2}$	**Symmetric** $\frac{M(M+1)}{2}$	**Bianchi** $\binom{n}{4}$	**Independent** N_{Riem}
1	1	0	0	0	**0**
2	16	1	1	0	**1**
3	81	3	6	0	**6**
4	256	6	21	1	**20**
5	625	10	55	5	**50**
n	n^4	$\frac{n(n-1)}{2}$	$\frac{n(n-1)(n^2-n+2)}{8}$	$\frac{n(n-1)(n-2)(n-3)}{24}$	$\mathbf{\frac{n^2(n^2-1)}{12}}$

Table 14.1: Systematic algebraic reduction and independent component counting of the Riemann curvature tensor across various spacetime dimensions.

Foundational Question: Can the Metric be Uniquely Determined from the Riemann Tensor?

In $n = 4$ spacetime dimensions, the metric $g_{\mu\nu}$ possesses 10 independent components, while the Riemann curvature tensor $R^a{}_{bcd}$ possesses 20 independent components:

- **Curvature as Second Derivatives:** The Riemann tensor is constructed from the second partial derivatives and products of first derivatives of the metric: $R \sim \partial^2 g + (\partial g)^2$. Specifying the Riemann tensor at a point fixes the 20 irreducible second derivatives $\partial_\mu \partial_\nu g_{\alpha\beta}$ that cannot be eliminated by coordinate transformations. However, it does *not* fix the 10 zeroth-order components $g_{\mu\nu}(P)$ or the first derivatives $\partial_\lambda g_{\mu\nu}(P)$, which are set by the choice of coordinate frame.
- **Differential Overdetermination and Integrability:** The system $R^\rho{}_{\sigma\mu\nu}[g] =$ given is a system of 20 non-linear second-order PDEs for 10 metric functions $g_{\mu\nu}(x)$. For a general tensor field $R^\rho{}_{\sigma\mu\nu}(x)$, no metric solution exists unless R satisfies the integrability conditions—which are precisely the Differential Bianchi Identities ($\nabla_{[e} R^\rho{}_{|\sigma|\mu\nu]} = 0$).
- **Field Equations and Gauge Redundancy:** In General Relativity, the field equations equate the contracted curvature (the Einstein tensor $G_{\mu\nu}$, 10 components) to the stress-energy tensor $T_{\mu\nu}$ (10 components). The 4 Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$ reduce the 10 equations to 6 independent dynamical equations, exactly matching the $10 - 4 = 6$ physical degrees of freedom of the metric modulo 4 coordinate diffeomorphisms.

14.2 Rigorous Proof of the First Algebraic Bianchi Identity

While the proof in Section 14.1 relied on evaluating second derivatives of the metric in Riemann Normal Coordinates, it is profoundly instructive to prove the First Algebraic Bianchi Identity purely from the commutator algebra of covariant derivatives acting on an arbitrary scalar field.

Theorem: First Bianchi Identity from Commutator Algebra

Let $(\mathcal{M}, g)$ be a smooth pseudo-Riemannian manifold equipped with a torsion-free Levi-Civita connection ∇. Then for any smooth vector field $A_a = \nabla_a \Phi$, the cyclic permutation of covariant commutators vanishes identically:

$$[\nabla_{[c}, \nabla_d]\nabla_{b]}\Phi \equiv 0 \implies R^a{}_{[bcd]} = 0. \tag{14.16}$$

Proof. Recall the definition of the Riemann curvature tensor acting on a covariant 1-form field A_b:

$$[\nabla_c, \nabla_d]A_b = -R^i{}_{bcd}A_i. \tag{14.17}$$

Let $\Phi \in C^\infty(\mathcal{M})$ be an arbitrary smooth scalar field, and choose the 1-form to be the exact gradient $A_b \equiv \nabla_b\Phi = \partial_b\Phi$.

Substituting $A_b = \nabla_b\Phi$ into Eq. (14.17):

$$[\nabla_c, \nabla_d]\nabla_b\Phi = -R^i{}_{bcd}\nabla_i\Phi. \tag{14.18}$$

Taking the totally antisymmetric cyclic sum over the three indices $[b, c, d]$:

$$\begin{aligned}[\nabla_{[c}, \nabla_d]\nabla_{b]}\Phi = \frac{1}{3!}\Big([\nabla_c, \nabla_d]\nabla_b\Phi + [\nabla_d, \nabla_b]\nabla_c\Phi + [\nabla_b, \nabla_c]\nabla_d\Phi \\ - [\nabla_d, \nabla_c]\nabla_b\Phi - [\nabla_b, \nabla_d]\nabla_c\Phi - [\nabla_c, \nabla_b]\nabla_d\Phi\Big).\end{aligned} \tag{14.19}$$

Using the antisymmetry of the commutator $[\nabla_c, \nabla_d] = -[\nabla_d, \nabla_c]$, the six terms combine into three duplicate pairs:

$$\begin{aligned}[\nabla_{[c}, \nabla_d]\nabla_{b]}\Phi &= \frac{2}{6}\Big([\nabla_c, \nabla_d]\nabla_b\Phi + [\nabla_d, \nabla_b]\nabla_c\Phi + [\nabla_b, \nabla_c]\nabla_d\Phi\Big) \\ &= \frac{1}{3}\Big(-R^i{}_{bcd}\nabla_i\Phi - R^i{}_{cdb}\nabla_i\Phi - R^i{}_{dbc}\nabla_i\Phi\Big) \\ &= -\frac{1}{3}\left(R^i{}_{bcd} + R^i{}_{cdb} + R^i{}_{dbc}\right)\nabla_i\Phi \\ &= -R^i{}_{[bcd]}\nabla_i\Phi.\end{aligned} \tag{14.20}$$

Now, let us evaluate the left-hand side $[\nabla_{[c}, \nabla_d]\nabla_{b]}\Phi$ directly by expanding the covariant derivative operators:

$$\begin{aligned}[\nabla_{[c}, \nabla_d]\nabla_{b]}\Phi = \frac{1}{3!}\Big(\nabla_c\nabla_d\nabla_b\Phi - \nabla_d\nabla_c\nabla_b\Phi + \nabla_d\nabla_b\nabla_c\Phi \\ - \nabla_b\nabla_d\nabla_c\Phi + \nabla_b\nabla_c\nabla_d\Phi - \nabla_c\nabla_b\nabla_d\Phi\Big).\end{aligned} \tag{14.21}$$

Because the Levi-Civita connection is metric-compatible and torsion-free ($T^a{}_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} = 0$), second covariant derivatives on scalar fields commute:

$$[\nabla_b, \nabla_c]\Phi = \nabla_b\nabla_c\Phi - \nabla_c\nabla_b\Phi = 0 \implies \nabla_b\nabla_c\Phi = \nabla_c\nabla_b\Phi. \tag{14.22}$$

Applying this commutativity to the inner pairs of derivatives:

- $\nabla_c(\nabla_d\nabla_b\Phi) = \nabla_c(\nabla_b\nabla_d\Phi)$ cancels against $-\nabla_c\nabla_b\nabla_d\Phi$.
- $-\nabla_d(\nabla_c\nabla_b\Phi) = -\nabla_d(\nabla_b\nabla_c\Phi)$ cancels against $+\nabla_d\nabla_b\nabla_c\Phi$.
- $-\nabla_b(\nabla_d\nabla_c\Phi) = -\nabla_b(\nabla_c\nabla_d\Phi)$ cancels against $+\nabla_b\nabla_c\nabla_d\Phi$.

Every single term cancels identically in pairs:

$$[\nabla_{[c}, \nabla_d]\nabla_{b]}\Phi \equiv 0. \tag{14.23}$$

Equating this to our previous expression yields:

$$-R^i{}_{[bcd]}\nabla_i\Phi = 0. \tag{14.24}$$

Choosing $\Phi(x) = x^a$ in local coordinates around P, so that $\nabla_i\Phi = \partial_i x^a = \delta^a_i$:

$$R^i{}_{[bcd]}\delta^a_i = R^a{}_{[bcd]} \equiv 0. \tag{14.25}$$

□

14.3 The Second (Differential) Bianchi Identity

The **Second Bianchi Identity** is a fundamental differential conservation law governing the gradient of spacetime curvature.

Theorem: The Second (Differential) Bianchi Identity

For any pseudo-Riemannian manifold $(\mathcal{M}, g)$ equipped with a Levi-Civita connection, the covariant derivative of the Riemann curvature tensor satisfies:

$$\nabla_e R^a{}_{bcd} + \nabla_c R^a{}_{bde} + \nabla_d R^a{}_{bec} \equiv 0, \tag{14.26}$$

or in totally antisymmetric cyclic bracket notation:

$$\nabla_{[e}R^a{}_{|b|cd]} \equiv 0. \tag{14.27}$$

Proof via Riemann Normal Coordinates. Let $P \in \mathcal{M}$ be an arbitrary point. In Riemann Normal Coordinates centered at P, the connection coefficients vanish at the origin: $\Gamma^\lambda_{\mu\nu}\Big|_P = 0$.

Recall the full expression for the Riemann tensor in terms of Christoffel symbols:

$$R^n{}_{ikl} = \partial_k\Gamma^n_{il} - \partial_l\Gamma^n_{ik} + \Gamma^n_{rk}\Gamma^r_{il} - \Gamma^n_{rl}\Gamma^r_{ik}. \tag{14.28}$$

Taking the covariant derivative $\nabla_m R^n{}_{ikl}$ and evaluating it at P:

$$\begin{aligned}\nabla_m R^n{}_{ikl}|_P &= \partial_m R^n{}_{ikl}|_P + \cancelto{0}{\Gamma^n_{mr}R^r{}_{ikl}} - \cancelto{0}{\Gamma^r_{mi}R^n{}_{rkl}} - \cancelto{0}{\Gamma^r_{mk}R^n{}_{irl}} - \cancelto{0}{\Gamma^r_{ml}R^n{}_{ikr}} \\ &= \partial_m R^n{}_{ikl}|_P\,.\end{aligned} \tag{14.29}$$

Furthermore, differentiating connection products $\partial_m(\Gamma\cdot\Gamma)$ vanishes at P because each term contains an undifferentiated $\Gamma|_P = 0$. Therefore:

$$\nabla_m R^n{}_{ikl}|_P = \partial_m\partial_k\Gamma^n_{il} - \partial_m\partial_l\Gamma^n_{ik}. \tag{14.30}$$

Now, consider the cyclic sum over (m, k, l):

$$(\nabla_m R^n{}_{ikl} + \nabla_k R^n{}_{ilm} + \nabla_l R^n{}_{imk})|_P = (\partial_m\partial_k\Gamma^n_{il} - \partial_m\partial_l\Gamma^n_{ik})$$

$$+ (\partial_k \partial_l \Gamma^n_{im} - \partial_k \partial_m \Gamma^n_{il}) + (\partial_l \partial_m \Gamma^n_{ik} - \partial_l \partial_k \Gamma^n_{im}) \,. \tag{14.31}$$

By the equality of mixed partial derivatives ($\partial_m \partial_k = \partial_k \partial_m$), all six terms cancel pairwise:

$$(\partial_m \partial_k \Gamma^n_{il} - \partial_k \partial_m \Gamma^n_{il}) + (-\partial_m \partial_l \Gamma^n_{ik} + \partial_l \partial_m \Gamma^n_{ik}) + (\partial_k \partial_l \Gamma^n_{im} - \partial_l \partial_k \Gamma^n_{im}) = 0. \tag{14.32}$$

Because this tensor equation holds at an arbitrary point P in an arbitrary Riemann Normal Coordinate system, it holds globally on $\mathcal{M}$:

$$\nabla_m R^n{}_{ikl} + \nabla_k R^n{}_{ilm} + \nabla_l R^n{}_{imk} \equiv 0. \tag{14.33}$$

□

Differential Forms Interpretation ($D\Omega = 0$): In the Cartan formalism of moving frames, the curvature 2-form is $\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$. Taking the exterior covariant derivative $D\Omega$:

$$D\Omega = \mathrm{d}\Omega + \omega \wedge \Omega - \Omega \wedge \omega \equiv 0. \tag{14.34}$$

Thus, the Second Bianchi Identity is the non-Abelian geometric analogue of the Maxwell identity $\mathrm{d}F = 0$.

14.4 Traces of Curvature: Ricci Tensor and Ricci Scalar

The Riemann curvature tensor contains 20 independent components in 4D. To extract lower-rank geometric quantities that can couple directly to physical sources, we perform metric contractions.

14.4.1 Uniqueness of the Non-Trivial Trace

Let us systematically check all possible pairwise index contractions of R_{abcd}:

1. $g^{ab} R_{abcd} = 0$ (since $R_{abcd} = -R_{bacd}$).
2. $g^{cd} R_{abcd} = 0$ (since $R_{abcd} = -R_{abdc}$).
3. $g^{ac} R_{abcd} \equiv R_{bd}$ (**Ricci Tensor**).
4. $g^{bd} R_{abcd} = g^{bd} R_{cdab} = R_{ac} = R_{ca}$.
5. $g^{ad} R_{abcd} = -g^{ad} R_{abdc} = -R_{bc}$.

Thus, up to an overall sign, **there is only one unique non-trivial single contraction** of the Riemann curvature tensor:

1. The **Ricci Tensor** $R_{\mu\nu}$ is defined by:

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}. \tag{14.35}$$

   Using $R_{abcd} = R_{cdab}$, the Ricci tensor is strictly symmetric: $R_{\mu\nu} = R_{\nu\mu}$.

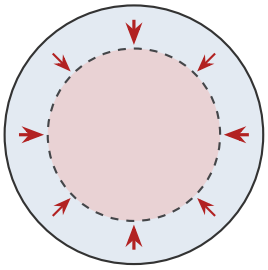
2. The **Ricci Scalar** (or Scalar Curvature) R is the complete metric trace:

$$R \equiv g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\alpha\beta} R_{\alpha\mu\beta\nu}. \tag{14.36}$$

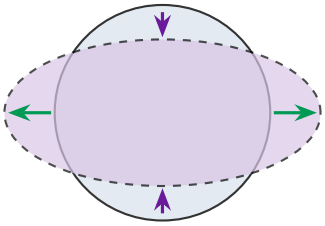
14.4.2 Geometric Decomposition: Ricci vs. Weyl Curvature

In $n \geq 4$ spacetime dimensions, the Riemann tensor decomposes into two parts:

$$R_{abcd} = \underbrace{C_{abcd}}_{\text{Weyl (Conformal) Curvature}} + \underbrace{\frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) - \frac{2}{(n-1)(n-2)}R\, g_{a[c}g_{d]b}}_{\text{Ricci Curvature (Matter-Coupled)}}. \tag{14.37}$$



(a) Ricci Curvature ($R_{\mu\nu}$)
Net volume contraction $\Delta V < 0$
Directly coupled to matter: $T_{\mu\nu} \neq 0$



(b) Weyl Curvature ($C_{\mu\nu\rho\sigma}$)
Pure shape distortion (tidal shear)
Volume preserved: $\Delta V = 0$
Propagates in vacuum ($R_{\mu\nu} = 0$)

Figure 14.1: Curvature decomposition on a cloud of freely falling particles: (a) Ricci volume compression vs. (b) Weyl tidal shape deformation.

- **Ricci Tensor ($R_{\mu\nu}$, 10 components):** Governs the **rate of change of volume** of a small ball of freely falling test particles. Via Einstein's field equations, the Ricci tensor is determined locally by matter at that point.
- **Weyl Tensor (C_{abcd}, 10 components):** Completely traceless (${C^{\mu}}_{\alpha\mu\beta} = 0$). It measures the **pure shape distortion (tidal shear)** of a test cloud with zero change in volume. It represents the *free gravitational field*—carrying gravitational waves through the vacuum ($R_{\mu\nu} = 0$).

14.5 The Contracted Bianchi Identity and the Einstein Tensor

We perform two successive metric contractions on the Second Differential Bianchi Identity:

$$\nabla_m {R^n}_{ikl} + \nabla_k {R^n}_{ilm} + \nabla_l {R^n}_{imk} = 0. \tag{14.38}$$

14.5.1 First Contraction: Setting $n = k$

Contracting the upper index n with k:

$$\nabla_m {R^k}_{ikl} + \nabla_k {R^k}_{ilm} + \nabla_l {R^k}_{imk} = 0. \tag{14.39}$$

Using ${R^k}_{ikl} = R_{il}$ and ${R^k}_{imk} = -R_{im}$, this yields the **Once-Contracted Bianchi Identity**:

$$\nabla_m R_{il} + \nabla_k {R^k}_{ilm} - \nabla_l R_{im} = 0. \tag{14.40}$$

14.5.2 Second Contraction: Contracting with g^{il}

Contracting with g^{il} (using $\nabla_k g^{il} = 0$):

$$g^{il}\nabla_m R_{il} + g^{il}\nabla_k {R^k}_{ilm} - g^{il}\nabla_l R_{im} = 0 \implies \nabla_m R - \nabla_k {R^k}_m - \nabla_l {R^l}_m = 0. \tag{14.41}$$

Relabeling $l \to k$:

$$\nabla_m R - 2\nabla_k R^k{}_m = 0 \iff \nabla_k \left(R^k{}_m - \frac{1}{2}\delta^k_m R \right) \equiv 0. \tag{14.42}$$

Raising the index m:

$$\nabla_k \left(R^{km} - \frac{1}{2} g^{km} R \right) \equiv 0. \tag{14.43}$$

Theorem: The Einstein Tensor and the Twice-Contracted Bianchi Identity

The **Einstein Tensor** $G_{\mu\nu}$ is defined as:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{14.44}$$

It is identically symmetric and automatically divergence-free:

$$G_{\mu\nu} = G_{\nu\mu}, \qquad \nabla^\mu G_{\mu\nu} \equiv 0. \tag{14.45}$$

Geometric Conservation Identity: The vanishing of the covariant divergence $\nabla^\mu G_{\mu\nu} \equiv 0$ is a **pure geometric identity** arising from the Jacobi identity of the curvature tensor. It holds identically on **any** Riemannian or pseudo-Riemannian manifold.

14.5.3 The Inevitability of Einstein's Field Equations

In special and general relativity, the local conservation of energy, momentum, and stress is expressed as $\nabla^\mu T_{\mu\nu} = 0$.

- Proposing $R_{\mu\nu} = \kappa T_{\mu\nu}$ leads to $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R = 0$, requiring $R = \text{const}$, which is unphysical.
- Consistency with energy-momentum conservation **demands** that the geometric side of the field equations be identically divergence-free.

This leads directly to **Einstein's Field Equations**:

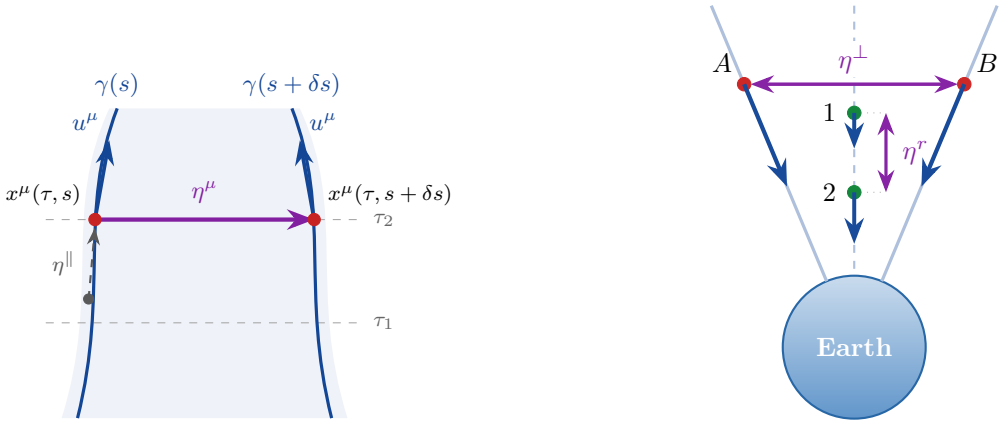
$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{14.46}$$

where $\nabla^\mu G_{\mu\nu} \equiv 0 \iff \nabla^\mu T_{\mu\nu} = 0$ is automatically guaranteed.

14.6 Pedagogical Reflections and Foundational Inquiries

1. Holonomy, Loop Dependence, and Point Curvature: When defining the Riemann curvature tensor as the change in a vector after parallel transport around a loop, the finite Wilson loop $U(\gamma) = \mathcal{P} \exp\left(- \oint_\gamma \Gamma \right)$ depends on the full loop geometry. However, in the **infinitesimal limit**, the leading-order rotation $\Delta v^\alpha = \frac{1}{2} R^\alpha{}_{\beta\mu\nu}(P) v^\beta \Delta A^{\mu\nu}$ is multilinear in the area element $\Delta A^{\mu\nu}$, and the tensor $R^\alpha{}_{\beta\mu\nu}(P)$ depends **only on the base point** P.

2. Geodesic Deviation vs. Test Particle Proper Acceleration: An individual freely falling observer has zero proper acceleration ($a^\mu = u^\nu \nabla_\nu u^\mu = 0$). However, two adjacent observers separated by η^μ experience relative acceleration governed by $\frac{D^2 \eta^\mu}{\mathrm{d}\tau^2} = R^\mu{}_{\nu\alpha\beta} u^\nu u^\alpha \eta^\beta$.



(a) Geodesic congruence in spacetime. (b) Tidal deviation in radial field.

Figure 14.2: Geodesic deviation: (a) congruence geometry, and (b) tidal deviation in Earth's radial field.

For two observers falling radially toward Earth:

- **Transverse Separation ($\eta^\perp$):** Trajectories point toward the center of mass, producing convergence: $\frac{\mathrm{d}^2\eta^\perp}{\mathrm{d}t^2} = -\frac{GM}{r^3}\eta^\perp$.
- **Longitudinal Separation ($\eta^\parallel$):** The lower observer feels stronger gravity, accelerating away from the upper observer: $\frac{\mathrm{d}^2\eta^\parallel}{\mathrm{d}t^2} = +\frac{2GM}{r^3}\eta^\parallel$.

3. Local Flatness vs. Non-Zero Curvature: "Locally flat" means that in a Local Inertial Frame at P, $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}(P) = 0$ (first derivatives vanish). It does *not* mean second derivatives vanish. The 20 irreducible second derivatives form $R^\rho{}_{\sigma\mu\nu}(P) \neq 0$. Flatness in a finite neighborhood ($R = 0$) is Minkowski space; in curved space, flatness is strictly a first-order tangent space approximation.

4. Weinberg's Analysis for Extended Spinning Bodies: In Weinberg (page 133), the equations of motion are compared:

$$\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + fR^\lambda{}_{\mu\nu k}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}S^k = 0, \qquad \frac{\mathrm{d}^2x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0. \tag{14.47}$$

For structureless point particles ($\ell \to 0$), the Strong Equivalence Principle uniquely selects the standard geodesic equation. For extended spinning bodies of size ℓ, tidal forces couple spin $S^{\mu\nu}$ to curvature via the **Mathisson–Papapetrou–Dixon (MPD) equations**: $\frac{DP^\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}u^\nu S^{\alpha\beta}$. In the point particle limit $\ell/L \ll 1$, this non-minimal coupling vanishes.

5. Geodesic Grids and the Global Breakdown of Coordinates: In flat space, parallel geodesics never intersect, allowing a global Cartesian coordinate grid. In curved spacetime, geodesic congruences focus and develop caustics / conjugate points where the Jacobian determinant vanishes. Thus, Riemann curvature is the fundamental obstruction to a global geodesic coordinate grid.

Chapter 15

Hypersurface Geometry, Weyl Curvature, and the Gravitational Action Principle

15.1 Hypersurface Geometry and Projected Curvature

15.1.1 Kinematics vs. Dynamics: Accidental Dynamics Fallacy

In Chapter 14, we derived the contracted Bianchi identity:

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = \nabla_\mu G^{\mu\nu} \equiv 0. \tag{15.1}$$

A naive argument frequently encountered in introductory discussions claims that Einstein's Field Equations,

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}, \tag{15.2}$$

are simply the "solution" or automatic algebraic consequence of equating two identically conserved symmetric rank-2 tensors, since local energy-momentum conservation states $\nabla_\mu T^{\mu\nu} = 0$.

This argument is conceptually flawed. Equation (15.1) is an **off-shell geometric identity** valid for *any* smooth metric $g_{\mu\nu}$, completely independent of physics, matter sources, or field equations. Diffeomorphism invariance guarantees $\nabla_\mu G^{\mu\nu} \equiv 0$ via Noether's second theorem. Merely pairing two divergence-free tensors does not constitute a dynamical theory of gravity; it represents an "accidental kinematics-to-dynamics jump."

True gravitational dynamics requires an independent physical principle—either a variational action principle (the Einstein–Hilbert action) or a physical statement relating curvature to stress-energy calibrated by the Newtonian limit.

15.1.2 Hypersurface Projections and the Induced Metric

Let $(\mathcal{M}, g_{ab})$ be a 4-dimensional pseudo-Riemannian manifold. Consider a 3-dimensional hypersurface $\mathcal{H} \subset \mathcal{M}$ equipped with a unit normal 1-form n_a and vector $n^a = g^{ab} n_b$. For a timelike hypersurface (spacelike normal), $n_a n^a = +1$. For a spacelike hypersurface (representing an instantaneous spatial Cauchy slice of spacetime), the unit normal is timelike, $u_a = n_a$, satisfying:

$$u_a u^a = -1. \tag{15.3}$$

Projection Operator / Induced Metric: We define the spatial projection tensor h_{ab} onto $\mathcal{H}$ as:

$$h_{ab} \equiv g_{ab} + u_a u_b, \qquad h^a{}_b \equiv g^a{}_b + u^a u_b = \delta^a_b + u^a u_b. \tag{15.4}$$

Let us verify that $h^a{}_b$ is an idempotent projection operator ($P^2 = P$) orthogonal to u^a:

$$h^a{}_b u^b = (\delta^a_b + u^a u_b)u^b = u^a + u^a(u_b u^b) = u^a + u^a(-1) = 0, \tag{15.5}$$

$$\begin{aligned} h^a{}_b h^b{}_c &= (\delta^a_b + u^a u_b)(\delta^b_c + u^b u_c) = \delta^a_c + u^a u_c + u^a u_c + u^a(u_b u^b)u_c \\ &= \delta^a_c + 2u^a u_c - u^a u_c = \delta^a_c + u^a u_c = h^a{}_c. \end{aligned} \tag{15.6}$$

The tensor h_{ab} acts as the induced 3-metric on the spatial hypersurface $\mathcal{H}$. Any 4-vector $V^a \in T_p\mathcal{M}$ decomposes uniquely into a normal component and a spatial component tangential to $\mathcal{H}$:

$$V^a = (-u_b V^b)u^a + h^a{}_b V^b = V^a_{\parallel} + V^a_{\perp}. \tag{15.7}$$

15.1.3 Projection of the 4D Riemann Tensor

We can project all four indices of the 4D Riemann curvature tensor $R^a{}_{bcd}$ onto the hypersurface $\mathcal{H}$:

$${}^{\mathcal{H}}R^i{}_{jk\ell} \equiv h^i{}_a h^b{}_j h^c{}_k h^d{}_\ell R^a{}_{bcd}. \tag{15.8}$$

This projected tensor ${}^{\mathcal{H}}R^i{}_{jk\ell}$ is expressed in the 4D coordinate basis, but it contains only the components of curvature purely tangential to $\mathcal{H}$.

Coordinate Basis vs. Intrinsic Hypersurface Basis: To describe intrinsic 3-dimensional geometry on $\mathcal{H}$, let y^α ($\alpha = 1, 2, 3$) be intrinsic coordinates on $\mathcal{H}$, parameterized by spacetime embedding functions $x^a = x^a(y^\alpha)$. The projection vectors (tangent basis vectors) are:

$$e^a_\alpha \equiv \frac{\partial x^a}{\partial y^\alpha}. \tag{15.9}$$

The intrinsic 3-metric $\gamma_{\alpha\beta}$ and the intrinsic 3D Riemann tensor ${}^{(3)}R^\alpha{}_{\beta\gamma\delta}$ are given by:

$$\gamma_{\alpha\beta} = e^a_\alpha e^b_\beta g_{ab} = e^a_\alpha e^b_\beta h_{ab}, \tag{15.10}$$

$${}^{(3)}R^\alpha{}_{\beta\gamma\delta} = e^\alpha_a e^b_\beta e^c_\gamma e^{d\,(3)}_\delta R^a{}_{bcd}, \tag{15.11}$$

where $e^\alpha_a = \gamma^{\alpha\beta} g_{ab} e^b_\beta$. Thus, ${}^{(3)}R^\alpha{}_{\beta\gamma\delta}$ is a $3 \times 3 \times 3 \times 3$ object in intrinsic coordinates, while ${}^{\mathcal{H}}R^i{}_{jk\ell}$ is represented as a $4 \times 4 \times 4 \times 4$ matrix whose contraction with the normal u^i vanishes identically.

15.1.4 Contraction and the Hamiltonian Curvature Identity

Let us calculate the fully projected Ricci scalar curvature ${}^{\mathcal{H}}R \equiv h^{ac}h^{bd}R_{abcd}$.

Theorem: Hypersurface Curvature and the Einstein Tensor

Let u^a be a unit timelike normal vector field ($u_a u^a = -1$) with associated spatial projection operator $h_{ab} = g_{ab} + u_a u_b$. The fully projected 4D Riemann curvature scalar is related directly to the Einstein tensor G_{ab} by:

$${}^{\mathcal{H}}R \equiv h^{ac}h^{bd}R_{abcd} = 2u^a u^b G_{ab}. \tag{15.12}$$

Proof. We substitute the explicit projection operator $h^{ac} = g^{ac} + u^a u^c$ and $h^{bd} = g^{bd} + u^b u^d$ into ${}^{\mathcal{H}}R$:

$$\begin{aligned} {}^{\mathcal{H}}R &= (g^{ac} + u^a u^c)\left(g^{bd} + u^b u^d\right) R_{abcd} \\ &= g^{ac}g^{bd}R_{abcd} + g^{ac}u^b u^d R_{abcd} + u^a u^c g^{bd} R_{abcd} + u^a u^c u^b u^d R_{abcd}. \end{aligned} \tag{15.13}$$

Let us analyze each term in Equation (15.13):

1. First term: $g^{ac}g^{bd}R_{abcd} = g^{bd}R^{c}{}_{bcd} = g^{bd}R_{bd} = R$.
2. Second term: $g^{ac}u^b u^d R_{abcd} = u^b u^d R^{c}{}_{bcd} = u^b u^d R_{bd}$.
3. Third term: $u^a u^c g^{bd} R_{abcd} = u^a u^c(-g^{bd}R_{bacd}) = -u^a u^c R^{d}{}_{acd} = -u^a u^c(-R_{ac}) = u^a u^c R_{ac}$.
4. Fourth term: $u^a u^b u^c u^d R_{abcd}$. By the algebraic antisymmetry of the Riemann tensor in its last two indices, $R_{ab(cd)} = 0$, contracting with the symmetric product $u^c u^d$ yields:

$$u^c u^d R_{abcd} = 0 \implies u^a u^b u^c u^d R_{abcd} \equiv 0. \tag{15.14}$$

Combining these four terms gives:

$$^{\mathcal{H}}R = R + u^b u^d R_{bd} + u^a u^c R_{ac} = R + 2u^a u^b R_{ab}. \tag{15.15}$$

Now consider the projection of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ along $u^a u^b$:

$$2u^a u^b G_{ab} = 2u^a u^b \left(R_{ab} - \frac{1}{2}g_{ab}R\right) = 2u^a u^b R_{ab} - (u_a u^a)R. \tag{15.16}$$

Since $u_a u^a = -1$, the last term becomes $-(-1)R = +R$:

$$2u^a u^b G_{ab} = 2u^a u^b R_{ab} + R = {}^{\mathcal{H}}R. \tag{15.17}$$

This completes the proof. □

Physical Significance in $3+1$ Gravitational Dynamics: Under the Einstein equations $G_{ab} = 8\pi G T_{ab}$, the contraction $u^a u^b G_{ab} = 8\pi G u^a u^b T_{ab} = 8\pi G\rho$ represents the local energy density ρ measured by an observer with 4-velocity u^a. Via the Gauss equation relating $^{\mathcal{H}}R$ to the intrinsic scalar curvature $^{(3)}R$ and extrinsic curvature K_{ab}:

$$^{(3)}R + K^2 - K_{ab}K^{ab} = 2u^a u^b G_{ab} = 16\pi G\rho. \tag{15.18}$$

Equation (15.18) is precisely the fundamental **Hamiltonian Constraint** of General Relativity (the initial value constraint equation on spatial hypersurfaces in the ADM formalism).

15.2 The Weyl Curvature Tensor and Conformal Geometry

15.2.1 Irreducible Decomposition of Curvature in $N \geq 3$ Dimensions

In Chapter 14, we noted that in $N = 4$ dimensions, the Riemann curvature tensor R_{abcd} possesses 20 algebraically independent components, while the symmetric Ricci tensor R_{ab} contains 10 independent components. The trace of R_{ab} is the Ricci scalar R.

The 10 remaining degrees of freedom of the Riemann tensor constitute the **Weyl Conformal Curvature Tensor** C_{abcd}. In $N \geq 3$ spacetime dimensions, the Riemann tensor decomposes uniquely into its trace parts (Ricci) and its totally trace-free part (Weyl):

$$R_{abcd} = C_{abcd} + \frac{2}{N-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) - \frac{2}{(N-1)(N-2)}R\, g_{a[c}g_{d]b}, \tag{15.19}$$

where index antisymmetrization is defined as $A_{[a}B_{b]} \equiv \frac{1}{2}(A_a B_b - A_b B_a)$.

In $N = 4$ dimensions, Equation (15.19) simplifies to:

$$C_{abcd} = R_{abcd} - \left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{1}{3}R\, g_{a[c}g_{d]b}. \tag{15.20}$$

15.2.2 Algebraic Symmetries and Tracelessness

The Weyl tensor C_{abcd} inherits all the algebraic symmetries of the Riemann tensor:

1. Antisymmetry: $C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab}$.
2. First Algebraic Bianchi Identity: $C_{a[bcd]} \equiv 0$.
3. **Total Tracelessness:** Every single trace contraction vanishes identically:

$$g^{ac}C_{abcd} = C^{a}{}_{bad} \equiv 0. \tag{15.21}$$

Equation (15.21) imposes $\frac{N(N+1)}{2}$ independent constraints (10 constraints in 4D). Therefore, in 4D:

$$N_{\text{Weyl}} = N_{\text{Riemann}} - N_{\text{Ricci}} = 20 - 10 = 10 \text{ independent components.} \tag{15.22}$$

Physical Interpretation: Free Gravitational Radiation and Tidal Shear: In a matter-free vacuum region ($T_{ab} = 0$), Einstein's equations dictate $R_{ab} = 0$ and $R = 0$. In vacuum, Equation (15.20) reduces to:

$$R_{abcd} = C_{abcd}. \tag{15.23}$$

Thus, while the Ricci tensor R_{ab} represents the *local, instantaneous* distortion of volume produced directly by local matter-energy, the Weyl tensor C_{abcd} represents *tidal shear, gravitational waves, and free gravitational fields* that propagate through empty vacuum from distant sources.

15.2.3 Conformal Invariance and Conformal Flatness

A **conformal transformation** is a local, smooth rescaling of the spacetime metric that preserves angles and light cones:

$$\tilde{g}_{ab}(x) = \Omega^2(x)g_{ab}(x), \qquad \tilde{g}^{ab}(x) = \Omega^{-2}(x)g^{ab}(x), \tag{15.24}$$

where $\Omega(x) > 0$ is a strictly positive, differentiable conformal factor.

Transformation of Curvature: Under a conformal transformation, the Levi-Civita connection and Riemann tensor transform non-trivially with gradient terms $\nabla_a \ln \Omega$. Specifically:

$$\tilde{\Gamma}^{c}_{ab} = \Gamma^{c}_{ab} + \Omega^{-1}\left(\delta^{c}_{a}\partial_b\Omega + \delta^{c}_{b}\partial_a\Omega - g_{ab}g^{cd}\partial_d\Omega\right), \tag{15.25}$$

$$\tilde{R}_{abcd} = \Omega^2 R_{abcd} + \ldots \quad \text{(terms involving } \nabla\Omega \text{ and } \nabla^2\Omega\text{)}. \tag{15.26}$$

Remarkably, the non-linear gradient terms appearing in the Ricci and scalar curvature transformations precisely cancel out in the trace-free combination defining the Weyl tensor!

Weyl Conformal Invariance: The $(1,3)$-valence Weyl tensor is strictly **conformally invariant**:

$$\tilde{C}^{a}{}_{bcd}(x) = C^{a}{}_{bcd}(x). \tag{15.27}$$

Lowering the first index with $\tilde{g}_{ae} = \Omega^2 g_{ae}$ gives:

$$\tilde{C}_{abcd}(x) = \Omega^2(x)C_{abcd}(x). \tag{15.28}$$

Theorem on Conformal Flatness: A spacetime is said to be **conformally flat** if its metric can be globally or locally transformed into the flat Minkowski metric η_{ab} via a conformal transformation:

$$g_{ab}(x) = \Omega^2(x)\eta_{ab}. \tag{15.29}$$

- For dimensions $N \geq 4$: A spacetime is conformally flat if and only if its Weyl tensor vanishes identically:
$$C_{abcd} \equiv 0 \iff g_{ab} = \Omega^2(x)\eta_{ab}. \tag{15.30}$$
- For $N = 3$: The Weyl tensor vanishes identically in all 3D geometries ($C_{abcd} \equiv 0$). Conformal flatness in 3D is governed by the vanishing of the 3D **Cotton tensor** $C_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} + \frac{1}{4}(g_{ac}\nabla_b R - g_{ab}\nabla_c R) = 0$.

Standard examples of conformally flat spacetimes in general relativity include the Robertson–Walker cosmological metrics (FLRW), anti-de Sitter space (AdS_4), and the spatial slices of Schwarzschild geometry in isotropic coordinates.

15.3 Integrability, Geodesics, and Energy-Momentum Balance

15.3.1 Integrability and Uniqueness in the Geodesic Differential Equation

A foundational question in curved geometry asks: *To integrate the geodesic differential equation and obtain a unique trajectory, do we require the differential equation to be an exact differential, and where is the missing uniqueness information stored?*

The geodesic equation,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0, \tag{15.31}$$

is a non-linear system of second-order ordinary differential equations. By the Picard–Lindelöf theorem, given an initial event $x^\mu(0)$ and an initial tangent velocity $u^\mu(0) = \left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right|_0$, a unique geodesic trajectory $x^\mu(\lambda)$ exists locally in an affine neighborhood.

Initial-Value vs. Boundary-Value Problems:

1. **Initial Value Formulation:** Specifying $(x^\mu(0), \dot{x}^\mu(0))$ guarantees local existence and uniqueness.
2. **Two-Point Boundary Value Formulation:** If we instead fix two spacetime events $x_1^\mu = x^\mu(\lambda_1)$ and $x_2^\mu = x^\mu(\lambda_2)$, curvature can focus geodesics, leading to the formation of **conjugate points** (e.g., antipodal points on S^2 connected by infinitely many geodesics). In such cases, boundary conditions alone do not select a unique solution without specifying the homotopy class or initial launch angle.

15.3.2 Why $\nabla_\mu T^{\mu\nu} = 0$ is Not a True Global Conservation Law

In flat Minkowski space, the conservation of energy and momentum is expressed by the vanishing ordinary divergence $\partial_\mu T^{\mu\nu} = 0$. Integrating over a spatial volume $\mathcal{V}$ between Cauchy surfaces t_1 and t_2 gives exact time-independent conserved charges:

$$P^\nu = \int_{\Sigma_t} \mathrm{d}^3x\, T^{0\nu} = \text{constant}. \tag{15.32}$$

In curved spacetime, minimal coupling promotes the divergence to a covariant one:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{15.33}$$

Let us analyze whether Equation (15.33) leads to conserved integrated quantities.

Explicit Volume Integral Expansion: Recall the identity for the covariant divergence of a symmetric rank-2 tensor:

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^{\mu\nu}\right) + \Gamma^\nu_{\alpha\beta}T^{\alpha\beta}. \tag{15.34}$$

Lowering the index $\nu \to i$ (where $i = 1, 2, 3$ denotes spatial components) and using $\Gamma_{i,\alpha\beta} = \frac{1}{2}(\partial_\alpha g_{i\beta} + \partial_\beta g_{i\alpha} - \partial_i g_{\alpha\beta})$ with $T^{\alpha\beta} = T^{\beta\alpha}$:

$$\Gamma^i_{\alpha\beta}T^{\alpha\beta} = g^{ik}\Gamma_{k,\alpha\beta}T^{\alpha\beta} = -\frac{1}{2}T^{\alpha\beta}\partial_i g_{\alpha\beta} + \partial_\alpha g_{i\beta}T^{\alpha\beta}. \tag{15.35}$$

Thus, multiplying by the invariant volume element $\sqrt{-g}$ and integrating over a 4D spacetime domain $\mathcal{M} = [t_1, t_2] \times \Sigma$:

$$\begin{aligned} 0 &= \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu T^\mu{}_i \\ &= \int_{t_2} \mathrm{d}^3x\sqrt{-g}\,T^0{}_i - \int_{t_1} \mathrm{d}^3x\sqrt{-g}\,T^0{}_i + \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,T^{\alpha\beta}\partial_i g_{\alpha\beta}. \end{aligned} \tag{15.36}$$

Physical Conclusion: Gravitational Momentum Exchange: The non-vanishing volume term,

$$\mathcal{F}^{\mathrm{grav}}_i \equiv \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,T^{\alpha\beta}\partial_i g_{\alpha\beta}, \tag{15.37}$$

acts as an explicit gravitational source/sink of momentum. It represents the physical force exerted by the dynamical metric field on matter. Therefore, in general relativity:

- **No Local Conservation of Matter Stress-Energy Alone:** Matter constantly exchanges energy and momentum with the spacetime geometry.
- **Killing Vector Symmetries Restore True Conservation:** If spacetime admits a Killing vector field ξ^μ ($\nabla_{(\mu}\xi_{\nu)} = 0$), then the current $J^\mu \equiv T^{\mu\nu}\xi_\nu$ is strictly conserved:

$$\nabla_\mu J^\mu = \nabla_\mu(T^{\mu\nu}\xi_\nu) = (\nabla_\mu T^{\mu\nu})\xi_\nu + T^{\mu\nu}\nabla_\mu\xi_\nu = 0 + T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)} = 0. \tag{15.38}$$

- **Asymptotic Flatness and Global Charges:** Setting $\partial_i g_{\alpha\beta} \to 0$ asymptotically at spatial infinity allows one to define global conserved charges (the ADM mass M_{ADM}, ADM momentum P^i, and angular momentum J^{ij}) as surface flux integrals at infinity, but local conservation inside the bulk remains non-existent.

15.4 The Variational Action Principle for Gravitation

15.4.1 Why the Gravitational Action Requires Metric Second Derivatives

To formulate General Relativity as a dynamical field theory, we postulate a total action:

$$S_{\mathrm{total}} = S_{\mathrm{gravity}} + S_{\mathrm{matter}}, \tag{15.39}$$

where $S_{\mathrm{matter}} = \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}_{\mathrm{matter}}(\phi, \nabla\phi, g_{ab})$ and S_{gravity} must be invariant under spacetime diffeomorphisms.

Consider what functional form the gravitational Lagrangian density $\mathcal{L}_{\mathrm{gravity}}$ can take:

1. **Hypothesis: Lagrangian depending only on $(g_{ab}, \partial_c g_{ab})$:** Suppose $\mathcal{L}_{\text{gravity}} = \mathcal{L}(g_{ab}, \partial_c g_{ab})$. In a Local Inertial Coordinate (LIC) system centered at an arbitrary event P:

$$g_{ab}|_P = \eta_{ab}, \qquad \partial_c g_{ab}|_P = 0. \tag{15.40}$$

Therefore, at event P, the Lagrangian evaluates to a pure constant $\mathcal{L}(\eta_{ab}, 0) = \text{const}$. Because $\mathcal{L}_{\text{gravity}}$ is constructed as a diffeomorphism scalar, if it is constant in one coordinate frame at P, it must be constant at P in *all* coordinate frames! Since P was arbitrary, $\mathcal{L}$ is constant everywhere:

$$\mathcal{L}_{\text{gravity}}(g_{ab}, \partial_c g_{ab}) \equiv \text{constant} \implies \text{No dynamical equations of motion!} \tag{15.41}$$

2. **Necessity of Second Derivatives:** Spacetime curvature is intrinsically characterized by the non-vanishing second derivatives $\partial_c \partial_d g_{ab}$. Therefore, to generate non-trivial gravitational dynamics, the gravitational Lagrangian **must depend explicitly on second derivatives of the metric**:

$$\mathcal{L}_{\text{gravity}} = \mathcal{L}(g_{ab}, \partial_c g_{ab}, \partial_c \partial_d g_{ab}). \tag{15.42}$$

15.4.2 Higher Derivatives, the Ostrogradsky Instability, and Action Splitting

In classical field theory, Lagrangians containing second derivatives $\ddot{q}$ or $\partial^2\phi$ generally lead to higher-order Euler–Lagrange equations (4th order ODEs/PDEs). By the **Ostrogradsky Theorem**, non-degenerate higher-derivative Lagrangians possess Hamiltonians that are unbounded from below, giving rise to catastrophic ghost instabilities.

Toy Model with a Total Derivative Second Order Term: Consider the one-dimensional particle action:

$$S = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{1}{2}\dot{q}^2 + \ddot{q}\right) = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{1}{2}\dot{q}^2\right) + \Big[\dot{q}(t_2) - \dot{q}(t_1)\Big]. \tag{15.43}$$

Varying the action with respect to $q(t)$ gives:

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t\, (-\ddot{q})\, \delta q + \Big[\dot{q}\, \delta q\Big]_{t_1}^{t_2} + \Big[\delta \dot{q}\Big]_{t_1}^{t_2}. \tag{15.44}$$

To obtain the standard equation of motion $\ddot{q} = 0$, we would be forced to impose four boundary conditions:

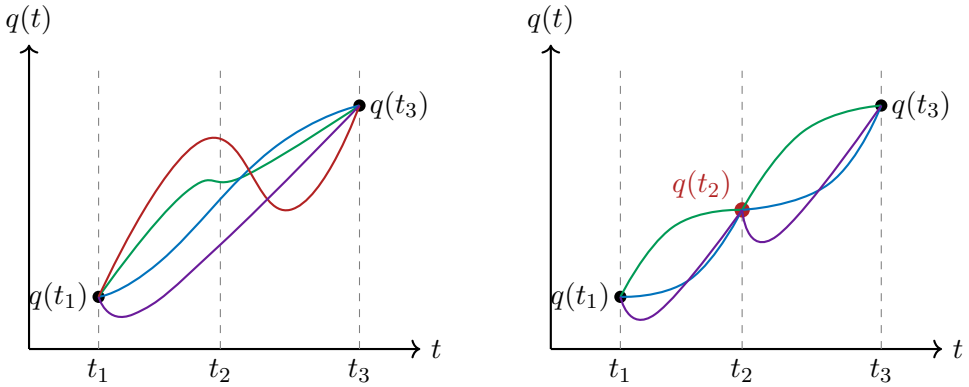
$$\delta q(t_1) = \delta q(t_2) = 0, \qquad \delta \dot{q}(t_1) = \delta \dot{q}(t_2) = 0. \tag{15.45}$$

The Inconsistency / Over-Determination Problem: The equation of motion $\ddot{q} = 0$ is a second-order ODE, which requires exactly *two* boundary conditions to specify a unique solution. Demanding four boundary conditions severely over-determines the system, leading to mathematical inconsistency for general boundary data.

Action Additivity and Path Splitting: Consider decomposing the total action over an intermediate time $t_2 \in (t_1, t_3)$:

$$S[t_1, t_3] = \int_{t_1}^{t_2} L\, \mathrm{d}t + \int_{t_2}^{t_3} L\, \mathrm{d}t. \tag{15.46}$$

When performing variations $\delta S = 0$, the intermediate boundary variations at t_2 must cancel seamlessly across the boundary interface:



(a) Variational family across full interval **(b)** Independent splitting: $\dot{q}(t_2^-) \neq \dot{q}(t_2^+)$

Figure 15.1: Action decomposition across intermediate boundary t_2. If derivative variations $\delta\dot{q}$ are arbitrarily constrained at intermediate surfaces, non-differentiable trajectory kinks ($\dot{q}(t_2^-) \neq \dot{q}(t_2^+)$) are artificially introduced.

15.4.3 The Gibbons–Hawking–York (GHY) Boundary Term

How does General Relativity circumvent this over-determination problem despite depending on second derivatives $\partial^2 g_{ab}$?

Linearity in Second Derivatives (Lovelock's Theorem): In $N = 4$ dimensions, the Ricci scalar R is uniquely distinguished: it depends **linearly** on the second derivatives $\partial^2 g_{ab}$. Quadratic curvature invariants like the Kretschmann scalar $K = R_{abcd}R^{abcd}$ or R^2 contain squared second derivatives $(\partial^2 g)^2$, which inevitably yield 4th-order field equations and fatal Ostrogradsky ghost modes. The linearity of R ensures that the resulting field equations (Einstein equations) remain strictly second-order in metric derivatives!

Varying the Einstein–Hilbert Action: Let $\mathcal{M}$ be a 4-manifold with boundary $\partial\mathcal{M}$. The pure Einstein–Hilbert action is:

$$S_{\rm EH} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,R, \qquad (\kappa \equiv 8\pi G/c^4). \tag{15.47}$$

Under an arbitrary metric variation δg^{ab}:

$$\delta S_{\rm EH} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,G_{ab}\delta g^{ab} + \frac{1}{2\kappa}\oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,n_c v^c, \tag{15.48}$$

where the boundary vector field $v^c \equiv g^{ab}\delta\Gamma^c_{ab} - g^{cb}\delta\Gamma^a_{ab}$ contains normal derivatives of the metric variation $\partial_n(\delta g_{ab})$.

To have a well-posed Dirichlet boundary problem, we must specify only the induced metric on the boundary, $\delta g_{ab}|_{\partial\mathcal{M}} = 0$, without constraining the normal derivatives $\partial_n(\delta g_{ab})|_{\partial\mathcal{M}}$.

The GHY / Brown–York Boundary Term: To eliminate the unwanted $\partial_n(\delta g_{ab})$ boundary variations, York (1972) and Gibbons & Hawking (1977) introduced the surface boundary action:

$$S_{\rm GHY} = \frac{\epsilon}{\kappa}\oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,K, \tag{15.49}$$

where $\epsilon = n_a n^a = \pm 1$ is the normal signature and $K = h^{ab}\nabla_a n_b$ is the trace of the extrinsic curvature of $\partial\mathcal{M}$.

Postulate / Principle: The Complete Well-Posed Gravitational Action with GHY Boundary Term

The complete, mathematically well-posed gravitational action principle for a bounded spacetime manifold $(\mathcal{M}, \partial\mathcal{M})$ is:

$$S_{\text{grav}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,R + \frac{\epsilon}{\kappa}\oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,(K - K_0), \tag{15.50}$$

where K_0 is a reference extrinsic curvature for flat spacetime subtraction. Varying S_{grav} with Dirichlet boundary conditions $\delta g_{ab}|_{\partial\mathcal{M}} = 0$ yields:

$$\delta S_{\text{grav}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,G_{ab}\delta g^{ab} + 0, \tag{15.51}$$

with the boundary variation of normal derivatives canceled identically.

15.5 Conceptual Synthesis and Inquiries

Foundational Question: Diffeomorphism Invariance vs. Noetherian Conserved Quantities

Is diffeomorphism invariance a physical symmetry in the Noether sense that generates a conserved matter current?

- **Global vs. Local (Gauge) Symmetries:** Global symmetries (e.g., $U(1)$ global phase rotation in field theory, or global time translation in Newtonian mechanics) relate distinct physical configurations and generate non-trivial physical conservation laws via **Noether's First Theorem**.
- **Local Gauge Redundancies and Noether's Second Theorem:** Diffeomorphism invariance states that spacetime coordinates x^μ are purely arbitrary mathematical labels. A diffeomorphism maps a physical field configuration to an *identical, gauge-equivalent* physical state. By Noether's Second Theorem, local continuous gauge symmetries do not produce conserved currents on-shell; instead, they generate **off-shell mathematical identities** (the contracted Bianchi identity $\nabla_\mu G^{\mu\nu} \equiv 0$).
- **Physical Conserved Charges:** True conserved charges in general relativity require global spacetime isometries generated by Killing vector fields ξ^μ.

Foundational Question: Boundary Terms and Ghost Degrees of Freedom

Does the boundary term S_{GHY} eliminate ghost degrees of freedom? Yes. The second derivatives $\partial^2 g_{ab}$ in the Ricci scalar R appear purely as a total divergence:

$$\sqrt{-g}\,R = \sqrt{-g}\,\mathcal{L}_{\text{first-order}}(g, \partial g) + \partial_\mu \mathcal{V}^\mu(g, \partial g). \tag{15.52}$$

The boundary term S_{GHY} precisely subtracts the surface integral arising from $\partial_\mu\mathcal{V}^\mu$, reducing the physical dynamical content to a strictly first-order canonical Hamiltonian system with 2 physical propagating tensor polarization states (h_+, $h_\times$), completely removing potential Ostrogradsky ghost modes.

3. Parallel Transport Path-Dependence vs. Einstein Equation Integration: Why does parallel transport suffer from non-unique path dependence, while integrating the Einstein field equations produces a well-posed spacetime geometry?

- **Path-Dependence in Parallel Transport:** Parallel transport moves an individual tangent vector V^a along 1D curves within the tangent bundle. Because the connection has non-zero curvature $R^a{}_{bcd} \neq 0$, the holonomy along different curves produces different vectors: the tangent space rotates along the journey.
- **Hyperbolic Initial Value Formulation of GR:** The Einstein field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ are not 1D curve equations; they form a closed system of non-linear hyperbolic partial differential equations on the 4D manifold. By the **Choquet-Bruhat–Geroch Theorem**, given initial Cauchy data (Σ, h_{ab}, K_{ab}) satisfying the Hamiltonian and momentum constraint equations, there exists a unique (up to diffeomorphism) maximal Cauchy development $(\mathcal{M}, g_{ab})$. Thus, the evolution of spacetime geometry itself is completely deterministic and well-posed.

Chapter 16

Variational Derivations of Einstein's Equations: Metric, Palatini, and Lovelock Formalisms

16.1 The Gravitational Action and Variational Paradigms

16.1.1 The Einstein–Hilbert Lagrangian with Cosmological Constant

In General Relativity, the gravitational field is governed by the action:

$$S_g = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}_g \left(g_{ab}, R^a{}_{bcd}\right), \tag{16.1}$$

where the standard Einstein–Hilbert Lagrangian density with a cosmological constant Λ is:

$$\begin{aligned} \mathcal{L}_g &= R - 2\Lambda = g^{ab} R_{ab} - 2\Lambda \\ &= g^{ab} R^c{}_{acb} - 2\Lambda = g^{ab} g^{cd} R_{acbd} - 2\Lambda. \end{aligned} \tag{16.2}$$

16.1.2 Three Distinct Variational Formulations of General Relativity

Depending on the choice of independent dynamical variables, the variational principle for General Relativity can be formulated in three distinct ways:

(i) Hilbert's Metric Formulation: The metric tensor $g_{ab}(x)$ is chosen as the sole independent field variable. The affine connection $\Gamma^a_{bc}(g, \partial g)$ is fixed *a priori* to be the metric-compatible, torsion-free Levi-Civita connection. This is a second-order variational principle requiring the Gibbons–Hawking–York (GHY) boundary term to cancel metric second derivatives at the boundary.

(ii) Palatini (First-Order) Formalism: The metric $g_{ab}(x)$ and the affine connection $\Gamma^a_{bc}(x)$ are treated as **independent variables**. The Lagrangian $\mathcal{L}(g, \Gamma)$ contains only first derivatives of Γ. Varying with respect to Γ^a_{bc} dynamically enforces metric compatibility ($\nabla_c g_{ab} = 0$), while varying with respect to g_{ab} produces Einstein's equations.

(iii) Lanczos–Lovelock / Padmanabhan Algebraic Curvature Formulation: The inverse metric g^{ab} and the Riemann curvature tensor $R^a{}_{bcd}$ are treated algebraically as independent arguments of the Lagrangian:

$$\mathcal{L} = \mathcal{L}\left(g^{ab}, R^a{}_{bcd}\right). \tag{16.3}$$

The total variation of the gravitational action density decomposes into metric and curvature derivatives:

$$\delta\left(\sqrt{-g}\,\mathcal{L}\right) = \frac{\partial\left(\sqrt{-g}\,\mathcal{L}\right)}{\partial g^{ab}}\, \delta g^{ab} + \frac{\partial\left(\sqrt{-g}\,\mathcal{L}\right)}{\partial R^a{}_{bcd}}\, \delta R^a{}_{bcd}. \tag{16.4}$$

We adopt the algebraic curvature prescription **(iii)**, which unifies the derivation of Einstein's General Relativity with higher-order Lanczos–Lovelock gravity theories.

16.2 The Palatini Identity and the Tensor Nature of Connection Variations

16.2.1 The Variation of the Riemann Tensor

Let us define the auxiliary curvature derivative tensor:

$$P_a{}^{bcd} \equiv \frac{\partial \mathcal{L}}{\partial R^a{}_{bcd}}. \tag{16.5}$$

Since the metric volume element $\sqrt{-g}$ has no explicit dependence on the Riemann tensor ($\frac{\partial \sqrt{-g}}{\partial R^a{}_{bcd}} = 0$), the second term in Equation (16.4) becomes:

$$\frac{\partial \left(\sqrt{-g}\,\mathcal{L}\right)}{\partial R^a{}_{bcd}} = \sqrt{-g}\,\frac{\partial \mathcal{L}}{\partial R^a{}_{bcd}} = \sqrt{-g}\,P_a{}^{bcd}. \tag{16.6}$$

Thus:

$$\delta\left(\sqrt{-g}\,\mathcal{L}\right) = \frac{\partial \left(\sqrt{-g}\,\mathcal{L}\right)}{\partial g^{ab}}\,\delta g^{ab} + \sqrt{-g}\,P_a{}^{bcd}\,\delta R^a{}_{bcd}. \tag{16.7}$$

To evaluate $\delta R^a{}_{bcd}$, we must express it in terms of variations of the metric and connection.

Theorem: The Palatini Identity and Tensor Nature of Connection Variations

Let Γ^a_{bc} and $\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta\Gamma^a_{bc}$ be two arbitrary torsion-free affine connections on a smooth manifold $\mathcal{M}$.

1. The difference $\delta\Gamma^a_{bc}$ transforms as a **genuine** $(1,2)$ **tensor field**.
2. The variation of the Riemann curvature tensor is given universally by the **Palatini Identity**:
$$\delta R^a{}_{bcd} = \nabla_c\left(\delta\Gamma^a_{db}\right) - \nabla_d\left(\delta\Gamma^a_{cb}\right). \tag{16.8}$$

Proof. **Step 1: Proof that $\delta\Gamma^a_{bc}$ is a True Tensor:**

Recall the inhomogeneous coordinate transformation law for the Christoffel connection under $x \to x'$:

$$\Gamma'^i_{k\ell} = \frac{\partial x'^i}{\partial x^a}\frac{\partial x^b}{\partial x'^k}\frac{\partial x^c}{\partial x'^\ell}\Gamma^a_{bc} + \frac{\partial^2 x^m}{\partial x'^k \partial x'^\ell}\frac{\partial x'^i}{\partial x^m}. \tag{16.9}$$

Now consider a perturbed connection $\Gamma^a_{bc} + \delta\Gamma^a_{bc}$. In the primed coordinate chart:

$$\Gamma'^i_{k\ell} + \delta\Gamma'^i_{k\ell} = \frac{\partial x'^i}{\partial x^a}\frac{\partial x^b}{\partial x'^k}\frac{\partial x^c}{\partial x'^\ell}\left(\Gamma^a_{bc} + \delta\Gamma^a_{bc}\right) + \frac{\partial^2 x^m}{\partial x'^k \partial x'^\ell}\frac{\partial x'^i}{\partial x^m}. \tag{16.10}$$

Subtracting Equation (16.9) from Equation (16.10), the non-tensorial inhomogeneous second-derivative term cancels identically:

$$\delta\Gamma'^i_{k\ell} = \frac{\partial x'^i}{\partial x^a}\frac{\partial x^b}{\partial x'^k}\frac{\partial x^c}{\partial x'^\ell}\,\delta\Gamma^a_{bc}. \tag{16.11}$$

Therefore, while the connection Γ^a_{bc} is not a tensor, the variation $\delta\Gamma^a_{bc}$ transforms as a true rank-$(1,2)$ tensor.

Step 2: Proof of the Palatini Identity:

In a Local Inertial Frame (LIF) centered at an arbitrary event P, the background connection vanishes: $\Gamma^a_{bc}|_P = 0$. The Riemann tensor definition reduces at P to:

$$R^a{}_{bcd}|_P = (\partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb})_P \,. \tag{16.12}$$

Taking the variation:

$$\delta R^a{}_{bcd}|_P = \partial_c \left(\delta\Gamma^a_{db}\right) - \partial_d \left(\delta\Gamma^a_{cb}\right). \tag{16.13}$$

Because $\delta\Gamma^a_{db}$ is a tensor, the partial derivatives ∂_c at the origin of a Local Inertial Frame coincide with covariant derivatives ∇_c:

$$\delta R^a{}_{bcd}|_P = \left[\nabla_c \left(\delta\Gamma^a_{db}\right) - \nabla_d \left(\delta\Gamma^a_{cb}\right)\right]_P . \tag{16.14}$$

Since this is an equality between genuine tensor fields valid at an arbitrary point P, it holds universally in all coordinate charts:

$$\delta R^a{}_{bcd} = \nabla_c \left(\delta\Gamma^a_{db}\right) - \nabla_d \left(\delta\Gamma^a_{cb}\right). \tag{16.15}$$

This establishes the Palatini Identity. □

16.2.2 Connection Variation in Terms of Metric Variations

For a Levi-Civita connection, the Christoffel symbol is:

$$\Gamma^a_{db} = \frac{1}{2} g^{ai} \left(\partial_d g_{ib} + \partial_b g_{di} - \partial_i g_{db}\right). \tag{16.16}$$

Evaluating the variation in a Local Inertial Frame where $\partial_c g_{ab}|_P = 0$:

$$\delta\Gamma^a_{db}|_P = \frac{1}{2} g^{ai} \left(\partial_d \delta g_{ib} + \partial_b \delta g_{di} - \partial_i \delta g_{db}\right). \tag{16.17}$$

Promoting partial derivatives to covariant derivatives yields the exact tensorial expression:

$$\delta\Gamma^a_{db} = \frac{1}{2} g^{ai} \left(\nabla_d \delta g_{ib} + \nabla_b \delta g_{di} - \nabla_i \delta g_{db}\right). \tag{16.18}$$

Substituting Equation (16.18) into the Palatini Identity (16.8):

$$\delta R^a{}_{bcd} = \frac{1}{2} \nabla_c \left[g^{ai} \left(\nabla_d \delta g_{ib} + \nabla_b \delta g_{di} - \nabla_i \delta g_{db}\right)\right] - (c \leftrightarrow d). \tag{16.19}$$

16.3 The General Field Equations for Curvature-Dependent Actions

16.3.1 Contracting the Curvature Variation with $P_a{}^{bcd}$

Let us now contract the curvature derivative tensor $P_a{}^{bcd}$ with $\delta R^a{}_{bcd}$. Using metric compatibility ($\nabla_c g^{ai} = 0$) and raising the first index $P^{ibcd} \equiv g^{ai} P_a{}^{bcd}$:

$$P_a{}^{bcd}\, \delta R^a{}_{bcd} = \frac{1}{2} \left\{ P^{ibcd} \nabla_c \nabla_d \delta g_{ib} + P^{ibcd} \nabla_c \nabla_b \delta g_{di} - P^{ibcd} \nabla_c \nabla_i \delta g_{db} \right\} - (c \leftrightarrow d). \tag{16.20}$$

Algebraic Simplification via Symmetries of P^{ibcd}: Recall that $P^{ibcd} = \frac{\partial \mathcal{L}}{\partial R_{ibcd}}$ inherits the index symmetries of the Riemann tensor:

$$P^{ibcd} = -P^{bicd} = -P^{ibdc} = P^{cdib}. \tag{16.21}$$

Let us analyze the three terms in Equation (16.20):

1. **First Term ($P^{ibcd}\nabla_c\nabla_d\delta g_{ib}$):** P^{ibcd} is antisymmetric under $i \leftrightarrow b$, whereas the metric variation $\delta g_{ib} = \delta g_{bi}$ is symmetric. Therefore:

$$P^{ibcd}\nabla_c\nabla_d\delta g_{ib} = 0. \tag{16.22}$$

2. **Third Term ($-P^{ibcd}\nabla_c\nabla_i\delta g_{db}$):** Relabeling the dummy indices $i \leftrightarrow b$:

$$-P^{ibcd}\nabla_c\nabla_i\delta g_{db} \xrightarrow{i\leftrightarrow b} -P^{bicd}\nabla_c\nabla_b\delta g_{di}. \tag{16.23}$$

Using the antisymmetry $P^{bicd} = -P^{ibcd}$:

$$-P^{bicd}\nabla_c\nabla_b\delta g_{di} = +P^{ibcd}\nabla_c\nabla_b\delta g_{di}. \tag{16.24}$$

This is identical to the second term!

Combining these terms, Equation (16.20) simplifies to:

$$P_a{}^{bcd}\,\delta R^a{}_{bcd} = P^{ibcd}\nabla_c\nabla_b\delta g_{di} - (c \leftrightarrow d) = 2P^{ibcd}\nabla_c\nabla_b\delta g_{di}. \tag{16.25}$$

16.3.2 Double Integration by Parts and Boundary Currents

To isolate the un-derivated variation δg_{di}, we integrate by parts twice:

$$\begin{aligned} 2P^{ibcd}\nabla_c\nabla_b\delta g_{di} &= 2\nabla_c\left(P^{ibcd}\nabla_b\delta g_{di}\right) - 2\left(\nabla_c P^{ibcd}\right)\nabla_b\delta g_{di} \\ &= 2\nabla_c\left(P^{ibcd}\nabla_b\delta g_{di}\right) - 2\nabla_b\left(\delta g_{di}\nabla_c P^{ibcd}\right) + 2\,\delta g_{di}\left(\nabla_b\nabla_c P^{ibcd}\right) \\ &= 2\nabla_c\left\{P^{ibcd}\nabla_b\delta g_{di} - \delta g_{di}\nabla_b P^{icbd}\right\} - 2\,\delta g^{di}\left(\nabla^b\nabla^c P_{ibcd}\right). \end{aligned} \tag{16.26}$$

Relabeling indices on the bulk term $i \to a$ and $d \to b$:

$$-2\,\delta g^{di}\left(\nabla^b\nabla^c P_{ibcd}\right) = -2\,\delta g^{ab}\left(\nabla^m\nabla^n P_{amnb}\right). \tag{16.27}$$

Defining the boundary current vector field:

$$\delta v^c \equiv 2\left(P^{ibcd}\nabla_b\delta g_{di} - \delta g_{di}\nabla_b P^{icbd}\right), \tag{16.28}$$

the total action variation becomes:

$$\delta\left(\sqrt{-g}\,\mathcal{L}\right) = \sqrt{-g}\left[\frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\,\mathcal{L}\right)}{\partial g^{ab}} - 2\nabla^m\nabla^n P_{amnb}\right]\delta g^{ab} + \sqrt{-g}\,\nabla_c\delta v^c. \tag{16.29}$$

Definition: Euler–Lagrange Equations for Curvature-Dependent Theories

For any diffeomorphism-invariant action $S = \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}(g^{ab}, R^a{}_{bcd})$, the dynamical equations of motion with respect to the metric g^{ab} are:

$$\mathcal{E}_{ab} \equiv \frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\,\mathcal{L}\right)}{\partial g^{ab}} - 2\nabla^m\nabla^n P_{amnb} = 0, \tag{16.30}$$

where $P_{amnb} \equiv \frac{\partial \mathcal{L}}{\partial R^{amnb}}$.

16.4 Specialization to the Einstein–Hilbert Action

16.4.1 Metric Variation of the Volume Element and Jacobi's Formula

Let us define the algebraic tensor:

$$E_{ab} \equiv \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\,\mathcal{L}\right)}{\partial g^{ab}} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial g^{ab}}\,\mathcal{L} + \frac{\partial \mathcal{L}}{\partial g^{ab}}. \tag{16.31}$$

Recall Jacobi's formula for the variation of the metric determinant:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{mn}\, \delta g^{mn}. \tag{16.32}$$

Taking the functional derivative with respect to the symmetric variable g^{ab}:

$$\frac{\partial \sqrt{-g}}{\partial g^{ab}} = -\frac{1}{2}\sqrt{-g}\, g_{mn} \left[\frac{1}{2}\left(\delta^m_a \delta^n_b + \delta^m_b \delta^n_a\right)\right] = -\frac{1}{2}\sqrt{-g}\, g_{ab}. \tag{16.33}$$

Substituting into Equation (16.31):

$$E_{ab} = -\frac{1}{2} g_{ab}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial g^{ab}}. \tag{16.34}$$

16.4.2 Evaluating $\partial\mathcal{L}/\partial g^{ab}$ for the Ricci Scalar

Consider the Einstein–Hilbert Lagrangian $\mathcal{L} = R - 2\Lambda$. We write the Ricci scalar R in fully covariant form treating the Riemann tensor $R_{ikm\ell}$ as algebraically independent of the metric factors raising indices:

$$R = g^{k\ell} g^{mi} R_{ikm\ell}. \tag{16.35}$$

Taking account of the antisymmetry $R_{ikm\ell} = -R_{kim\ell} = -R_{ik\ell m}$, we express R symmetrically as:

$$R = \frac{1}{2}\left(g^{k\ell}g^{mi} - g^{km}g^{\ell i}\right) R_{ikm\ell}. \tag{16.36}$$

Differentiating with respect to g^{ab}:

$$\begin{aligned}
\frac{\partial R}{\partial g^{ab}} &= \frac{1}{2}\frac{\partial}{\partial g^{ab}}\left[g^{k\ell}g^{mi} - g^{km}g^{\ell i}\right] R_{ikm\ell} \\
&= \frac{1}{2}\left[\delta^k_a \delta^\ell_b g^{mi} + g^{k\ell}\delta^m_a \delta^i_b - \delta^k_a \delta^m_b g^{\ell i} - g^{km}\delta^\ell_a \delta^i_b\right] R_{ikm\ell} \\
&= \frac{1}{2}\left[g^{mi}R_{iamb} + g^{k\ell}R_{bkam} - g^{\ell i}R_{iab\ell} - g^{km}R_{bkma}\right] \\
&= \frac{1}{2}\left[R_{ab} + R_{ba} + R_{ab} + R_{ba}\right] = 2R_{ab}\cdot\frac{1}{2} = R_{ab}.
\end{aligned} \tag{16.37}$$

Therefore, the first term E_{ab} evaluates exactly to:

$$E_{ab} = -\frac{1}{2}g_{ab}(R - 2\Lambda) + R_{ab} = R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = G_{ab} + \Lambda g_{ab}. \tag{16.38}$$

16.4.3 Evaluation of P_{abcd} and the Vanishing of Higher Derivatives

Now let us evaluate the curvature derivative tensor $P_{abcd} \equiv \frac{\partial \mathcal{L}}{\partial R^{abcd}}$ for $\mathcal{L} = R$:

$$R = g^{ac} g^{bd} R_{abcd} = \frac{1}{2} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) R^{abcd}. \tag{16.39}$$

Differentiating with respect to R^{abcd}:

$$P_{abcd} \equiv \frac{\partial R}{\partial R^{abcd}} = \frac{1}{2} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right). \tag{16.40}$$

Covariant Divergence of P_{abcd}: Notice that P_{abcd} in Equation (16.40) is constructed **entirely from metric tensors** g_{ab}. By metric compatibility ($\nabla_m g_{ab} \equiv 0$):

$$\nabla_m P_{abcd} = \frac{1}{2} \left(g_{ac} \nabla_m g_{bd} + g_{bd} \nabla_m g_{ac} - g_{ad} \nabla_m g_{bc} - g_{bc} \nabla_m g_{ad} \right) \equiv 0. \tag{16.41}$$

Consequently, the higher-derivative term in Equation (16.30) vanishes identically:

$$\nabla^m \nabla^n P_{amnb} \equiv 0. \tag{16.42}$$

16.4.4 Recovery of Einstein's Field Equations

Substituting Equations (16.38) and (16.42) into the master field equation (16.30):

$$\mathcal{E}_{ab} = E_{ab} - 2 \nabla^m \nabla^n P_{amnb} = \left(R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} \right) - 0 = 0. \tag{16.43}$$

Coupling to matter with action S_{matter} where $T_{ab} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{ab}}$:

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{16.44}$$

This rigorously completes the variational derivation of Einstein's Field Equations.

Chapter 17

Stress-Energy Tensors, Diffeomorphism Invariance, and Matter Coupling in General Relativity

17.1 The Total Action and the Hilbert Stress-Energy Tensor

17.1.1 Coupling Gravity to Matter Fields

In the preceding chapter, we derived the vacuum Einstein equations from the gravitational action S_g. To describe the dynamical interaction between spacetime geometry and physical matter, we formulate the total action:

$$S_{\text{total}}[g, \Phi] = S_g[g] + S_m[g, \Phi], \tag{17.1}$$

where the Einstein–Hilbert action and matter action are given respectively by

$$S_g = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, (R - 2\Lambda), \tag{17.2}$$

$$S_m = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}_m(g_{ab}, \Phi, \nabla\Phi). \tag{17.3}$$

Under an arbitrary variation of the inverse metric δg^{ab} with Dirichlet boundary conditions ($\delta g^{ab}\big|_{\partial\mathcal{M}} = 0$):

$$\delta S_{\text{total}} = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} \left(R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} \right) \delta g^{ab} + \int_{\mathcal{M}} \mathrm{d}^4x \frac{\partial \left(\sqrt{-g}\, \mathcal{L}_m \right)}{\partial g^{ab}}\, \delta g^{ab}. \tag{17.4}$$

17.1.2 Definition of the Hilbert (Dynamical) Stress-Energy Tensor

Definition: The Hilbert (Dynamical) Stress-Energy Tensor

The **Hilbert (dynamical) stress-energy-momentum tensor** T_{ab} is defined as the functional derivative of the matter action with respect to the spacetime metric:

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{ab}} = -\frac{2}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\, \mathcal{L}_m \right)}{\partial g^{ab}}. \tag{17.5}$$

Equivalently, expanding $\sqrt{-g}$:

$$T_{ab} = -2\frac{\partial \mathcal{L}_m}{\partial g^{ab}} + g_{ab}\mathcal{L}_m. \tag{17.6}$$

Fundamental Properties of T_{ab}:

1. **Symmetry:** Because g^{ab} is symmetric ($g^{ab} = g^{ba}$), T_{ab} is automatically symmetric:

$$T_{ab} = T_{ba}. \tag{17.7}$$

   Unlike the canonical Noether stress tensor of flat space (which is generally asymmetric and requires the Belinfante–Rosenfeld symmetrization procedure), the Hilbert tensor is intrinsically symmetric by construction.

2. **Tensorial Nature:** T_{ab} transforms as a genuine rank-$(0,2)$ spacetime tensor field under arbitrary coordinate diffeomorphisms.

3. **Source of Gravitational Curvature:** Demanding stationary action $\delta S_{\text{total}} = 0$ yields the full Einstein Field Equations:

$$\frac{1}{16\pi G}\left(R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab}\right) - \frac{1}{2}T_{ab} = 0 \implies G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{17.8}$$

17.2 Diffeomorphism Invariance and Stress-Energy Conservation

17.2.1 Off-Shell Bianchi Identity vs. On-Shell Matter Conservation

By the contracted Bianchi identity, $\nabla^a G_{ab} \equiv 0$ holds identically for any smooth metric. Taking the covariant divergence of Einstein's equations (17.8) immediately yields:

$$\nabla^a T_{ab} = 0. \tag{17.9}$$

This raises a fundamental question: *Is $\nabla^a T_{ab} = 0$ an accident of Einstein's equations, or does it hold independently for any diffeomorphism-invariant matter theory?*

Theorem: Diffeomorphism Invariance and Local Stress-Energy Conservation

Let $S_m[g, \Phi]$ be any matter action that is invariant under spacetime diffeomorphisms. When the matter fields Φ satisfy their Euler–Lagrange equations of motion ($\frac{\delta S_m}{\delta \Phi} = 0$), the Hilbert stress-energy tensor is **identically divergence-free on-shell**:

$$\nabla^a T_{ab} = 0, \tag{17.10}$$

regardless of whether the metric g_{ab} satisfies Einstein's field equations.

Proof. Consider an infinitesimal active diffeomorphism generated by a smooth vector field $\xi^a(x)$:

$$x^a \to x'^a = x^a + \xi^a(x). \tag{17.11}$$

Under this transformation, the variations of the metric and matter fields are given by their respective Lie derivatives:

$$\delta_\xi g^{ab} = \mathcal{L}_\xi g^{ab} = -\left(\nabla^a \xi^b + \nabla^b \xi^a\right) = -2\nabla^{(a}\xi^{b)}, \tag{17.12}$$

$$\delta_\xi \Phi = \mathcal{L}_\xi \Phi = \xi^a \nabla_a \Phi. \tag{17.13}$$

Method 1: Variation via the Stress-Energy Tensor Definition: Because the matter fields Φ satisfy their equations of motion on-shell ($\frac{\delta S_m}{\delta \Phi} = 0$), the variation of S_m arises entirely from the metric variation:

$$\begin{aligned}\delta_\xi S_m &= \int_{\mathcal{M}} \mathrm{d}^4x \frac{\partial\left(\sqrt{-g}\,\mathcal{L}_m\right)}{\partial g^{ab}}\,\delta_\xi g^{ab} \\ &= -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,T_{ab}\left[-2\nabla^{(a}\xi^{b)}\right] \\ &= \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,T_{ab}\nabla^a\xi^b. \end{aligned} \tag{17.14}$$

Using the Leibniz rule $\nabla^a(T_{ab}\xi^b) = (\nabla^a T_{ab})\xi^b + T_{ab}\nabla^a\xi^b$:

$$\delta_\xi S_m = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\nabla_a\left(T^a{}_b\xi^b\right) - \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\xi^b\left(\nabla_a T^a{}_b\right). \tag{17.15}$$

The first term is a total spacetime divergence. By Gauss's divergence theorem:

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\nabla_a\left(T^a{}_b\xi^b\right) = \oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,n_a T^a{}_b\xi^b. \tag{17.16}$$

Choosing the test vector field ξ^a to have compact support inside $\mathcal{M}$ (i.e., $\xi^a|_{\partial\mathcal{M}} = 0$ on the timelike and spacelike boundaries), the boundary integral vanishes:

$$\delta_\xi S_m = -\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\xi^b\left(\nabla_a T^a{}_b\right). \tag{17.17}$$

Method 2: Diffeomorphism Invariance of the Scalar Density: On the other hand, the Lagrangian density $\sqrt{-g}\,\mathcal{L}_m$ is a scalar density of weight $+1$. Its Lie derivative under ξ is a pure total divergence:

$$\begin{aligned}\mathcal{L}_\xi\left(\sqrt{-g}\,\mathcal{L}_m\right) &= \left(\mathcal{L}_\xi\sqrt{-g}\right)\mathcal{L}_m + \sqrt{-g}\left(\mathcal{L}_\xi\mathcal{L}_m\right) \\ &= \left(\sqrt{-g}\,\nabla_a\xi^a\right)\mathcal{L}_m + \sqrt{-g}\left(\xi^a\nabla_a\mathcal{L}_m\right) \\ &= \sqrt{-g}\,\nabla_a\left(\xi^a\mathcal{L}_m\right) = \partial_a\left(\sqrt{-g}\,\xi^a\mathcal{L}_m\right). \end{aligned} \tag{17.18}$$

Integrating over $\mathcal{M}$ with $\xi^a|_{\partial\mathcal{M}} = 0$:

$$\delta_\xi S_m = \int_{\mathcal{M}} \mathrm{d}^4x\,\partial_a\left(\sqrt{-g}\,\xi^a\mathcal{L}_m\right) = \oint_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,n_a\xi^a\mathcal{L}_m \equiv 0. \tag{17.19}$$

Equating the Variations: Equating Equation (17.17) and Equation (17.19):

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\xi^b\left(\nabla_a T^a{}_b\right) = 0. \tag{17.20}$$

Since $\xi^b(x)$ is an entirely arbitrary smooth vector field, the integrand must vanish identically:

$$\nabla_a T^a{}_b = 0 \iff \nabla_a T^{ab} = 0. \tag{17.21}$$

This completes the proof. □

Physical Insight: Equations of Motion Encoded in Stress-Energy: This theorem reveals a profound property of General Relativity: the equations of motion for matter fields are not independent postulates; they are **algebraically and geometrically encoded** in the vanishing covariant divergence of the stress-energy tensor ($\nabla_\mu T^{\mu\nu} = 0$).

17.3 Commutator Identities on Higher-Rank Antisymmetric Tensors

Let U^{abc} be a rank-3 tensor that is completely antisymmetric in its first two indices ($U^{abc} = -U^{bac}$). Let us compute the double covariant divergence $\nabla_a \nabla_b U^{abc}$.

Proof. Because U^{abc} is antisymmetric in $a \leftrightarrow b$, we can rewrite the double derivative as half of a commutator:

$$\nabla_a \nabla_b U^{abc} = \frac{1}{2} \left(\nabla_a \nabla_b - \nabla_b \nabla_a \right) U^{abc} = \frac{1}{2} [\nabla_a, \nabla_b] U^{abc}. \tag{17.22}$$

Applying the Ricci commutator identity for each index of U^{abc}:

$$[\nabla_a, \nabla_b] U^{abc} = R^a{}_{iab} U^{ibc} + R^b{}_{iab} U^{aic} + R^c{}_{iab} U^{abi}. \tag{17.23}$$

Let us evaluate each term:

1. First term: $R^a{}_{iab} U^{ibc} = -R_{ia} U^{ibc} = -R_{ib} U^{ibc}$.
2. Second term: $R^b{}_{iab} U^{aic} = R_{ia} U^{aic} = R_{ib} U^{bic} = -R_{ib} U^{ibc}$. These two terms cancel against the antisymmetry of U^{ibc}:
$$R^a{}_{iab} U^{ibc} + R^b{}_{iab} U^{aic} = -R_{ib} U^{ibc} + R_{ib} U^{ibc} = 0. \tag{17.24}$$
3. Third term: $R^c{}_{iab} U^{abi}$.

By the First Bianchi Identity ($R^c{}_{[iab]} = 0$), contracting with the antisymmetric tensor U^{abi} vanishes identically:

$$R^c{}_{iab} U^{abi} = 0. \tag{17.25}$$

Therefore:

$$\nabla_a \nabla_b U^{abc} \equiv 0. \tag{17.26}$$

□

17.4 Explicit Derivations of Stress-Energy Tensors

17.4.1 Real Scalar Field with Self-Interaction Potential $V(\Phi)$

Consider a minimally coupled real scalar field $\Phi(x)$ with potential $V(\Phi)$:

$$\begin{aligned} S_\Phi &= \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{2} \nabla_a \Phi \nabla^a \Phi - V(\Phi) \right] \\ &= \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi - V(\Phi) \right]. \end{aligned} \tag{17.27}$$

Key Concept: Stress-Energy Tensor of a Scalar Field

To compute $T^{(\Phi)}_{ab}$, we vary S_Φ with respect to the inverse metric g^{ab} while keeping Φ fixed:

$$\begin{aligned} \delta S_\Phi|_\Phi &= \int \mathrm{d}^4x \left(\delta\sqrt{-g} \right) \left[-\frac{1}{2} \nabla_c \Phi \nabla^c \Phi - V(\Phi) \right] + \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{2} \left(\delta g^{ab} \right) \partial_a \Phi \partial_b \Phi \right] \\ &= \int \mathrm{d}^4x \left(-\frac{1}{2} \sqrt{-g}\, g_{ab} \delta g^{ab} \right) \left[-\frac{1}{2} \nabla_c \Phi \nabla^c \Phi - V(\Phi) \right] - \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g}\, \nabla_a \Phi \nabla_b \Phi\, \delta g^{ab} \end{aligned}$$

$$= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left\{\nabla_a\Phi\nabla_b\Phi - g_{ab}\left[\frac{1}{2}\nabla_c\Phi\nabla^c\Phi + V(\Phi)\right]\right\}\delta g^{ab}. \tag{17.28}$$

Comparing with $\delta S_m = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T_{ab}\delta g^{ab}$, we obtain:

$$T^{(\Phi)}_{ab} = \nabla_a\Phi\nabla_b\Phi - g_{ab}\left[\frac{1}{2}\nabla_c\Phi\nabla^c\Phi + V(\Phi)\right]. \tag{17.29}$$

Conservation Check: Taking the covariant divergence of Equation (17.29):

$$\begin{aligned}\nabla^a T^{(\Phi)}_{ab} &= \nabla^a\left(\nabla_a\Phi\nabla_b\Phi\right) - \nabla_b\left[\frac{1}{2}\nabla_c\Phi\nabla^c\Phi + V(\Phi)\right]\\ &= (\Box\Phi)\nabla_b\Phi + \nabla_a\Phi\nabla^a\nabla_b\Phi - \nabla_c\Phi\nabla_b\nabla^c\Phi - V'(\Phi)\nabla_b\Phi\\ &= \left[\Box\Phi - V'(\Phi)\right]\nabla_b\Phi + \nabla_a\Phi\left(\nabla^a\nabla_b\Phi - \nabla_b\nabla^a\Phi\right).\end{aligned} \tag{17.30}$$

Since covariant derivatives commute on scalar fields ($[\nabla^a, \nabla_b]\Phi = 0$):

$$\nabla^a T^{(\Phi)}_{ab} = \left[\Box\Phi - V'(\Phi)\right]\nabla_b\Phi = 0, \tag{17.31}$$

which vanishes identically when Φ satisfies the **Klein–Gordon equation** $\Box\Phi - V'(\Phi) = 0$.

17.4.2 Electromagnetic Maxwell Field

The covariant action for the Maxwell electromagnetic field in curved spacetime is:

$$S_{\mathrm{EM}} = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\,F_{ab}F^{ab} = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\,g^{am}g^{bn}F_{ab}F_{mn}, \tag{17.32}$$

where $F_{ab} \equiv \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a$ is the gauge-invariant electromagnetic field strength tensor.

Key Concept: Stress-Energy Tensor of the Electromagnetic Field

Varying S_{EM} with respect to g^{ab} at fixed gauge potential A_μ:

$$\begin{aligned}\delta S_{\mathrm{EM}}|_A &= -\frac{1}{16\pi}\int \mathrm{d}^4x\left[\left(\delta\sqrt{-g}\right)g^{am}g^{bn}F_{ab}F_{mn} + \sqrt{-g}\left(\delta g^{am}\right)g^{bn}F_{ab}F_{mn}\right.\\ &\qquad\left. + \sqrt{-g}\,g^{am}\left(\delta g^{bn}\right)F_{ab}F_{mn}\right]\\ &= -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g_{ab}\delta g^{ab}F_{cd}F^{cd} + \delta g^{am}g^{bn}F_{ab}F_{mn} + \delta g^{bn}g^{am}F_{ab}F_{mn}\right].\end{aligned} \tag{17.33}$$

Relabeling indices in the third term ($a \leftrightarrow b$ and $m \leftrightarrow n$) and using the antisymmetry $F_{ba} = -F_{ab}$ and $F_{nm} = -F_{mn}$:

$$\delta g^{bn}g^{am}F_{ab}F_{mn} = \delta g^{am}g^{bn}F_{ba}F_{nm} = \delta g^{am}g^{bn}(-F_{ab})(-F_{mn}) = \delta g^{am}g^{bn}F_{ab}F_{mn}. \tag{17.34}$$

The two metric variation terms combine:

$$\delta S_{\mathrm{EM}}|_A = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g_{ab}\left(F_{cd}F^{cd}\right) + 2g^{mn}F_{am}F_{bn}\right]\delta g^{ab}. \tag{17.35}$$

Factoring out $-\frac{1}{2}$:

$$\delta S_{\mathrm{EM}}|_A = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{4\pi}\left[F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right]\right\}\delta g^{ab}. \tag{17.36}$$

Comparing with $\delta S_m = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T_{ab}\delta g^{ab}$, we obtain the standard **Maxwell Stress-Energy Tensor**:

$$T^{(\mathrm{EM})}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right). \tag{17.37}$$

Tracelessness of Electromagnetism in 4 Dimensions: Let us evaluate the trace $T \equiv g^{ab}T^{(\mathrm{EM})}_{ab}$ in $n = 4$ spacetime dimensions:

$$g^{ab}T^{(\mathrm{EM})}_{ab} = \frac{1}{4\pi}\left(g^{ab}F_{ac}F_b{}^c - \frac{1}{4}(g^{ab}g_{ab})F_{cd}F^{cd}\right) = \frac{1}{4\pi}\left(F_{bc}F^{bc} - \frac{4}{4}F_{cd}F^{cd}\right) \equiv 0. \tag{17.38}$$

The vanishing trace $T = 0$ reflects the exact **classical conformal invariance** of Maxwell's electrodynamics in four spacetime dimensions.

Chapter 18

Stress-Energy of Fluids, the Newtonian Limit, Degrees of Freedom, and Linearized Gravity

18.1 Stress-Energy of Relativistic Particles and Continuous Fluids

18.1.1 Point Particle Action and Worldline Density

In Chapter 17, we established the Hilbert stress-energy tensor for continuous fields. Let us now compute the stress-energy tensor for a point mass m following a timelike trajectory $z^\mu(\tau)$ in curved spacetime.

The standard reparametrization-invariant action for a free massive point particle is proportional to its proper time:

$$S_p[z, g] = -m \int \mathrm{d}\tau \sqrt{-g_{\mu\nu}(z(\tau))\, \dot{z}^\mu \dot{z}^\nu}, \tag{18.1}$$

where $\dot{z}^\mu = \frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}$ and τ is an arbitrary worldline parameter.

Under an arbitrary variation of the metric $g_{\mu\nu}(x)$, the variation of the particle action evaluates to:

$$\begin{aligned}
\delta S_p &= -m \int \mathrm{d}\tau \frac{-\delta g_{\mu\nu}(z)\, \dot{z}^\mu \dot{z}^\nu}{2\sqrt{-g_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta}} \\
&= \frac{m}{2} \int \mathrm{d}\tau\, \delta g_{\mu\nu}(z)\, \frac{\dot{z}^\mu}{\sqrt{-\dot{z}^2}} \frac{\dot{z}^\nu}{\sqrt{-\dot{z}^2}} \sqrt{-\dot{z}^2} \\
&= \frac{m}{2} \int \mathrm{d}\tau\, \delta g_{\mu\nu}(z(\tau))\, u^\mu(\tau) u^\nu(\tau),
\end{aligned} \tag{18.2}$$

where $u^\mu(\tau) \equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}s} = \frac{\dot{z}^\mu}{\sqrt{-\dot{z}^2}}$ is the unit 4-velocity satisfying $g_{\mu\nu}u^\mu u^\nu = -1$, and $\mathrm{d}s = \sqrt{-\dot{z}^2}\,\mathrm{d}\tau$ is the proper time differential.

Using the metric variation identity $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}$:

$$\delta S_p = -\frac{m}{2} \int \mathrm{d}\tau\, u_\mu(\tau) u_\nu(\tau)\, \delta g^{\mu\nu}(z(\tau)). \tag{18.3}$$

18.1.2 Transition from Worldline to 4D Spacetime Density

The particle variation (18.3) is expressed as a 1-dimensional line integral along the trajectory $z^\mu(\tau)$, whereas the Hilbert definition of $T_{\mu\nu}$ requires a 4-dimensional volume integral over

spacetime $\mathcal{M}$:

$$\delta S_m = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, T_{\mu\nu}(x)\, \delta g^{\mu\nu}(x). \tag{18.4}$$

To equate these two expressions, we insert a 4-dimensional invariant Dirac delta distribution $\delta^{(4)}(x - z(\tau))$:

$$\begin{aligned}\delta S_p &= -\frac{m}{2}\int \mathrm{d}\tau \int_{\mathcal{M}} \mathrm{d}^4x\, \delta^{(4)}(x - z(\tau))\, u_\mu(\tau)u_\nu(\tau)\, \delta g^{\mu\nu}(x) \\ &= -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[m\int \mathrm{d}\tau\, \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}\, u_\mu(\tau)u_\nu(\tau)\right]\delta g^{\mu\nu}(x).\end{aligned} \tag{18.5}$$

Comparing directly with Equation (18.4), the stress-energy tensor of a single point particle is:

$$T_{\mu\nu}(x) = m\int \mathrm{d}\tau\, \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}\, u_\mu(\tau)u_\nu(\tau). \tag{18.6}$$

Definition: Stress-Energy Tensor of Relativistic Dust

For a continuous distribution (a congruence) of non-interacting massive particles (pressureless **dust**) with invariant rest-mass density $\rho(x) \equiv \sum_k m_k \int \mathrm{d}\tau_k \frac{\delta^{(4)}(x - z_k(\tau_k))}{\sqrt{-g}}$, the stress-energy tensor is:

$$T^{(\mathrm{dust})}_{\mu\nu} = \rho(x)\, u_\mu(x)u_\nu(x). \tag{18.7}$$

Key Concept: Geodesic Motion from Local Energy Conservation

Taking the covariant divergence of $T^{\mu\nu}_{(\mathrm{dust})} = \rho u^\mu u^\nu$:

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho u^\mu u^\nu) = \nabla_\mu(\rho u^\mu)\, u^\nu + \rho u^\mu \nabla_\mu u^\nu = 0. \tag{18.8}$$

Projecting parallel and orthogonal to u_ν (using $u_\nu u^\nu = -1 \implies u_\nu \nabla_\mu u^\nu = 0$):

$$u_\nu \nabla_\mu T^{\mu\nu} = 0 \implies \nabla_\mu(\rho u^\mu) = 0 \quad \text{(Continuity / mass conservation)}, \tag{18.9}$$

$$h^\alpha{}_\nu \nabla_\mu T^{\mu\nu} = 0 \implies u^\mu \nabla_\mu u^\alpha = 0 \quad \text{(Geodesic equation of motion)}. \tag{18.10}$$

Thus, local stress-energy conservation $\nabla_\mu T^{\mu\nu} = 0$ automatically enforces that test dust particles travel along spacetime geodesics without requiring any additional kinematic assumptions!

18.2 Irreducible Decomposition of General Relativistic Fluids

18.2.1 Observer Velocity and Hypersurface Projector

For an arbitrary continuous medium with timelike 4-velocity field u^μ normalized such that:

$$u^\mu u_\mu = -1, \tag{18.11}$$

we define the spatial projection tensor (the induced metric on hypersurfaces orthogonal to u^μ):

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu. \tag{18.12}$$

The tensor $h_{\mu\nu}$ satisfies the standard projection properties:

$$h^\mu{}_\alpha h^\alpha{}_\nu = h^\mu{}_\nu, \qquad h_{\mu\nu}u^\nu = 0, \qquad h^\mu{}_\mu = g^{\mu\nu}(g_{\mu\nu} + u_\mu u_\nu) = 4 - 1 = 3. \tag{18.13}$$

18.2.2 Decomposition into Physical Fluid Components

Definition: Irreducible Decomposition of Relativistic Fluids

Any symmetric rank-$(0,2)$ stress-energy tensor $T_{\mu\nu}$ can be uniquely decomposed into components parallel and orthogonal to a timelike flow u^μ:

$$T_{\mu\nu} = \rho\, u_\mu u_\nu + p\, h_{\mu\nu} + 2\, q_{(\mu} u_{\nu)} + \Pi_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\, g_{\mu\nu} + 2\, q_{(\mu} u_{\nu)} + \Pi_{\mu\nu}, \quad (18.14)$$

where $q_{(\mu}u_{\nu)} \equiv \frac{1}{2}(q_\mu u_\nu + q_\nu u_\mu)$.

Physical Meanings of the Decomposed Quantities:

1. **Energy Density ρ:** The total energy density measured by a comoving observer:
$$\rho \equiv T_{\mu\nu} u^\mu u^\nu. \quad (18.15)$$

2. **Isotropic Pressure p:** The average normal spatial stress (pressure):
$$p \equiv \frac{1}{3} T_{\mu\nu} h^{\mu\nu}. \quad (18.16)$$

3. **Energy Flux / Heat Conduction Vector q_μ:** The spatial flux of energy relative to u^μ:
$$q_\mu \equiv -h_\mu{}^\alpha T_{\alpha\beta} u^\beta, \qquad q_\mu u^\mu = 0. \quad (18.17)$$

4. **Anisotropic Shear Stress Tensor $\Pi_{\mu\nu}$:** The trace-free symmetric spatial stress tensor describing viscosity, shear, and anisotropic pressures:
$$\Pi_{\mu\nu} \equiv \left(h_\mu{}^\alpha h_\nu{}^\beta - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta} \right) T_{\alpha\beta}, \quad (18.18)$$
satisfying the orthogonality and tracelessness conditions:
$$\Pi_{\mu\nu} u^\nu = 0, \qquad \Pi^\mu{}_\mu = g^{\mu\nu}\Pi_{\mu\nu} = 0, \qquad \Pi_{\mu\nu} = \Pi_{\nu\mu}. \quad (18.19)$$

Trace of the Stress-Energy Tensor: Taking the metric contraction of Equation (18.14):

$$T \equiv g^{\mu\nu} T_{\mu\nu} = \rho(u^\mu u_\mu) + p(h^\mu{}_\mu) + 2 q_\mu u^\mu + \Pi^\mu{}_\mu = -\rho + 3p. \quad (18.20)$$

Thus, for an ideal fluid ($\Pi_{\mu\nu} = 0, q_\mu = 0$), $T = 3p - \rho$. For dust ($p = 0$), $T = -\rho$. For electromagnetic radiation ($p = \frac{1}{3}\rho$), $T = 0$.

Hypersurface Curvature Projection: Projecting the Einstein tensor along $u^\mu u^\nu$, the contracted Gauss identity yields:

$$^{\mathcal{H}}R = 2 G_{\mu\nu} u^\mu u^\nu = 16\pi G\, T_{\mu\nu} u^\mu u^\nu = 16\pi G\, \rho, \quad (18.21)$$

confirming that the spatial Ricci curvature $^{\mathcal{H}}R$ of a comoving Cauchy slice is determined entirely by the local energy density ρ.

18.3 Cosmological Constant as Dark Energy and Trace-Reversed Einstein Equations

18.3.1 The Cosmological Constant as an Ideal Fluid

The Einstein field equations with a cosmological constant Λ are given by:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{18.22}$$

Moving the cosmological term to the right-hand side, we define an effective dark energy stress-energy tensor $T^{(\Lambda)}_{\mu\nu}$:

$$G_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}\right), \qquad T^{(\Lambda)}_{\mu\nu} \equiv -\frac{\Lambda}{8\pi G}\, g_{\mu\nu}. \tag{18.23}$$

Matching $T^{(\Lambda)}_{\mu\nu}$ with the ideal fluid template $T_{\mu\nu} = (\rho_\Lambda + p_\Lambda)u_\mu u_\nu + p_\Lambda g_{\mu\nu}$:

$$\rho_\Lambda = T^{(\Lambda)}_{\mu\nu}u^\mu u^\nu = -\frac{\Lambda}{8\pi G}(g_{\mu\nu}u^\mu u^\nu) = \frac{\Lambda}{8\pi G}, \tag{18.24}$$

$$p_\Lambda = \frac{1}{3}T^{(\Lambda)}_{\mu\nu}h^{\mu\nu} = -\frac{\Lambda}{24\pi G}(3) = -\frac{\Lambda}{8\pi G} = -\rho_\Lambda. \tag{18.25}$$

Hence, a positive cosmological constant ($\Lambda > 0$) corresponds precisely to an isotropic fluid with positive vacuum energy density $\rho_\Lambda > 0$ and constant negative pressure $p_\Lambda = -\rho_\Lambda$, yielding the equation of state parameter:

$$w \equiv \frac{p_\Lambda}{\rho_\Lambda} = -1. \tag{18.26}$$

18.3.2 Trace-Reversed Form of Einstein's Equations

To express the Ricci tensor $R_{\mu\nu}$ directly in terms of matter sources, we contract Equation (18.22) with $g^{\mu\nu}$:

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right) = 8\pi G\, g^{\mu\nu}T_{\mu\nu} \implies R - \frac{1}{2}(4)R + 4\Lambda = 8\pi G T. \tag{18.27}$$

Simplifying the left-hand side:

$$-R + 4\Lambda = 8\pi G T \implies R = 4\Lambda - 8\pi G T. \tag{18.28}$$

Substituting Equation (18.28) back into Einstein's equations:

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}(4\Lambda - 8\pi G T) - \Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu} \\ &= 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}. \end{aligned} \tag{18.29}$$

Equation (18.29) is the **trace-reversed Einstein Field Equation**. In vacuum ($T_{\mu\nu} = 0$) with $\Lambda = 0$, it simplifies to $R_{\mu\nu} = 0$, demonstrating that vacuum spacetimes are Ricci-flat.

18.4 The Newtonian Limit of General Relativity

We now demonstrate how Newtonian gravitation $\nabla^2\Phi = 4\pi G\rho$ emerges rigorously from the full Einstein field equations (18.29) (with $\Lambda = 0$) in the non-relativistic, weak-field regime.

18.4.1 The Three Physical Postulates of the Newtonian Limit

1. **Slow Motion / Non-Relativistic Velocities:** All material velocities v are negligible compared to the speed of light:
$$\frac{v}{c} \ll 1 \implies \frac{\mathrm{d}x^i}{\mathrm{d}t} \ll c. \tag{18.30}$$
Consequently, the 4-velocity of matter is dominated by its timelike component: $u^\mu \approx (c, 0, 0, 0)$.

2. **Weak Gravitational Field:** The spacetime metric is a small perturbation around flat Minkowski space $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$:
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{18.31}$$
From geodesic motion in a static weak field (derived in Chapter 1), the 00-component of the metric is directly related to the Newtonian gravitational potential Φ:
$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right), \qquad g^{00} \approx -\left(1 - \frac{2\Phi}{c^2}\right). \tag{18.32}$$

3. **Quasi-Static Field Configuration:** Time derivatives of the gravitational field are negligible compared to spatial gradients:
$$\frac{1}{c}\frac{\partial g_{\mu\nu}}{\partial t} \ll \left|\frac{\partial g_{\mu\nu}}{\partial x^i}\right| \implies \partial_0 g_{\mu\nu} \approx 0. \tag{18.33}$$

18.4.2 Computation of the Geometric Curvature R_{00}

In the static weak-field limit, the Christoffel symbols reduce to first order in derivatives of $h_{\mu\nu}$:
$$\Gamma^\alpha_{\mu\nu} \approx \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right). \tag{18.34}$$

The Ricci tensor component R_{00} is given by:
$$\begin{aligned} R_{00} &= \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} \\ &\approx \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{0i} + O(h^2) \\ &\approx \partial_i\left[\frac{1}{2}\eta^{ij}\left(2\partial_0 h_{j0} - \partial_j h_{00}\right)\right] - 0 \\ &= -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} = -\frac{1}{2}\nabla^2 h_{00}, \end{aligned} \tag{18.35}$$
where $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the flat 3-dimensional Laplacian.

Substituting $h_{00} = -\frac{2\Phi}{c^2}$ from Equation (18.32):
$$R_{00} = -\frac{1}{2}\nabla^2\left(-\frac{2\Phi}{c^2}\right) = \frac{1}{c^2}\nabla^2\Phi. \tag{18.36}$$

18.4.3 Evaluation of the Matter Source Term

For non-relativistic matter (dust with density ρ), the components of the stress-energy tensor are:
$$T_{00} = \rho c^2, \qquad T_{0i} \sim \rho c v_i \ll T_{00}, \qquad T_{ij} \sim \rho v_i v_j \ll T_{00}. \tag{18.37}$$

The trace is:
$$T = g^{\mu\nu}T_{\mu\nu} \approx \eta^{00}T_{00} = -T_{00} = -\rho c^2. \tag{18.38}$$

The source combination appearing in the trace-reversed Einstein equation (18.29) is:
$$T_{00} - \frac{1}{2}g_{00}T \approx \rho c^2 - \frac{1}{2}(-1)(-\rho c^2) = \rho c^2 - \frac{1}{2}\rho c^2 = \frac{1}{2}\rho c^2. \tag{18.39}$$

18.4.4 Derivation of Poisson's Equation

Theorem: Newtonian Gravitation as the Weak-Field Limit of GR

Equating the geometric curvature $R_{00} \approx \frac{1}{c^2}\nabla^2\Phi$ and the matter source $T_{00} - \frac{1}{2}g_{00}T \approx \frac{1}{2}\rho c^2$ in the 00-component of the trace-reversed Einstein equations:

$$R_{00} = \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2}g_{00}T\right) \implies \frac{1}{c^2}\nabla^2\Phi = \frac{8\pi G}{c^4}\left(\frac{1}{2}\rho c^2\right). \tag{18.40}$$

Canceling the powers of c^2 on both sides yields the classical **Newtonian Poisson Equation**:

$$\nabla^2\Phi = 4\pi G\rho. \tag{18.41}$$

This rigorously establishes that Newtonian gravitation is the exact non-relativistic, static, weak-field limit of General Relativity, fixing Einstein's gravitational coupling constant to $\kappa = \frac{8\pi G}{c^4}$.

18.5 Degrees of Freedom and Gauge Invariance in General Relativity

A crucial question in classical field theory is determining the number of true, physical propagating degrees of freedom (polarizations) in the gravitational field.

18.5.1 Algebraic vs. Differential Constraints

1. **Metric Components:** In $D = 4$ spacetime dimensions, the symmetric metric tensor $g_{\mu\nu}$ contains $N_{\text{metric}} = \frac{4\times 5}{2} = 10$ independent components.
2. **Field Equations:** Einstein's equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ furnish 10 second-order partial differential equations.
3. **Contracted Bianchi Identities:** The metric satisfies $\nabla_\mu G^\mu{}_\nu \equiv 0$ (4 differential relations), so only $10 - 4 = 6$ equations are dynamically independent.
4. **Diffeomorphism Gauge Freedom:** Spacetime coordinate transformations $x^\mu \to x^\mu + \xi^\mu(x)$ eliminate 4 gauge components.

Key Concept: Physical Degrees of Freedom of the Gravitational Field

In the $3+1$ Cauchy (ADM) formulation of General Relativity:

- The 4 components $G^0{}_\mu = 8\pi G T^0{}_\mu$ ($G^0{}_0$ and $G^0{}_i$) contain no second time derivatives $\partial_0^2 g_{\mu\nu}$. They are **initial-value constraints** (the Hamiltonian and momentum constraints) restricting initial Cauchy data.
- The remaining 6 components $G_{ij} = 8\pi G T_{ij}$ contain genuine second time derivatives $\partial_0^2 g_{ij}$ and represent **dynamical evolution equations**.

Subtracting both gauge redundancies and initial constraints:

$$\begin{aligned} N_{\text{DOF}} &= (\text{Metric}) - (\text{Gauge}) - (\text{Constraints}) \\ &= 10 - 4 - 4 = 2 \quad \text{physical propagating polarizations per point.} \end{aligned} \tag{18.42}$$

These 2 physical modes correspond to the transverse-traceless polarization states of gravitational waves: the **plus mode** (h_+) and the **cross mode** ($h_\times$).

18.6 Linearized Gravity in the Weak-Field Regime

18.6.1 The Linearized Metric and Inverse Perturbation

Consider a spacetime whose geometry is everywhere arbitrarily close to flat Minkowski space:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \qquad |h_{\mu\nu}| \ll 1, \tag{18.43}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the background Minkowski metric.

Inverse Metric to First Order: Let $g^{\mu\nu} = \eta^{\mu\nu} + c\, h^{\mu\nu} + O(h^2)$. Demanding $g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu$:

$$(\eta_{\mu\alpha} + h_{\mu\alpha})(\eta^{\alpha\nu} + c\, h^{\alpha\nu}) = \delta^\nu_\mu + (1 + c)h_\mu{}^\nu + O(h^2) = \delta^\nu_\mu \implies c = -1. \tag{18.44}$$

Therefore:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2), \tag{18.45}$$

where indices on $h_{\mu\nu}$ and all perturbation quantities are raised and lowered using the background Minkowski metric:

$$h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \qquad h^\mu{}_\nu \equiv \eta^{\mu\alpha}h_{\alpha\nu}, \qquad h \equiv \eta^{\mu\nu}h_{\mu\nu}. \tag{18.46}$$

18.6.2 Linearized Curvature Tensors

1. Linearized Christoffel Connection: Substituting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and keeping terms to first order in h:

$$\begin{aligned}
\Gamma^\sigma_{\mu\nu} &= \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) \\
&= \frac{1}{2}\eta^{\sigma\rho}\left(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right) + O(h^2) \\
&= \frac{1}{2}\left(\partial_\mu h^\sigma{}_\nu + \partial_\nu h_\mu{}^\sigma - \partial^\sigma h_{\mu\nu}\right).
\end{aligned} \tag{18.47}$$

2. Linearized Riemann Curvature Tensor: The quadratic connection products $\Gamma\Gamma \sim O(h^2)$ are neglected:

$$\begin{aligned}
R^\sigma{}_{\mu\nu\rho} &= \partial_\nu\Gamma^\sigma_{\mu\rho} - \partial_\rho\Gamma^\sigma_{\mu\nu} + O(h^2) \\
&= \frac{1}{2}\partial_\nu\left(\partial_\mu h^\sigma{}_\rho + \partial_\rho h_\mu{}^\sigma - \partial^\sigma h_{\mu\rho}\right) - \frac{1}{2}\partial_\rho\left(\partial_\mu h^\sigma{}_\nu + \partial_\nu h_\mu{}^\sigma - \partial^\sigma h_{\mu\nu}\right) \\
&= \frac{1}{2}\left(\partial_\nu\partial_\mu h^\sigma{}_\rho + \partial_\rho\partial^\sigma h_{\mu\nu} - \partial_\nu\partial^\sigma h_{\mu\rho} - \partial_\rho\partial_\mu h^\sigma{}_\nu\right).
\end{aligned} \tag{18.48}$$

Lowering the upper index with $\eta_{\alpha\sigma}$:

$$R_{\alpha\mu\nu\rho} = \frac{1}{2}\left(\partial_\nu\partial_\mu h_{\alpha\rho} + \partial_\rho\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\alpha h_{\mu\rho} - \partial_\rho\partial_\mu h_{\alpha\nu}\right). \tag{18.49}$$

3. Linearized Ricci Tensor and Ricci Scalar: Contracting σ with ν in Equation (18.48):

$$\begin{aligned} R_{\mu\rho} \equiv R^{\sigma}{}_{\mu\sigma\rho} &= \frac{1}{2}\left(\partial_\sigma\partial_\mu h^{\sigma}{}_{\rho} + \partial_\rho\partial^{\sigma} h_{\mu\sigma} - \partial_\sigma\partial^{\sigma} h_{\mu\rho} - \partial_\rho\partial_\mu h^{\sigma}{}_{\sigma}\right) \\ &= -\frac{1}{2}\left(\Box h_{\mu\rho} - \partial_\mu\partial_\sigma h^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma h^{\sigma}{}_{\mu} + \partial_\mu\partial_\rho h\right), \end{aligned} \tag{18.50}$$

where $\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\partial_0^2 + \nabla^2$ is the flat d'Alembertian wave operator.

Contracting with $\eta^{\mu\rho}$ yields the linearized Ricci scalar:

$$R \equiv \eta^{\mu\rho} R_{\mu\rho} = -\Box h + \partial_\mu\partial_\nu h^{\mu\nu}. \tag{18.51}$$

4. Linearized Einstein Tensor: Constructing $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$:

$$G_{\mu\nu} = -\frac{1}{2}\left(\Box h_{\mu\nu} - \partial_\mu\partial_\alpha h^{\alpha}{}_{\nu} - \partial_\nu\partial_\alpha h^{\alpha}{}_{\mu} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}\right). \tag{18.52}$$

18.6.3 The Trace-Reversed Metric Perturbation

Einstein's equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ written in terms of $h_{\mu\nu}$ contains multiple cross-derivative and trace terms. To streamline the equation, we introduce the **trace-reversed perturbation** $\bar{h}_{\mu\nu}$:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{18.53}$$

Key Algebraic Properties of $\bar{h}_{\mu\nu}$:

1. **Reversed Trace:**
$$\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{\mu\nu}\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = h - \frac{1}{2}(4)h = -h. \tag{18.54}$$

2. **Involutive Inversion:** Inverting the relation:
$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) - \frac{1}{2}\eta_{\mu\nu}(-h) = h_{\mu\nu}. \tag{18.55}$$

3. **Divergence Relation:**
$$\partial_\alpha\bar{h}^{\alpha}{}_{\mu} = \partial_\alpha h^{\alpha}{}_{\mu} - \frac{1}{2}\partial_\mu h. \tag{18.56}$$

18.6.4 The Compact Linearized Field Equations

Expressing the linearized Einstein tensor (18.52) directly in terms of $\bar{h}_{\mu\nu}$:

$$\begin{aligned} G_{\mu\nu} = -\frac{1}{2}\Bigg[&\Box\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) - \partial_\mu\left(\partial_\alpha\bar{h}^{\alpha}{}_{\nu} - \frac{1}{2}\partial_\nu\bar{h}\right) - \partial_\nu\left(\partial_\alpha\bar{h}^{\alpha}{}_{\mu} - \frac{1}{2}\partial_\mu\bar{h}\right) \\ &-\partial_\mu\partial_\nu\bar{h} + \eta_{\mu\nu}\Box\bar{h} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\left(\bar{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h}\right)\Bigg] \\ = -\frac{1}{2}&\left(\Box\bar{h}_{\mu\nu} - \partial_\mu\partial_\alpha\bar{h}^{\alpha}{}_{\nu} - \partial_\nu\partial_\alpha\bar{h}^{\alpha}{}_{\mu} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta}\right). \end{aligned} \tag{18.57}$$

Notice how all explicit trace terms $\bar{h}$ have completely canceled out!

Setting $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, the full linearized Einstein field equations assume the remarkably elegant form:

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} - \partial_\mu\partial_\alpha\bar{h}^{\alpha}{}_{\nu} - \partial_\nu\partial_\alpha\bar{h}^{\alpha}{}_{\mu} = -16\pi G T_{\mu\nu}. \tag{18.58}$$

This compact wave-like equation is the master equation of linearized General Relativity. In the following chapter, we will exploit the gauge freedom of coordinate transformations to set the Lorenz gauge $\partial_\alpha\bar{h}^{\alpha}{}_{\mu} = 0$, uncoupling the equation into the pure gravitational wave wave equation $\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$.

Chapter 19

Linearized Gauge Invariance, Energy-Momentum Pseudotensors, Superpotentials, and ADM Mass

Physical Foundations: Non-Linearity and the Decoupling of Degrees of Freedom

Einstein's field equations,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{19.1}$$

are fundamentally **non-linear** partial differential equations. Unlike linear field theories (such as Maxwell's electrodynamics), gravity gravitates: the gravitational field carries energy and momentum that itself acts as a source of further spacetime curvature. In the full non-linear theory, exact analytic solutions exist only in spacetimes with high degrees of symmetry (such as Schwarzschild, Kerr, or FLRW geometries).

Crucially, in the full non-linear theory, the **true physical degrees of freedom** (the dynamical radiative modes of spacetime curvature) and the **unphysical coordinate degrees of freedom** (gauge artifacts arising from arbitrary choices of spacetime coordinate grids via diffeomorphisms) are deeply and non-linearly entangled. It is impossible, in general, to separate them into clean, non-interacting sectors.

To gain profound physical insight into gravitational physics—such as the propagation of gravitational waves, the Newtonian limit, frame dragging, and gravitational energy transport—we must **linearize the theory** around a background spacetime (Minkowski spacetime $\eta_{\mu\nu}$):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{19.2}$$

At first order in perturbation theory, a remarkable mathematical simplification occurs: **the physical and unphysical degrees of freedom completely disentangle and separate out**.

19.1 Warm-Up: Gauge Invariance and Physical Degrees of Freedom in Electromagnetism

To understand how physical degrees of freedom separate from gauge redundancies, consider Maxwell's electrodynamics in flat Minkowski spacetime as an illuminating pedagogical template.

19.1.1 Maxwell's Equations and Gauge Transformations

Maxwell's equations for the 4-vector potential $A^\mu = (\Phi, \vec{A})$ (where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]}$) are:

$$\partial_\mu F^{\nu\mu} = \Box A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = J^\nu, \qquad \Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2. \tag{19.3}$$

A priori, the vector potential A^μ has 4 components. However, under an arbitrary gauge transformation parameterized by a scalar function $f(t, \vec{x})$:

$$A_\mu \longrightarrow A_\mu + \partial_\mu f \iff \begin{cases} A_0 \longrightarrow A_0 - \partial_t f, \\ \vec{A} \longrightarrow \vec{A} + \vec{\nabla} f, \end{cases} \tag{19.4}$$

the field strength tensor $F_{\mu\nu}$ and all physical observables (e.g., the Lorentz force on charged particles) remain strictly invariant. Thus, not all 4 components of A^μ represent independent physical degrees of freedom.

19.1.2 Helmholtz Decomposition and Gauge-Invariant Variables

Under spatial $SO(3)$ rotations, A_0 is a spatial scalar, while the 3-vector $\vec{A}(t, \vec{x})$ can be uniquely decomposed via the Helmholtz theorem into the gradient of a longitudinal scalar ψ and the curl of a transverse (divergence-free) vector $\vec{V}$:

$$\vec{A}(t, \vec{x}) = \vec{\nabla}\psi + \vec{\nabla} \times \vec{V}, \qquad \vec{\nabla} \cdot \vec{V} = 0. \tag{19.5}$$

In spatial Fourier space ($\vec{A}(t, \vec{k}) = \int \mathrm{d}^3x\, e^{-i\vec{k}\cdot\vec{x}} \vec{A}(t, \vec{x})$):

$$\vec{A}(t, \vec{k}) = i\vec{k}\,\psi + i\vec{k} \times \vec{V}, \qquad \vec{k} \cdot \vec{V} = 0. \tag{19.6}$$

The transverse vector $\vec{V}$ is perpendicular to the wavevector $\vec{k}$ and contains **2 independent degrees of freedom**. The fields ψ and $\vec{V}$ are uniquely determined by:

$$\nabla^2 \psi = \vec{\nabla} \cdot \vec{A}, \qquad \nabla^2 \vec{V} = -\vec{\nabla} \times \vec{A}. \tag{19.7}$$

Thus, the 4 initial components of A^μ are reorganized under spatial rotations as:

$$4 \text{ components} = \underbrace{2 \text{ scalars}}_{(A_0, \psi)} + \underbrace{1 \text{ transverse vector}}_{\vec{V} \text{ (2 components)}}. \tag{19.8}$$

Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu f$:

- Because the curl of a gradient vanishes identically ($\vec{\nabla} \times \vec{\nabla} f = 0$), the transverse vector $\vec{V}$ is **strictly gauge-invariant**:
$$\vec{V} \longrightarrow \vec{V}. \tag{19.9}$$
- The scalar functions transform as $\psi \to \psi + f$ and $A_0 \to A_0 - \partial_t f$. Consequently, the linear combination:
$$\varphi \equiv A_0 + \partial_t \psi \tag{19.10}$$
is **strictly gauge-invariant** ($\varphi \to A_0 - \partial_t f + \partial_t(\psi + f) = \varphi$).

19.1.3 Physical Meaning: Unphysical Polarizations, Off-Shell Modes, and On-Shell Radiative Degrees of Freedom

Expressing Maxwell's equations in terms of these gauge-invariant variables reveals their fundamental physical structure:

$$J^0 = \Box A_0 - \partial_0 \left(-\partial_t A_0 + \vec{\nabla}\cdot\vec{A}\right) = \nabla^2 \left(A_0 + \partial_t \psi\right) = \nabla^2 \varphi, \tag{19.11}$$

$$\vec{\nabla}\times\vec{J} = \vec{\nabla}\times\left(\Box\vec{A}\right) = \Box\left(\vec{\nabla}\times\vec{A}\right) = -\nabla^2\left(\Box\vec{V}\right). \tag{19.12}$$

1. **Unphysical Timelike Polarization and Gauge Redundancy:** The timelike component A_0 has no conjugate momentum ($\Pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0$) because the Lagrangian contains no $\partial_0 A_0$ time derivatives. The timelike polarization is strictly unphysical and represents a non-dynamical gauge redundancy.

2. **Off-Shell Constraint Mode:** Equation (19.11) is Poisson's equation $\nabla^2\varphi = J^0$. It contains no second-order time derivatives and acts as an instantaneous elliptic constraint. The longitudinal mode is strictly **off-shell**, determined entirely by the instantaneous charge configuration rather than propagating independently.

3. **On-Shell Dynamical Radiative Modes:** Equation (19.12) governs the transverse vector potential $\vec{V}$. In vacuum ($J^\mu = 0$), the constraint forces $\varphi = 0$, while $\vec{V}$ satisfies the hyperbolic wave equation $\Box\vec{V} = 0$.

Thus, out of the 4 original components of A_μ, the timelike polarization is unphysical, the longitudinal scalar is off-shell, and there are **only 2 on-shell dynamical degrees of freedom** corresponding to the two physical transverse polarization states of propagating electromagnetic waves (photons).

19.2 Linearized Gravity: Diffeomorphisms and Gauge Transformations

19.2.1 Gauge Transformations from Coordinate Diffeomorphisms

We now apply this decomposition framework to general relativity. In linearized gravity around Minkowski spacetime:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \qquad |h_{\mu\nu}| \ll 1. \tag{19.13}$$

Indices on $h_{\mu\nu}$ are raised and lowered using the flat background metric $\eta_{\mu\nu}$. Under an infinitesimal coordinate transformation (diffeomorphism) generated by a vector field $\xi^\mu(x) = O(h)$:

$$x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu(x), \tag{19.14}$$

the metric transforms tensorially:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) = \left(\delta^\alpha_\mu - \partial_\mu \xi^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu \xi^\beta\right)\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right). \tag{19.15}$$

Evaluating both sides at the same coordinate point:

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu = h_{\mu\nu}(x) - 2\partial_{(\mu}\xi_{\nu)}. \tag{19.16}$$

Key Concept: Linearized Gravity as a Spin-2 Gauge Theory

Equation (19.16) establishes that linearized gravity is a **spin-2 abelian gauge theory** with the exact dictionary:

$$\begin{aligned} &\textbf{Electromagnetism (Spin-1)} \longleftrightarrow \textbf{Linearized Gravity (Spin-2)} \\ A_\mu &\longleftrightarrow h_{\mu\nu} \\ A_\mu \to A_\mu + \partial_\mu f &\longleftrightarrow h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} \quad \text{(Gauge Transformation)} \\ F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} &\longleftrightarrow R_{\mu\nu\rho\sigma} = 2\partial_{[\mu}\partial_{[\rho}h_{\sigma]\nu]} \quad \text{(Gauge-Invariant Curvature)} \end{aligned} \tag{19.17}$$

Under background isometries (the 10-parameter Poincaré group where $\partial_{(\mu}\xi_{\nu)} = 0$), $h_{\mu\nu}$ transforms as a genuine rank-$(0,2)$ symmetric Lorentz tensor. Under general diffeomorphisms, it shifts by gauge gradients without altering physical curvature.

Theorem: Strict Gauge Invariance of Linearized Riemann Curvature

Under $h_{\mu\nu} \to h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$, the linearized Riemann curvature tensor is **identically invariant**:

$$R'_{\alpha\mu\nu\beta} = R_{\alpha\mu\nu\beta}. \tag{19.18}$$

Proof. Substituting $\Delta h_{\mu\nu} = -(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)$ into the linearized Riemann tensor:

$$\begin{aligned} \Delta R_{\alpha\mu\nu\beta} &= \frac{1}{2}\left(\partial_\nu\partial_\mu\Delta h_{\alpha\beta} + \partial_\beta\partial_\alpha\Delta h_{\mu\nu} - \partial_\nu\partial_\alpha\Delta h_{\mu\beta} - \partial_\beta\partial_\mu\Delta h_{\alpha\nu}\right) \\ &= -\frac{1}{2}\left[\partial_\nu\partial_\mu(\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha) + \partial_\beta\partial_\alpha(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)\right. \\ &\quad \left. -\partial_\nu\partial_\alpha(\partial_\mu\xi_\beta + \partial_\beta\xi_\mu) - \partial_\beta\partial_\mu(\partial_\alpha\xi_\nu + \partial_\nu\xi_\alpha)\right] \equiv 0, \end{aligned} \tag{19.19}$$

where all 8 partial derivative terms cancel pairwise because partial derivatives commute ($\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$). □

19.3 Scalar-Vector-Tensor (SVT) Decomposition of Metric Perturbations

Just as the 4 components of A_μ decomposed into scalars and transverse vectors under spatial rotations, the **10 independent components** of the symmetric metric perturbation $h_{\mu\nu}$ decompose uniquely under $SO(3)$ into scalar, vector, and tensor modes:

1. **Scalar Sector (4 Components)**:
   - h_{00} is a scalar under spatial rotations.
   - The spatial-temporal vector $\vec{h}_0 \equiv (h_{01}, h_{02}, h_{03})$ contains a longitudinal scalar part: $\vec{h}_0 = \vec{\nabla}\chi + \vec{\nabla} \times \vec{w}$.
   - The spatial trace $h_{kk} = \delta^{ij}h_{ij}$ is a spatial scalar.
   - The longitudinal double-divergence of the trace-free part of h_{ij} defines a scalar A:

$$A \equiv \frac{3}{2}\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)h_{ij}. \tag{19.20}$$

2. **Vector Sector (4 Components / 2 Transverse Vectors)**:
   - The transverse divergence-free vector $\vec{w}$ from h_{0i} ($\vec{\nabla}\cdot\vec{w}=0$, 2 components).
   - The transverse vector $\vec{B}$ obtained from the curl-gradient of h_{ij}:

$$B_q \equiv \hat{k}_i\hat{k}_p\epsilon_{jpq}h_{ij} \iff \vec{B} = \nabla^{-2}\left(\vec{\nabla}\times\vec{\nabla}\cdot h\right) \quad (\vec{k}\cdot\vec{B}=0,\ 2\text{ components}). \tag{19.21}$$

3. **Tensor Sector (2 Components / 1 Transverse-Traceless Tensor)**: The symmetric transverse-traceless (TT) tensor h_{ij}^{TT} defined by:

$$h_{ij}^{\mathrm{TT}} \equiv P_i{}^m P_j{}^n h_{mn} - \frac{1}{2}P_{ij}\left(P^{mn}h_{mn}\right), \qquad P_{ij}\equiv\delta_{ij}-\hat{k}_i\hat{k}_j, \tag{19.22}$$

which satisfies:

$$h_{ii}^{\mathrm{TT}}=0, \qquad \partial_j h_{ij}^{\mathrm{TT}}=0 \quad (\text{or } \hat{k}_j h_{ij}^{\mathrm{TT}}=0). \tag{19.23}$$

Theorem: Off-Shell vs. On-Shell Degree-of-Freedom Counting for the Graviton

Under an infinitesimal diffeomorphism $h_{\mu\nu}\to h_{\mu\nu}-2\partial_{(\mu}\xi_{\nu)}$ parameterized by the 4 gauge generators $\xi^\mu=(\xi^0,\vec{\xi}_\parallel+\vec{\xi}_\perp)$ (2 scalars $\xi^0,\xi_\parallel$ and 1 transverse vector $\vec{\xi}_\perp$):

1. **Gauge Redundancy (4 DOFs):** The 4 arbitrary gauge parameters $\xi^\mu(x)$ remove 4 unphysical coordinate redundancies from the 10 metric components.
2. **Off-Shell Graviton (6 DOFs):** Subtracting the 4 gauge redundancies from the 10 metric components leaves **6 off-shell degrees of freedom**:

$$10-4=6\ \text{DOFs}=\underbrace{2\ \text{scalars}}_{(\Phi,\Psi)}+\underbrace{2\ \text{vectors}}_{(\vec{B})}+\underbrace{2\ \text{tensors}}_{(h_+,h_\times)}. \tag{19.24}$$

   These 6 off-shell modes mediate all instantaneous Newtonian and gravitomagnetic forces in the presence of sources and appear in off-shell quantum gravitational amplitudes.
3. **On-Shell Constraint Elimination (4 constraints → 2 on-shell DOFs):** In vacuum ($T_{\mu\nu}=0$), Einstein's 4 field constraint equations:

$$G_{00}=0 \quad (\text{Hamiltonian constraint}), \qquad G_{0i}=0 \quad (\text{Momentum constraints}) \tag{19.25}$$

   completely eliminate the 2 gauge-invariant scalar potentials ($\Phi=\Psi=0$) and the 2 transverse vector potentials ($\vec{B}=0$).
4. **On-Shell Dynamical Graviton (2 DOFs):** The remaining transverse-traceless tensor h_{ij}^{TT} is strictly gauge-invariant and satisfies the homogeneous wave equation $\Box h_{ij}^{\mathrm{TT}}=0$. Thus, the physical on-shell graviton possesses **exactly 2 propagating dynamical degrees of freedom** (h_+ and $h_\times$).

Thus, linearizing general relativity provides deep physical insight: it cleanly separates the 4 unphysical gauge redundancies, the 4 off-shell elliptic constraint modes, and isolates the 2 true dynamical, transverse-traceless radiative degrees of freedom of gravitational waves.

19.4 Lorenz Gauge Fixing and the Linearized Conservation Paradox

19.4.1 Lorenz (Harmonic) Gauge Fixing

Taking the trace of the gauge transformation (19.16):

$$h' = \eta^{\mu\nu} h'_{\mu\nu} = h - 2\partial_\alpha \xi^\alpha. \tag{19.26}$$

For the trace-reversed perturbation $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$:

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' = (h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu) - \frac{1}{2}\eta_{\mu\nu}\left(h - 2\partial_\alpha \xi^\alpha\right), \tag{19.27}$$

which yields:

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial_\alpha \xi^\alpha. \tag{19.28}$$

Taking the divergence of $\bar{h}'^{\mu\nu}$:

$$\begin{aligned} \partial_\mu \bar{h}'^{\mu\nu} &= \partial_\mu \bar{h}^{\mu\nu} - \partial_\mu \partial^\mu \xi^\nu - \partial_\mu \partial^\nu \xi^\mu + \partial^\nu (\partial_\alpha \xi^\alpha) \\ &= \partial_\mu \bar{h}^{\mu\nu} - \Box \xi^\nu. \end{aligned} \tag{19.29}$$

We can always choose the gauge generator $\xi^\nu(x)$ to satisfy the inhomogeneous wave equation $\Box \xi^\nu = \partial_\mu \bar{h}^{\mu\nu}$, enforcing the **Lorenz gauge condition**:

$$\partial_\mu \bar{h}^{\mu\nu} = 0. \tag{19.30}$$

In Lorenz gauge, the linearized Einstein field equations reduce to the decoupled wave equation:

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}. \tag{19.31}$$

19.4.2 The Linearized Conservation Paradox

Taking the flat divergence of Eq. (19.31) in Lorenz gauge:

$$\partial^\mu \left(\Box \bar{h}_{\mu\nu}\right) = \Box \left(\partial^\mu \bar{h}_{\mu\nu}\right) = 0 \implies \partial^\mu T_{\mu\nu} = 0. \tag{19.32}$$

While $\partial^\mu T_{\mu\nu} = 0$ is the zeroth-order limit of covariant conservation $\nabla_\mu T^{\mu\nu} = 0$, it produces a profound physical paradox:

Foundational Question: The Falling Body Paradox in Linearized Gravity

Suppose an isolated compact object of mass M is placed at rest in Earth's external gravitational field. If the region surrounding the body is vacuum ($T^{\mu\nu}_{\text{ext}} = 0$), then Eq. (19.32) states that its total 3-momentum:

$$P^i \equiv \int_{\text{object}} T^{0i}\, \mathrm{d}^3x \tag{19.33}$$

has a time derivative given by boundary terms:

$$\dot{P}^i = \int_{\text{object}} \partial_0 T^{0i}\, \mathrm{d}^3x = -\int_{\text{object}} \partial_j T^{ji}\, \mathrm{d}^3x = -\oint_{\partial\text{object}} T^{ji}\, \hat{n}_j\, \mathrm{d}^2x = 0. \tag{19.34}$$

Taken literally, the object would maintain constant momentum $P^i = 0$ and would **never fall** in the gravitational field!

Key Concept: Resolution: Non-Linearity, Force Densities, and Pseudotensors

This apparent contradiction demonstrates that **linearized gravity alone is not self-consistent**. In full general relativity, the exact conservation law is covariant:

$$\nabla_\nu T^{\mu\nu} = 0 \iff \partial_\nu \left(\sqrt{-g}\, T^\mu{}_\nu\right) = \frac{1}{2}\sqrt{-g}\, (\partial^\mu g_{\alpha\beta})\, T^{\alpha\beta}. \tag{19.35}$$

The right-hand side represents the non-zero **gravitational force density** that transfers momentum from the gravitational field to the falling object.
Furthermore, because gravitational waves carry away energy-momentum from radiating systems, the gravitational field itself must possess an effective stress-energy. Since gravity can always be transformed away locally in a local inertial frame ($\Gamma^\mu_{\alpha\beta} = 0$), gravitational energy **cannot be represented by a local tensor**. Instead, it is captured by **energy-momentum pseudotensors** $t^{\mu\nu}$ and boundary **superpotentials** $U^{[\mu\nu]\alpha}$, giving well-defined, gauge-invariant global charges such as the **ADM mass**.

19.4.3 Retarded Green's Function Solution

Because Equation (19.31) is a decoupled system of flat-space wave equations for each component of $\bar{h}_{\mu\nu}$, its causal retarded solution is given by the standard Green's function integral:

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4G \int \mathrm{d}^3 y\, \frac{T_{\mu\nu}\left(t - \frac{|\vec{x}-\vec{y}|}{c}, \vec{y}\right)}{|\vec{x}-\vec{y}|}. \tag{19.36}$$

In the radiation zone far from the source ($r \equiv |\vec{x}| \gg |\vec{y}|$), this retarded integral directly determines the asymptotic transverse-traceless waveform h^{TT}_{ij} through the quadrupole formula.

19.5 Energy-Momentum Conservation and Pseudotensors

19.5.1 The Breakdown of Global Energy Conservation in Curved Spacetime

In flat Minkowski spacetime, the stress-energy tensor satisfies the ordinary continuity equation $\partial_\mu T^{\mu\nu} = 0$. Integrating over a spatial volume V bounded by a closed surface ∂V yields an exact, time-independent global 4-momentum:

$$P^\mu \equiv \int_V \mathrm{d}^3 x\, T^{0\mu} \implies \frac{\mathrm{d}P^\mu}{\mathrm{d}t} = -\oint_{\partial V} \mathrm{d}^2 S_i\, T^{i\mu} = 0 \quad \text{(for isolated systems)}. \tag{19.37}$$

In curved spacetime, the principle of general covariance replaces partial derivatives with covariant derivatives:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{19.38}$$

Expanding the covariant divergence using the connection coefficients:

$$\nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\, T^\mu{}_\nu\right) - \frac{1}{2}\partial_\nu(g_{\alpha\beta})T^{\alpha\beta} = 0. \tag{19.39}$$

Multiplying by $\sqrt{-g}$ and integrating over a spacetime region $\mathcal{M}$ bounded by spatial Cauchy hypersurfaces Σ_1 at t_1 and Σ_2 at t_2:

$$\int_{\mathcal{M}} \mathrm{d}^4 x\, \partial_\mu \left(\sqrt{-g}\, T^\mu{}_\nu\right) = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g}\, \partial_\nu(g_{\alpha\beta})T^{\alpha\beta}. \tag{19.40}$$

Applying Gauss's theorem to the left-hand side (assuming matter fields vanish at spatial infinity):

$$\int_{\Sigma_2} \mathrm{d}^3 x \sqrt{-g}\, T^0{}_\nu - \int_{\Sigma_1} \mathrm{d}^3 x \sqrt{-g}\, T^0{}_\nu = \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3 x \sqrt{-g}\, \partial_\nu(g_{\alpha\beta})T^{\alpha\beta}. \tag{19.41}$$

Physical Reason for Non-Conservation: The right-hand side of Equation (19.41) does not vanish in general! It represents the continuous exchange of energy and momentum between matter fields and the dynamical gravitational field.

19.5.2 Why Gravitational Energy Cannot be a Tensor

Can we define a gravitational stress-energy tensor $T_{\mu\nu}^{(\text{grav})}$ such that the total energy $\nabla_\mu(T^{\mu\nu} + T_{(\text{grav})}^{\mu\nu}) = 0$?

- By the **Einstein Equivalence Principle**, in the neighborhood of any point $p \in \mathcal{M}$, one can construct a Local Inertial Frame (LIF) where $g_{\mu\nu}|_p = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu}|_p = 0$.
- In this frame, all gravitational forces vanish locally, meaning any local energy density of the gravitational field must vanish at p.
- However, if a tensor vanishes in one coordinate frame, it vanishes in **all** coordinate frames ($T_{\mu\nu} = 0 \implies T'_{\alpha\beta} = 0$).

Therefore, **no non-zero rank-2 tensor $T_{\mu\nu}^{(\text{grav})}$ can represent the local energy-momentum density of gravity**. Gravitational energy cannot be localized; it is an intrinsically non-local, coordinate-dependent concept represented mathematically by an **energy-momentum pseudotensor** $t_{\mu\nu}$.

19.6 The Superpotential Formalism

19.6.1 Definition and Conservation of Superpotentials

To construct an exact conservation law $\partial_\mu \mathcal{T}^{\mu\nu} = 0$, we introduce a total effective energy-momentum complex $\mathcal{T}^{\mu\nu} \equiv \sqrt{-g}\,(T^{\mu\nu} + t^{\mu\nu})$.

Definition: The Superpotential Complex

A **superpotential** $H^{\mu\alpha\nu\beta}$ is a rank-4 spacetime object that is antisymmetric in its second and fourth indices:

$$H^{\mu\alpha\nu\beta} = -H^{\mu\alpha\beta\nu}. \tag{19.42}$$

The total conserved energy-momentum complex is generated as the double partial derivative of the superpotential:

$$\mathcal{T}^{\mu\nu} = \frac{1}{16\pi G}\partial_\alpha\partial_\beta H^{\mu\alpha\nu\beta}. \tag{19.43}$$

Automatic Conservation: Taking the ordinary partial divergence of $\mathcal{T}^{\mu\nu}$:

$$\partial_\nu \mathcal{T}^{\mu\nu} = \frac{1}{16\pi G}\partial_\nu\partial_\alpha\partial_\beta H^{\mu\alpha\nu\beta} = \frac{1}{16\pi G}\partial_\alpha\left(\partial_\nu\partial_\beta H^{\mu\alpha\nu\beta}\right). \tag{19.44}$$

Because $\partial_\nu\partial_\beta$ is completely symmetric in $\nu \leftrightarrow \beta$ while $H^{\mu\alpha\nu\beta}$ is completely antisymmetric in $\nu \leftrightarrow \beta$, the contraction vanishes identically:

$$\partial_\nu\partial_\beta H^{\mu\alpha\nu\beta} \equiv 0 \implies \partial_\nu \mathcal{T}^{\mu\nu} = 0. \tag{19.45}$$

19.6.2 The Linearized Landau–Lifshitz / Freud Superpotential

Let us construct the explicit superpotential for linearized General Relativity. Recall the linearized Einstein equations from Chapter 18:

$$2G_{\mu\nu} = -\left[\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} - \partial_\mu\partial_\alpha\bar{h}^{\alpha}{}_{\nu} - \partial_\nu\partial_\alpha\bar{h}^{\alpha}{}_{\mu}\right]. \tag{19.46}$$

Rewriting the d'Alembertian and cross derivatives in terms of explicit second derivatives $\partial_\alpha\partial_\beta$:

$$\begin{aligned} 2G^{\mu\nu} &= -\left[\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}^{\mu\nu} + \eta^{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} - \delta^\alpha_\beta\partial_\alpha\partial_\nu\bar{h}^{\beta\mu} - \delta^\alpha_\beta\partial_\alpha\partial_\mu\bar{h}^{\beta\nu}\right] \\ &= \partial_\alpha\partial_\beta\left[-\eta^{\alpha\beta}\bar{h}^{\mu\nu} - \eta^{\mu\nu}\bar{h}^{\alpha\beta} + \eta^{\mu\beta}\bar{h}^{\alpha\nu} + \eta^{\nu\alpha}\bar{h}^{\beta\mu}\right]. \end{aligned} \tag{19.47}$$

This allows us to identify the **linearized Landau–Lifshitz superpotential** $H^{\mu\alpha\nu\beta}$:

$$H^{\mu\alpha\nu\beta} \equiv -\left(\eta^{\alpha\beta}\bar{h}^{\mu\nu} + \eta^{\mu\nu}\bar{h}^{\alpha\beta} - \eta^{\mu\beta}\bar{h}^{\alpha\nu} - \eta^{\nu\alpha}\bar{h}^{\beta\mu}\right). \tag{19.48}$$

Algebraic Symmetries of $H^{\mu\alpha\nu\beta}$: The superpotential $H^{\mu\alpha\nu\beta}$ satisfies the complete algebraic symmetries of the Riemann curvature tensor:

1. **Pair Antisymmetry:**
$$H^{\mu\alpha\nu\beta} = -H^{\alpha\mu\nu\beta} = -H^{\mu\alpha\beta\nu} = H^{\alpha\mu\beta\nu}. \tag{19.49}$$
2. **Pair Exchange Symmetry:**
$$H^{\mu\alpha\nu\beta} = H^{\nu\beta\mu\alpha}. \tag{19.50}$$
3. **Cyclic Bianchi-like Identity:**
$$H^{\mu[\alpha\nu\beta]} \equiv H^{\mu\alpha\nu\beta} + H^{\mu\nu\beta\alpha} + H^{\mu\beta\alpha\nu} = 0. \tag{19.51}$$

19.7 Asymptotic Symmetries and the ADM Mass

19.7.1 Analogy with Maxwell Electrodynamics and Gauss's Law

In Maxwell electrodynamics, the total electric charge Q contained in a spatial volume V is given by integrating the charge density J^0:

$$Q = \int_V \mathrm{d}^3x\, J^0 = \int_V \mathrm{d}^3x\, \partial_\alpha F^{0\alpha} = \oint_{\partial V} \mathrm{d}^2S_i\, F^{0i}, \tag{19.52}$$

where the volume integral of the source J^0 converts via Gauss's divergence theorem into a 2D flux integral of the field strength $F^{0i} = E^i$ on the boundary ∂V.

19.7.2 The ADM 4-Momentum as a Boundary Surface Integral

In General Relativity, the total energy-momentum P^μ of an asymptotically flat, isolated gravitational system is similarly obtained by integrating the effective energy-momentum complex over a constant-time hypersurface:

$$P^\mu = \int_V \mathrm{d}^3x\, \mathcal{T}^{0\mu} = \frac{1}{16\pi G}\int_V \mathrm{d}^3x\, \partial_\alpha\partial_\beta H^{\mu\alpha0\beta}. \tag{19.53}$$

Because $H^{\mu000} = 0$ by antisymmetry, the time derivative term vanishes, and the divergence reduces to a pure spatial derivative:

$$P^\mu = \frac{1}{16\pi G}\int_V \mathrm{d}^3x\, \partial_j\left(\partial_\alpha H^{\mu\alpha0j}\right) = \frac{1}{16\pi G}\oint_{\partial V} \mathrm{d}^2S_j\, \partial_\alpha H^{\mu\alpha0j}. \tag{19.54}$$

19.7.3 Explicit Derivation of the ADM Energy (P^0)

Let us compute the total energy P^0 by evaluating the component $H^{0\alpha 0j}$. Setting $\mu = 0$ and $\beta = j$ in the superpotential formula (19.48):

$$\begin{aligned} H^{0\alpha 0j} &= -\left(\eta^{\alpha j}\bar{h}^{00} + \eta^{00}\bar{h}^{\alpha j} - \eta^{0j}\bar{h}^{\alpha 0} - \eta^{0\alpha}\bar{h}^{j0}\right) \\ &= -\left(\eta^{\alpha j}\bar{h}^{00} - \bar{h}^{\alpha j} - \eta^{0\alpha}\bar{h}^{j0}\right), \end{aligned} \tag{19.55}$$

where we used $\eta^{00} = -1$ and $\eta^{0j} = 0$.

Substituting the relations between $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$:

$$\bar{h}^{00} = h^{00} - \frac{1}{2}\eta^{00}h = h^{00} + \frac{1}{2}h, \tag{19.56}$$

$$\bar{h}^{\alpha j} = h^{\alpha j} - \frac{1}{2}\eta^{\alpha j}h, \tag{19.57}$$

$$\bar{h}^{j0} = h^{j0}. \tag{19.58}$$

Substituting these into Equation (19.55):

$$\begin{aligned} H^{0\alpha 0j} &= -\left[\eta^{\alpha j}\left(h^{00} + \frac{1}{2}h\right) - \left(h^{\alpha j} - \frac{1}{2}\eta^{\alpha j}h\right) - \eta^{0\alpha}h^{j0}\right] \\ &= -\eta^{\alpha j}h^{00} + h^{\alpha j} + \eta^{0\alpha}h^{j0}. \end{aligned} \tag{19.59}$$

Taking the spatial divergence ∂_α (noting that for static or isolated configurations, time derivatives $\partial_0 \sim 0$):

$$\partial_\alpha H^{0\alpha 0j} = -\partial^j h^{00} + \partial_i h^{ij} + \partial_0 h^{j0} \approx \partial_i h^{ij} - \partial_j h^{00}. \tag{19.60}$$

In isotropic coordinates for a static Newtonian source, $h_{00} = -2\Phi/c^2$ and $h_{ij} = -2\Phi/c^2\delta_{ij}$, giving $h_{jj} = 3h_{00}$ and $h^{00} = h_{jj} - h_{kk}$.

Theorem: The Arnowitt–Deser–Misner (ADM) Mass Formula

Substituting into Equation (19.54), we obtain the celebrated **ADM Mass Formula**:

$$M_{\mathrm{ADM}} \equiv \frac{P^0}{c^2} = \frac{1}{16\pi G}\oint_{S_\infty} \mathrm{d}^2 S_i \left(\partial_j h_{ij} - \partial_i h_{jj}\right), \tag{19.61}$$

or equivalently in terms of the full spatial metric $g_{ij} = \delta_{ij} + h_{ij}$:

$$M_{\mathrm{ADM}} = \frac{1}{16\pi G}\lim_{r\to\infty}\oint_{S_r} \mathrm{d}^2 S_i \left(\partial_j g_{ij} - \partial_i g_{jj}\right). \tag{19.62}$$

The ADM mass measures the total asymptotic gravitational mass of the spacetime—including all matter, radiation, and gravitational field binding energy—purely through the flux of metric gradients at spatial infinity.

Chapter 20

The Landau–Lifshitz Energy-Momentum Complex, Total Angular Momentum, and Geometric Transport Laws

In the preceding chapter, we established that the local conservation of matter energy-momentum ($\nabla_\mu T^{\mu\nu} = 0$) does not yield a conserved 4-momentum vector under simple spatial integration because of the non-tensorial nature of gravitational energy. We introduced the Freud superpotential in the linearized regime to define the ADM mass.

In this chapter, we generalize this formalism in two fundamental directions:

1. We derive the **conserved relativistic angular momentum complex** $J^{\mu\nu}$ for isolated gravitational systems, showing how the superpotential symmetries lead to exact cancellations of time-derivative terms, reducing angular momentum to a 2-sphere boundary flux.
2. We present the **exact non-perturbative Landau–Lifshitz energy-momentum complex** $(-g)T^{\mu\nu}_{\text{total}} = \partial_\alpha \partial_\beta \lambda^{\mu\alpha\nu\beta}$, derived rigorously from the full non-linear Einstein tensor in Locally Inertial Frames (LIF).
3. We examine the geometric foundations of **Lie transport versus Parallel transport**, contrasting connection-dependent parallel propagation with connection-independent Lie dragging along vector flows.

20.1 Asymptotic Boundary Flux and Surface Integrals

20.1.1 The Asymptotic Limit of Volume Divergences

Recall from Chapter 19 that the total 4-momentum P^μ of an isolated system is expressed as a spatial volume integral of a divergence:

$$P^\mu = \frac{1}{16\pi G} \int_V \mathrm{d}^3x \, \partial_\alpha \partial_\beta H^{\mu\alpha 0\beta} = \frac{1}{16\pi G} \oint_{\partial V} \mathrm{d}^2 S_j \, \partial_\alpha H^{\mu\alpha 0 j}. \tag{20.1}$$

For the energy component P^0:

$$P^0 = \frac{1}{16\pi G} \oint_{S_\infty} \mathrm{d}^2 S_\beta \left[\partial^\beta (\eta^{\mu\nu} g_{\mu\nu}) - \partial_\alpha g^{\alpha\beta} \right]. \tag{20.2}$$

Physical Insight & Motivation: Asymptotic Integrand Decay vs. Finite Boundary Flux

A critical conceptual subtlety in general-relativistic boundary integrals concerns the decay rate of the integrand:

- As $r \to \infty$, the metric perturbation for an isolated mass M behaves asymptotically as $h_{\mu\nu} \sim O(1/r)$.
- Consequently, the spatial derivatives decay as $\partial_k h_{\mu\nu} \sim O(1/r^2)$.
- The surface area element of a 2-sphere at radius r scales as $\mathrm{d}^2 S_i \sim r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{n}_i \sim O(r^2)$.

Thus, while the **integrand pointwise vanishes** as $r \to \infty$ (since $\lim_{r\to\infty} \partial_k h_{\mu\nu} = 0$), the integral itself:

$$\oint_{S_r} \mathrm{d}^2 S_i \, \partial_j h \sim \int (r^2) \left(\frac{1}{r^2}\right) \mathrm{d}\Omega = \text{finite constant} \neq 0! \tag{20.3}$$

Therefore, we **must evaluate the integral on a finite sphere S_r first, and take the limit $r \to \infty$ only at the very end of the calculation**.

20.2 Conserved Total Relativistic Angular Momentum

20.2.1 Angular Momentum Tensor Density

In special relativity and flat-space field theory, given a symmetric, conserved energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$, the conserved relativistic angular momentum tensor density is defined as:

$$\mathcal{M}^{\mu\nu\rho} \equiv x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}, \tag{20.4}$$

which is antisymmetric in its first two indices ($\mathcal{M}^{\mu\nu\rho} = -\mathcal{M}^{\nu\mu\rho}$) and divergence-free:

$$\partial_\rho \mathcal{M}^{\mu\nu\rho} = (\partial_\rho x^\mu) T^{\nu\rho} - (\partial_\rho x^\nu) T^{\mu\rho} + x^\mu \partial_\rho T^{\nu\rho} - x^\nu \partial_\rho T^{\mu\rho} = \delta^\mu_\rho T^{\nu\rho} - \delta^\nu_\rho T^{\mu\rho} + 0 = T^{\nu\mu} - T^{\mu\nu} = 0, \tag{20.5}$$

by virtue of the symmetry $T^{\mu\nu} = T^{\nu\mu}$.

In General Relativity, replacing $T^{\mu\nu}$ with the effective total energy-momentum complex $\mathcal{T}^{\mu\nu} = \frac{1}{16\pi G} \partial_a \partial_b H^{\mu a \nu b}$, the spatial angular momentum components are given by:

$$J^{mn} \equiv \int_V \mathrm{d}^3 x \, (x^m \mathcal{P}^n - x^n \mathcal{P}^m) = \int_V \mathrm{d}^3 x \left(x^m \mathcal{T}^{n0} - x^n \mathcal{T}^{m0} \right). \tag{20.6}$$

20.2.2 Superpotential Expansion and Integration by Parts

Substituting $\mathcal{T}^{n0} = \frac{1}{16\pi G} \partial_b \partial_a H^{na0b}$ into the first term of the integrand:

$$\begin{aligned} x^m \partial_b \partial_a H^{na0b} &= \partial_b \left(x^m \partial_a H^{na0b} \right) - (\partial_b x^m) \partial_a H^{na0b} \\ &= \partial_b \left(x^m \partial_a H^{na0b} \right) - \delta^m_b \partial_a H^{na0b} \\ &= \partial_b \left(x^m \partial_a H^{na0b} \right) - \partial_a H^{na0m}. \end{aligned} \tag{20.7}$$

Splitting the spacetime indices into temporal ($\mu = 0$) and spatial ($\mu = \alpha, \beta \in \{1, 2, 3\}$) parts:

1. The total divergence term splits as:

$$\partial_b \left(x^m \partial_a H^{na0b} \right) = \partial_0 \left(x^m \partial_a H^{na00} \right) + \partial_\alpha \left(x^m \partial_a H^{na0\alpha} \right) = \partial_\alpha \left(x^m \partial_a H^{na0\alpha} \right), \tag{20.8}$$

because $H^{na00} \equiv 0$ identically due to the antisymmetry $H^{\mu\alpha\nu\beta} = -H^{\mu\alpha\beta\nu}$ in the second pair of indices.

2. The second term splits as:

$$\partial_a H^{na0m} = \partial_0 H^{n00m} + \partial_\alpha H^{n\alpha0m} = \partial_\alpha H^{n\alpha0m} - \partial_0 H^{na0m}\Big|_{a=0} = \partial_\alpha H^{n\alpha0m} + \partial_0 H^{na0m}. \tag{20.9}$$

Thus, Eq. (20.7) becomes:

$$x^m \partial_b \partial_a H^{na0b} = \partial_\alpha \left(x^m \partial_a H^{na0\alpha} \right) - \partial_0 H^{na0m} - \partial_\alpha H^{na0m}. \tag{20.10}$$

20.2.3 Exact Cancellation of Time-Derivative Terms

Subtracting the $m \leftrightarrow n$ swapped term to form the full integrand of J^{mn}:

$$16\pi G\, J^{mn} = \int \mathrm{d}^3x \Big[\partial_\alpha \left(x^m \partial_a H^{na0\alpha} \right) - \partial_0 H^{na0m} - \partial_\alpha H^{na0m} - \partial_\alpha \left(x^n \partial_a H^{ma0\alpha} \right) + \partial_0 H^{ma0n} + \partial_\alpha H^{ma0n} \Big]. \tag{20.11}$$

Now we examine the time-derivative terms $-\partial_0 H^{na0m} + \partial_0 H^{ma0n}$. From the algebraic symmetries of the superpotential:

$$H^{\mu\alpha\nu\beta} = -H^{\alpha\mu\nu\beta} = -H^{\mu\alpha\beta\nu} = H^{\nu\beta\mu\alpha}. \tag{20.12}$$

For $a = 0$, evaluating H^{n00m} and H^{m00n}:

$$H^{m00n} = -H^{0m0n} = -H^{0n0m} = H^{n00m}. \tag{20.13}$$

Therefore:

$$\partial_0 H^{ma0n} - \partial_0 H^{na0m} \equiv 0. \tag{20.14}$$

Theorem: Boundary Formula for Total Angular Momentum

The time-derivative terms cancel identically, leaving J^{mn} as a **pure spatial divergence**:

$$J^{mn} = \frac{1}{16\pi G} \int_V \mathrm{d}^3x\, \partial_\alpha \left[x^m \partial_a H^{na0\alpha} - x^n \partial_a H^{ma0\alpha} - H^{na0m} + H^{ma0n} \right]. \tag{20.15}$$

Applying Gauss's theorem on a large spatial sphere S_∞:

$$J^{mn} = \frac{1}{16\pi G} \lim_{r\to\infty} \oint_{S_r} \mathrm{d}^2S_\alpha \left[x^m \partial_a H^{na0\alpha} - x^n \partial_a H^{ma0\alpha} - H^{na0m} + H^{ma0n} \right]. \tag{20.16}$$

The 3-dimensional angular momentum pseudo-vector J_α is obtained via the 3D Levi-Civita tensor:

$$J_\alpha \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma} J^{\beta\gamma}. \tag{20.17}$$

20.3 The Non-Perturbative Landau–Lifshitz Pseudotensor

20.3.1 Physical Demand for a Non-Perturbative Formulation

In the weak-field approximation, we used the flat background metric $\eta_{\mu\nu}$ to raise and lower indices. However, for strong gravitational fields (such as near neutron stars, black holes, or in cosmological dynamics), we require a conservation law:

$$\partial_m \left[(-g) \left(T^{mn}_{\text{matter}} + t^{mn}_{\text{LL}} \right) \right] \equiv \partial_m \mathcal{T}^{mn}_{\text{total}} = 0, \tag{20.18}$$

which is exact and holds to **all orders in curvature**.

To guarantee automatic conservation via index antisymmetry, we demand that the total energy-momentum complex $\mathcal{T}^{mn}_{\text{total}}$ be generated by a rank-4 superpotential λ^{manb}:

$$(-g)\,(T^{mn}_{\text{matter}} + t^{mn}_{\text{LL}}) = \frac{1}{16\pi G}\partial_a\partial_b\lambda^{manb}, \tag{20.19}$$

where $\lambda^{manb} = -\lambda^{amnb} = -\lambda^{manb} = \lambda^{nbma}$.

Using Einstein's equations $G^{mn} = 8\pi G\, T^{mn}_{\text{matter}}$, Eq. (20.19) can be rearranged as:

$$16\pi G\,(-g)\,t^{mn}_{\text{LL}} = \partial_a\partial_b\lambda^{manb} - 2(-g)G^{mn}. \tag{20.20}$$

20.3.2 Rigorous LIF Derivation of the Landau–Lifshitz Superpotential

To find the explicit form of λ^{manb}, we evaluate the Einstein tensor in a **Locally Inertial Frame (LIF)** at an arbitrary spacetime point P. In a LIF at P:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}, \qquad \partial_\alpha g_{\mu\nu}(P) = 0 \implies \Gamma^a_{bc}(P) = 0, \qquad \partial_\alpha\partial_\beta g_{\mu\nu}(P) \neq 0. \tag{20.21}$$

Step 1: Ricci Tensor in LIF: In general coordinates, the Ricci tensor is:

$$R_{ik} = \partial_n\Gamma^n_{ik} - \partial_k\Gamma^n_{in} + \Gamma^n_{ik}\Gamma^p_{np} - \Gamma^p_{in}\Gamma^n_{kp}. \tag{20.22}$$

In a LIF at P, the quadratic Christoffel terms vanish:

$$R_{ik} \overset{\text{LIF}}{=} \partial_n\Gamma^n_{ik} - \partial_k\Gamma^n_{in}. \tag{20.23}$$

Raising both indices using g^{im} and g^{kn} (which can be pulled inside the partial derivatives since $\partial g = 0$ in LIF):

$$R^{ik} = g^{im}g^{kn}R_{mn} \overset{\text{LIF}}{=} g^{im}g^{kn}\left(\partial_j\Gamma^j_{mn} - \partial_n\Gamma^j_{mj}\right) = \underbrace{g^{im}g^{kn}\partial_j\Gamma^j_{mn}}_{\text{Term (1)}} - \underbrace{g^{im}g^{kn}\partial_n\Gamma^j_{mj}}_{\text{Term (2)}}. \tag{20.24}$$

Step 2: Evaluating Term (2) (Trace Connection): Recall the well-known contraction identity for Christoffel symbols:

$$\Gamma^j_{mj} = \partial_m \ln\sqrt{-g} = \frac{1}{2g}\partial_m g = \frac{1}{-2g}\partial_m(-g). \tag{20.25}$$

Therefore, Term (2) becomes:

$$g^{im}g^{kn}\partial_n\Gamma^j_{mj} = \frac{1}{2}\partial_n\left[\frac{g^{im}g^{kn}}{-g}\partial_m(-g)\right]. \tag{20.26}$$

Step 3: Evaluating Term (1) (General Connection): Expanding the Christoffel symbol:

$$\Gamma^t_{mn} = \frac{1}{2}g^{ta}\left(\partial_m g_{an} + \partial_n g_{ma} - \partial_a g_{mn}\right). \tag{20.27}$$

Using the metric orthogonality derivative identity $g^{kn}\partial_m g_{an} = -\partial_m g^{kn} g_{an}$:

$$\begin{aligned} g^{im}g^{kn}\Gamma^t_{mn} &= \frac{1}{2}g^{ta}\left(g^{im}g^{kn}\partial_m g_{an} + g^{im}g^{kn}\partial_n g_{ma} - g^{im}g^{kn}\partial_a g_{mn}\right) \\ &= \frac{1}{2}g^{ta}\left(-g^{im}g_{an}\partial_m g^{kn} - g^{kn}g_{ma}\partial_n g^{im} + g^{im}g_{mn}\partial_a g^{kn}\right) \\ &= \frac{1}{2}\left(-g^{im}\delta^t_n\partial_m g^{kn} - g^{kn}\delta^t_m\partial_n g^{im} + g^{ta}\delta^i_n\partial_a g^{kn}\right). \end{aligned} \tag{20.28}$$

Taking the divergence ∂_t (setting index $t \to n$):

$$g^{im}g^{kn}\partial_j\Gamma^j_{mn} = \frac{1}{2}\partial_n\left(-g^{im}\partial_m g^{kn} - g^{km}\partial_m g^{in} + g^{mn}\partial_m g^{ik}\right). \tag{20.29}$$

Step 4: Assembling R^{ik} in LIF: Combining Terms (1) and (2):

$$\begin{aligned} R^{ik} &\overset{\text{LIF}}{=} \frac{1}{2}\partial_n\left[-g^{im}\partial_m g^{kn} - g^{km}\partial_m g^{in} + g^{mn}\partial_m g^{ik}\right] - \frac{1}{2}\partial_n\left[\frac{g^{im}g^{kn}}{-g}\partial_m(-g)\right] \\ &= \frac{1}{2}\partial_n\left[-\partial_m(g^{kn}g^{im}) + g^{kn}\partial_m g^{im} - g^{km}\partial_m g^{in} + g^{mn}\partial_m g^{ik}\right] \\ &\quad - \frac{1}{2}\partial_n\left[\frac{1}{-g}\partial_m(-g\,g^{im}g^{kn})\right] + \frac{1}{2}\partial_n\left[\partial_m(g^{im}g^{kn})\right]. \end{aligned} \tag{20.30}$$

Notice that the terms $\frac{1}{2}\partial_n[g^{kn}\partial_m g^{im} - g^{km}\partial_m g^{in}]$ cancel identically upon contracting with ∂_n, since:

$$g^{kn}\partial_n\partial_m g^{im} - g^{km}\partial_n\partial_m g^{in} = 0 \quad \text{(symmetric under } m \leftrightarrow n \text{ relabeling)}. \tag{20.31}$$

Furthermore, the term $-\partial_m(g^{kn}g^{im})$ cancels against $+\partial_m(g^{im}g^{kn})$. Thus, we arrive at the streamlined LIF Ricci tensor:

$$R^{ik} \overset{\text{LIF}}{=} \frac{1}{2}\partial_n\left(g^{mn}\partial_m g^{ik}\right) - \frac{1}{2}\partial_n\left[\frac{1}{-g}\partial_m(-g\,g^{im}g^{kn})\right]. \tag{20.32}$$

Step 5: Scalar Curvature R in LIF: Contracting Eq. (20.32) with g_{ik}:

$$R = g_{ik}R^{ik} \overset{\text{LIF}}{=} -\partial_n\left[\frac{1}{-g}\partial_m(-g\,g^{mn})\right]. \tag{20.33}$$

Step 6: Einstein Tensor $G^{ik} = R^{ik} - \frac{1}{2}g^{ik}R$ in LIF: Substituting Eqs. (20.32) and (20.33):

$$\begin{aligned} G^{ik} &\overset{\text{LIF}}{=} \frac{1}{2}\partial_n\left(g^{mn}\partial_m g^{ik}\right) - \frac{1}{2}\partial_n\left[\frac{1}{-g}\partial_m(-g\,g^{im}g^{kn})\right] + \frac{1}{2}g^{ik}\partial_n\left[\frac{1}{-g}\partial_m(-g\,g^{mn})\right] \\ &= \frac{1}{2}\partial_n\left[\frac{1}{-g}\partial_m\left\{(-g)\left(g^{ik}g^{mn} - g^{im}g^{kn}\right)\right\}\right]. \end{aligned} \tag{20.34}$$

Theorem: The Landau–Lifshitz Superpotential

Multiplying Eq. (20.34) by $(-g)/(8\pi G)$ and defining the **Landau–Lifshitz Superpotential** λ^{imkl}:

$$\lambda^{imkl} \equiv \frac{-g}{16\pi G}\left(g^{ik}g^{ml} - g^{il}g^{km}\right), \tag{20.35}$$

the total effective energy-momentum complex in LIF is a **pure double divergence**:

$$(-g)T^{ik}_{\text{total}} = \frac{1}{-g}\partial_l\partial_m\lambda^{imkl} = \frac{1}{16\pi G}\partial_l\partial_m\left[(-g)\left(g^{ik}g^{ml} - g^{il}g^{km}\right)\right]. \tag{20.36}$$

Defining the following notation

$$\lambda^{imkl} \equiv \frac{-g}{16\pi G}\,A^{imkl}, \qquad A^{imkl} \equiv g^{ik}g^{ml} - g^{il}g^{km}, \tag{20.37}$$

so that the pseudotensor given by (20.20) becomes

$$16\pi G\,t^{ik}_{\text{LL}} = \frac{1}{(-g)}\,\partial_l\partial_m\Big[(-g)\,A^{imkl}\Big] - 2G^{ik}. \tag{20.38}$$

Evaluating the right-hand side of (20.38) directly looks forbidding, but its structure is severely constrained: At any point the metric, its first derivatives and its second derivatives are

independent data, and both terms of (20.38) are polynomials linear in $\partial\partial g$. In a locally inertial frame at P, where $g_{\mu\nu}(P)=\eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu}(P)=0$, every term quadratic in ∂g vanishes, so only the linear sector survives; since $t^{ik}_{\mathrm{LL}}(P)=0$ there and $\partial_\rho\partial_\sigma g_{\mu\nu}(P)$ is unconstrained, the linear form multiplying it vanishes identically,

$$\left[\frac{1}{(-g)}\partial_l\partial_m\Big[(-g)A^{imkl}\Big]\right]_{\partial\partial g}-\big[2G^{ik}\big]_{\partial\partial g}=0, \tag{20.39}$$

and by homogeneity under the frame-preserving linear transformations this holds for every $g_{\mu\nu}$. What remains is the quadratic sector,

$$16\pi G\, t^{ik}_{\mathrm{LL}}=\left[\frac{1}{(-g)}\partial_l\partial_m\Big[(-g)A^{imkl}\Big]\right]_{\Gamma\Gamma}-\big[2G^{ik}\big]_{\Gamma\Gamma}. \tag{20.40}$$

Before either bracket in (20.40) can be collected, one must know on which side of the line $\partial_m\Gamma^a_{bc}$ falls, and the answer is that it falls on both. Write the connection with its first index lowered, $\Gamma_{d,bc}\equiv g_{de}\Gamma^e_{bc}=\frac12(\partial_b g_{dc}+\partial_c g_{bd}-\partial_d g_{bc})$, so $\Gamma^a_{bc}=g^{ad}\Gamma_{d,bc}$. Since $\partial_m\Gamma_{d,bc}$ is purely second order while $\partial_m g^{ad}$ is purely first,

$$\partial_m\Gamma^a_{bc}=\underbrace{g^{ad}\,\partial_m\Gamma_{d,bc}}_{\partial\partial g}+\underbrace{(\partial_m g^{ad})\Gamma_{d,bc}}_{\Gamma\Gamma}. \tag{20.41}$$

The first half is discarded under (20.39). The second, using $\partial_m g^{ad}=-\Gamma^a_{ms}g^{sd}-\Gamma^d_{ms}g^{as}$, is

$$\big[\partial_m\Gamma^a_{bc}\big]_{\Gamma\Gamma}=-\Gamma^a_{ms}\Gamma^s_{bc}-g^{as}\,\Gamma^d_{ms}\,\Gamma_{d,bc}. \tag{20.42}$$

The second term does *not* collapse to a trace; it genuinely carries the first-kind object $\Gamma_{d,bc}$, and that object survives until the very last stage of the calculation.

Applying this splitting to the Ricci tensor produces an immediate simplification. From $R_{ab}=\partial_c\Gamma^c_{ab}-\partial_b\Gamma^c_{ac}+\Gamma^c_{cd}\Gamma^d_{ab}-\Gamma^c_{bd}\Gamma^d_{ac}$, expanding the derivatives of the inverse metric in the first two terms,

$$(\partial_c g^{cd})\Gamma_{d,ab}=-\Gamma^c_{cs}\Gamma^s_{ab}-g^{cs}\Gamma^d_{cs}\Gamma_{d,ab},\qquad -(\partial_b g^{cd})\Gamma_{d,ac}=\Gamma^c_{bs}\Gamma^s_{ac}+g^{cs}\Gamma^d_{bs}\Gamma_{d,ac}, \tag{20.43}$$

the leading term of each cancels one of the two explicit quadratic terms, leaving

$$\big[R_{ab}\big]_{\Gamma\Gamma}=\cancel{-\Gamma^c_{cs}\Gamma^s_{ab}}-g^{cs}\Gamma^d_{cs}\Gamma_{d,ab}\cancel{+\Gamma^c_{bs}\Gamma^s_{ac}}+g^{cs}\Gamma^d_{bs}\Gamma_{d,ac}\cancel{+\Gamma^c_{cd}\Gamma^d_{ab}}\cancel{-\Gamma^c_{bd}\Gamma^d_{ac}} \tag{20.44}$$

$$=g^{cs}\Big(\Gamma^d_{bs}\Gamma_{d,ac}-\Gamma^d_{cs}\Gamma_{d,ab}\Big), \tag{20.45}$$

whence, contracting with $G^{ik}=g^{ia}g^{kb}R_{ab}-\frac12 g^{ik}g^{ab}R_{ab}$,

$$\big[2G^{ik}\big]_{\Gamma\Gamma}=\Big(2g^{ia}g^{kb}-g^{ik}g^{ab}\Big)g^{cs}\Big(\Gamma^d_{bs}\Gamma_{d,ac}-\Gamma^d_{cs}\Gamma_{d,ab}\Big). \tag{20.46}$$

Expanded, this is four terms, each carrying $\Gamma_{d,ac}$ or $\Gamma_{d,ab}$:

$$\begin{aligned}\big[2G^{ik}\big]_{\Gamma\Gamma}=\Gamma^p_{qr}\Gamma_{p,st}\,\Big(&+g^{qr}g^{st}g^{ik}-2\,g^{qr}g^{si}g^{tk}-g^{qs}g^{rt}g^{ik}\\&+2\,g^{qs}g^{rk}g^{ti}\Big).\end{aligned} \tag{20.47}$$

With the Einstein-tensor side collected, the remaining task is the double divergence of the superpotential itself, taken one derivative at a time. Write $B^{imk}\equiv(-g)^{-1}\partial_l[(-g)A^{imkl}]$. With $\partial_l(-g)=2(-g)\Gamma^r_{lr}$ and $\partial_l g^{ab}=-\Gamma^a_{ls}g^{sb}-\Gamma^b_{ls}g^{as}$, the elementary expansion is ten terms which, by cancelling the two Γ^k terms on relabelling $l\leftrightarrow s$ and pairing the four traces, collapse to six:

$$B^{imk}=2\,\Gamma^r_{lr}(g^{ik}g^{ml}-g^{il}g^{km})-(\Gamma^i_{ls}\,g^{sk}+\cancel{\Gamma^k_{ls}\,g^{is}})g^{ml}-g^{ik}(\Gamma^m_{ls}\,g^{sl}+\Gamma^l_{ls}\,g^{ms})$$

$$
\begin{aligned}
&+ (\Gamma^{i}_{ls}\, g^{sl} + \Gamma^{l}_{ls}\, g^{is}) g^{km} + g^{il}(\cancel{\Gamma^{k}_{ls}\, g^{sm}} + \Gamma^{m}_{ls}\, g^{ks}) \\
&= g^{ik}\Big(g^{ms}\Gamma^{l}_{ls} - g^{sl}\Gamma^{m}_{ls}\Big) - g^{km}\Big(g^{is}\Gamma^{l}_{ls} - g^{sl}\Gamma^{i}_{ls}\Big) + g^{ks}\Big(g^{il}\Gamma^{m}_{ls} - g^{ml}\Gamma^{i}_{ls}\Big),
\end{aligned}
\tag{20.48}
$$

each structure separately antisymmetric under $i \leftrightarrow m$, as it must be since A^{imkl} is.

The second divergence is taken next, and it is worth working through a single term of (20.48) explicitly before stating the result for all six. It is

$$
\begin{aligned}
\frac{\partial_m \partial_l[(-g)A^{imkl}]}{-g} &= \frac{1}{-g}\,\partial_m\big[(-g)B^{imk}\big] = \frac{1}{-g}\Big[\partial_m(-g)\,B^{imk} + (-g)\,\partial_m B^{imk}\Big] \\
&= \frac{\partial_m(-g)}{-g}\,B^{imk} + \partial_m B^{imk} = 2\Gamma^{r}_{mr}B^{imk} + \partial_m B^{imk} \\
&= 2\Gamma^{r}_{mr}[g^{ik}\Big(g^{mp}\Gamma^{n}_{np} - g^{pn}\Gamma^{m}_{np}\Big) - g^{km}\Big(g^{ip}\Gamma^{n}_{np} - g^{pn}\Gamma^{i}_{np}\Big) + g^{kp}\Big(g^{in}\Gamma^{m}_{np} - g^{mn}\Gamma^{i}_{np}\Big)] \\
&\quad + \partial_m[g^{ik}\Big(g^{mp}\Gamma^{n}_{np} - g^{pn}\Gamma^{m}_{np}\Big) - g^{km}\Big(g^{ip}\Gamma^{n}_{np} - g^{pn}\Gamma^{i}_{np}\Big) + g^{kp}\Big(g^{in}\Gamma^{m}_{np} - g^{mn}\Gamma^{i}_{np}\Big)]
\end{aligned}
\tag{20.49}
$$

The first piece is immediately in the $\Gamma\Gamma$ form, however the second piece requires the product rule on each of the six terms of (20.48), each a product of two inverse metrics and one connection.

Take the first term of the second piece, written as $\Gamma^{n}_{np}\, g^{pm}\, g^{ik}$, as an illustration; the other five follow the same pattern but are longer to display. There are three factors, and since the second factor g^{pm} already carries the differentiation index m among its own indices, differentiating it contributes two terms in its own right, exactly as differentiating g^{ik} or Γ^{n}_{np} does.

$$
(\partial_m g^{pm})\Gamma^{n}_{np}g^{ik} = (-\Gamma^{p}_{mq}g^{qm} - \Gamma^{m}_{mq}g^{pq})\Gamma^{n}_{np}g^{ik} = -\Gamma^{n}_{np}\Gamma^{p}_{qm}g^{qm}g^{ik} - \Gamma^{n}_{np}\Gamma^{m}_{qm}g^{pq}g^{ik}, \tag{20.50}
$$

$$
g^{pm}\Gamma^{n}_{np}(\partial_m g^{ik}) = g^{pm}\Gamma^{n}_{np}(-\Gamma^{i}_{qm}g^{qk} - \Gamma^{k}_{qm}g^{qi}) = -\Gamma^{n}_{np}\Gamma^{i}_{qm}g^{pm}g^{qk} - \Gamma^{n}_{np}\Gamma^{k}_{qm}g^{pm}g^{qi}, \tag{20.51}
$$

$$
g^{pm}g^{ik}[\partial_m\Gamma^{n}_{np}]_{\Gamma\Gamma} = g^{pm}g^{ik}(-\Gamma^{n}_{ms}\Gamma^{s}_{np} - g^{ns}\Gamma^{d}_{ms}\Gamma_{d,np}) = -\Gamma^{n}_{pq}\Gamma^{p}_{nm}g^{qm}g^{ik} - \Gamma^{n}_{pm}\Gamma_{n,qr}g^{pq}g^{rm}g^{ik}. \tag{20.52}
$$

Applying the product rule to all three factors gives six monomials,

$$
\begin{aligned}
\partial_m\Big[\Gamma^{n}_{np}\, g^{pm}\, g^{ik}\Big]_{\Gamma\Gamma} = &- \Gamma^{n}_{np}\Gamma^{p}_{qm}\, g^{qm}g^{ik} - \Gamma^{n}_{np}\Gamma^{m}_{qm}\, g^{pq}g^{ik} \\
&- \Gamma^{n}_{np}\Gamma^{i}_{qm}\, g^{pm}g^{qk} - \Gamma^{n}_{np}\Gamma^{k}_{qm}\, g^{pm}g^{qi} \\
&- \Gamma^{n}_{pq}\Gamma^{p}_{nm}\, g^{qm}g^{ik} - \Gamma^{n}_{pm}\Gamma_{n,qr}\, g^{pq}g^{rm}g^{ik},
\end{aligned}
\tag{20.53}
$$

the first two from differentiating the factor g^{pm} by the exact metric rule, the next two from differentiating g^{ik}, the fifth from applying (20.42) to the trace connection Γ^{n}_{np} itself, and the sixth — the one carrying the first-kind object $\Gamma_{n,qr}$ — from the second piece of that same rule.

The same mechanism applies to each of the remaining five terms of (20.48) — every one of the six is a product of two inverse metrics and one connection, differentiated by the same product rule: differentiate each metric factor by the exact rule $\partial_m g^{ab} = -\Gamma^{a}_{ms}g^{sb} - \Gamma^{b}_{ms}g^{as}$, and the connection factor by the $\Gamma\Gamma$-sector rule (20.42). Carried out for all six, the result is thirty-four raw monomials, six from each of four terms and five from each of the remaining two (the difference occurs because in the second and fourth of the six terms of (20.48) the connection factor itself carries the free index m, so differentiating it produces a self-contraction that absorbs what would otherwise be a separate term):

$$
\begin{aligned}
\partial_m\Big[\Gamma^{n}_{np}\, g^{pm}\, g^{ik}\Big]_{\Gamma\Gamma} = &- \Gamma^{n}_{np}\Gamma^{p}_{qm}\, g^{qm}g^{ik} - \Gamma^{n}_{np}\Gamma^{i}_{qm}\, g^{pm}g^{qk} - \Gamma^{n}_{np}\Gamma^{k}_{qm}\, g^{pm}g^{qi} \\
&- \Gamma^{n}_{np}\Gamma^{m}_{qm}\, g^{pq}g^{ik} - \Gamma^{n}_{pq}\Gamma^{p}_{nm}\, g^{qm}g^{ik} - \Gamma^{n}_{pm}\Gamma_{n,qr}\, g^{pq}g^{rm}g^{ik},
\end{aligned}
$$

$$
\begin{aligned}
\partial_m\Big[-\Gamma^m_{np}\,g^{np}\,g^{ik}\Big]_{\Gamma\Gamma} &= +\Gamma^n_{pq}\Gamma^m_{nm}\,g^{pq}g^{ik}+2\,\Gamma^n_{pm}\Gamma^m_{nq}\,g^{pq}g^{ik}+\Gamma^n_{pm}\Gamma_{n,qr}\,g^{pm}g^{qr}g^{ik}\\
&\quad+\Gamma^i_{nm}\Gamma^m_{pq}\,g^{nk}g^{pq}+\Gamma^k_{nm}\Gamma^m_{pq}\,g^{ni}g^{pq},\\
\partial_m\Big[-\Gamma^n_{np}\,g^{pi}\,g^{km}\Big]_{\Gamma\Gamma} &= +\Gamma^n_{np}\Gamma^p_{qm}\,g^{qi}g^{km}+\Gamma^n_{np}\Gamma^i_{qm}\,g^{pq}g^{km}+\Gamma^n_{np}\Gamma^k_{qm}\,g^{pi}g^{qm}\\
&\quad+\Gamma^n_{np}\Gamma^m_{qm}\,g^{pi}g^{qk}+\Gamma^n_{pq}\Gamma^p_{nm}\,g^{qi}g^{km}+\Gamma^n_{pm}\Gamma_{n,qr}\,g^{pq}g^{ri}g^{km},\\
\partial_m\Big[\Gamma^i_{np}\,g^{np}\,g^{km}\Big]_{\Gamma\Gamma} &= -\Gamma^n_{pq}\Gamma^i_{nm}\,g^{pq}g^{km}-2\,\Gamma^n_{pm}\Gamma^i_{nq}\,g^{pq}g^{km}-\Gamma^n_{pm}\Gamma_{n,qr}\,g^{pi}g^{qr}g^{km}\\
&\quad-\Gamma^i_{np}\Gamma^k_{qm}\,g^{np}g^{qm}-\Gamma^i_{np}\Gamma^m_{qm}\,g^{np}g^{qk},\\
\partial_m\Big[\Gamma^m_{np}\,g^{ni}\,g^{pk}\Big]_{\Gamma\Gamma} &= -\Gamma^n_{pq}\Gamma^m_{nm}\,g^{pi}g^{qk}-\Gamma^n_{pm}\Gamma^m_{nq}\,g^{pi}g^{qk}-\Gamma^n_{pm}\Gamma^m_{nq}\,g^{pk}g^{qi}\\
&\quad-\Gamma^n_{pm}\Gamma_{n,qr}\,g^{pm}g^{qi}g^{rk}-\Gamma^i_{nm}\Gamma^m_{pq}\,g^{np}g^{qk}-\Gamma^k_{nm}\Gamma^m_{pq}\,g^{np}g^{qi},\\
\partial_m\Big[-\Gamma^i_{np}\,g^{nk}\,g^{pm}\Big]_{\Gamma\Gamma} &= +\Gamma^n_{pq}\Gamma^i_{nm}\,g^{pk}g^{qm}+\Gamma^n_{pm}\Gamma^i_{nq}\,g^{pk}g^{qm}+\Gamma^n_{pm}\Gamma^i_{nq}\,g^{pm}g^{qk}\\
&\quad+\Gamma^n_{pm}\Gamma_{n,qr}\,g^{pi}g^{qk}g^{rm}+\Gamma^i_{np}\Gamma^k_{qm}\,g^{nq}g^{pm}+\Gamma^i_{np}\Gamma^m_{qm}\,g^{nq}g^{pk}.
\end{aligned}
$$

Thirty-four monomials come from $\partial_m B^{imk}$, and six more from $2\Gamma^r_{mr}B^{imk}$, giving forty in total. Reducing each to a canonical dummy order, eighteen of the forty cancel in pairs under the cyclic relabelling $n\to p,\ p\to q,\ q\to m,\ m\to n$, followed by $\Gamma^n_{pn}=\Gamma^n_{np}$ resulting in 22 distinct pairs. These remaining twenty-two split into two classes:

- sixteen built from $g^{-1}g^{-1}\Gamma\Gamma$ alone;
- six carrying the first-kind connection $\Gamma_{d,st}$, all with the common prefactor $\Gamma^p_{qr}\Gamma_{p,st}$ that also appears in the Einstein tensor's quadratic sector.

The six first-kind monomials are

$$
\begin{aligned}
\Gamma^p_{qr}\Gamma_{p,st}\,\Big(&+g^{qr}g^{st}g^{ik}-g^{qr}g^{si}g^{tk}-g^{qs}g^{rt}g^{ik}\\
&+g^{qs}g^{ri}g^{tk}+g^{qs}g^{rk}g^{ti}-g^{qi}g^{rk}g^{st}\Big),
\end{aligned}
\tag{20.54}
$$

and the sixteen, already the complete pure sector on the superpotential side alone, are

$$
\begin{aligned}
&-2\,\Gamma^l_{lm}\Gamma^m_{np}\,g^{np}g^{ik}+2\,\Gamma^l_{lm}\Gamma^m_{np}\,g^{ni}g^{pk}\\
&+\Gamma^l_{lm}\Gamma^n_{np}\,g^{mp}g^{ik}-\Gamma^l_{lm}\Gamma^n_{np}\,g^{mi}g^{pk}\\
&+\Gamma^l_{mn}\Gamma^m_{lp}\,g^{np}g^{ik}-\Gamma^l_{mn}\Gamma^m_{lp}\,g^{ni}g^{pk}\\
&-\Gamma^l_{lm}\Gamma^i_{np}\,g^{mn}g^{pk}+\Gamma^l_{lm}\Gamma^i_{np}\,g^{mk}g^{np}\\
&+\Gamma^l_{mn}\Gamma^i_{lp}\,g^{mn}g^{pk}-\Gamma^l_{mn}\Gamma^i_{lp}\,g^{mp}g^{nk}\\
&-\Gamma^l_{lm}\Gamma^k_{np}\,g^{mn}g^{pi}+\Gamma^l_{lm}\Gamma^k_{np}\,g^{mi}g^{np}\\
&+\Gamma^l_{mn}\Gamma^k_{lp}\,g^{mn}g^{pi}-\Gamma^l_{mn}\Gamma^k_{lp}\,g^{mp}g^{ni}\\
&-\Gamma^i_{lm}\Gamma^k_{np}\,g^{lm}g^{np}+\Gamma^i_{lm}\Gamma^k_{np}\,g^{ln}g^{mp}.
\end{aligned}
\tag{20.55}
$$

This is already, term for term, the final answer (20.59) below. The Einstein tensor's quadratic sector (20.47) consists entirely of first-kind monomials of which all four share the prefactor $\Gamma^p_{qr}\Gamma_{p,st}$ and has no $\Gamma\Gamma$ part. Therefore subtracting it can only act on the six first-kind monomials (20.54) of the superpotential side, and leaves the sixteen pure monomials (20.55) completely untouched.

Write the six superpotential monomials and the four Einstein monomials in parallel columns, both with the common prefactor $\Gamma^p_{qr}\Gamma_{p,st}$ stripped off:

(20.54)	(20.47)	result
$+g^{qr}g^{st}g^{ik}$	$+g^{qr}g^{st}g^{ik}$	0
$-g^{qr}g^{si}g^{tk}$	$-2g^{qr}g^{si}g^{tk}$	$+g^{qr}g^{si}g^{tk}$
$-g^{qs}g^{rt}g^{ik}$	$-g^{qs}g^{rt}g^{ik}$	0
$+g^{qs}g^{ri}g^{tk}$	0	$+g^{qs}g^{ri}g^{tk}$
$+g^{qs}g^{rk}g^{ti}$	$+2g^{qs}g^{rk}g^{ti}$	$-g^{qs}g^{rk}g^{ti}$
$-g^{qi}g^{rk}g^{st}$	0	$-g^{qi}g^{rk}g^{st}$

The whole remaining task is therefore to check that (20.54) minus (20.47) vanishes; doing so, the first and third terms match exactly and cancel outright, leaving the four-term identity

$$T_1 + T_2 - T_3 - T_4 = 0, \tag{20.56}$$

with

$$T_1 \equiv \Gamma^p_{qr}\Gamma_{p,st}\, g^{qr}g^{si}g^{tk}, \qquad T_2 \equiv \Gamma^p_{qr}\Gamma_{p,st}\, g^{qs}g^{ri}g^{tk},$$
$$T_3 \equiv \Gamma^p_{qr}\Gamma_{p,st}\, g^{qs}g^{rk}g^{ti}, \qquad T_4 \equiv \Gamma^p_{qr}\Gamma_{p,st}\, g^{qi}g^{rk}g^{st}.$$

This is not zero by dummy relabelling of a single term, but both pairs vanish by one and the same mechanism: for any index pairs (a,b) and (c,d),

$$\sum_p \Gamma^p_{ab}\,\Gamma_{p,cd} = \sum_p \Gamma^p_{cd}\,\Gamma_{p,ab}, \tag{20.57}$$

which follows from $\Gamma^p_{ab} = g^{pe}\Gamma_{e,ab}$ and the symmetry of g^{pe}: relabelling the dummy pair (p,e) reproduces the left side with (ab) and (cd) exchanged. Now relabel the summed pair (q,r) as (s,t) and (s,t) as (q,r) throughout T_3 and, separately, throughout T_4 — valid since all four indices are fully summed in each. This turns

$$T_3 = \Gamma^p_{qr}\Gamma_{p,st}\, g^{qs}g^{rk}g^{ti} \;\longrightarrow\; \Gamma^p_{st}\Gamma_{p,qr}\, g^{qs}g^{ri}g^{tk}, \qquad T_4 = \Gamma^p_{qr}\Gamma_{p,st}\, g^{qi}g^{rk}g^{st} \;\longrightarrow\; \Gamma^p_{st}\Gamma_{p,qr}\, g^{qr}g^{si}g^{tk}, \tag{20.58}$$

and applying (20.57) with $(a,b)=(s,t)$, $(c,d)=(q,r)$ turns $\Gamma^p_{st}\Gamma_{p,qr}$ back into $\Gamma^p_{qr}\Gamma_{p,st}$ in both. The relabelled T_3 is then exactly T_2, and the relabelled T_4 is exactly T_1; hence $T_2 = T_3$ and $T_1 = T_4$, and both pairs cancel — confirmed also by direct numerical substitution of g^{ab}, Γ^a_{bc} and the matrix inverse g_{ab} at independent random values, with $T_1 - T_4$ and $T_2 - T_3$ each vanishing separately to 10^{-40}.

With (20.56) vanishing, every first-kind term drops out of (20.40), and what survives is the sixteen-term, purely $\Gamma\Gamma\, g^{-1}g^{-1}$ expression:

$$\begin{aligned} 16\pi G\, t^{ik}_{\mathrm{LL}} = {} & -2\,\Gamma^l_{lm}\Gamma^m_{np}\, g^{np}g^{ik} + 2\,\Gamma^l_{lm}\Gamma^m_{np}\, g^{ni}g^{pk} \\ & + \Gamma^l_{lm}\Gamma^n_{np}\, g^{mp}g^{ik} - \Gamma^l_{lm}\Gamma^n_{np}\, g^{mi}g^{pk} \\ & + \Gamma^l_{mn}\Gamma^m_{lp}\, g^{np}g^{ik} - \Gamma^l_{mn}\Gamma^m_{lp}\, g^{ni}g^{pk} \\ & - \Gamma^l_{lm}\Gamma^i_{np}\, g^{mn}g^{pk} + \Gamma^l_{lm}\Gamma^i_{np}\, g^{mk}g^{np} \\ & + \Gamma^l_{mn}\Gamma^i_{lp}\, g^{mn}g^{pk} - \Gamma^l_{mn}\Gamma^i_{lp}\, g^{mp}g^{nk} \\ & - \Gamma^l_{lm}\Gamma^k_{np}\, g^{mn}g^{pi} + \Gamma^l_{lm}\Gamma^k_{np}\, g^{mi}g^{np} \\ & + \Gamma^l_{mn}\Gamma^k_{lp}\, g^{mn}g^{pi} - \Gamma^l_{mn}\Gamma^k_{lp}\, g^{mp}g^{ni} \\ & - \Gamma^i_{lm}\Gamma^k_{np}\, g^{lm}g^{np} + \Gamma^i_{lm}\Gamma^k_{np}\, g^{ln}g^{mp}. \end{aligned} \tag{20.59}$$

The six monomials with no free index on either connection and the two with a free index on both are symmetric under $i \leftrightarrow k$ term by term; the remaining eight fall into i-carrying and k-carrying pairs that exchange under it.

With the first-kind sector gone, the result can be returned to its conventional, more compact arrangement. Factoring the shared metric structures out of each class returns (20.59) to

$$\begin{aligned} 16\pi G\, t_{\mathrm{LL}}^{ik} = {} & \left(2\Gamma^{n}_{lm}\Gamma^{p}_{np} - \Gamma^{n}_{lp}\Gamma^{p}_{mn} - \Gamma^{n}_{ln}\Gamma^{p}_{mp}\right)\left(g^{il}g^{km} - g^{ik}g^{lm}\right) \\ & + g^{il}g^{mn}\left(\Gamma^{k}_{lp}\Gamma^{p}_{mn} + \Gamma^{k}_{mn}\Gamma^{p}_{lp} - \Gamma^{k}_{np}\Gamma^{p}_{lm} - \Gamma^{k}_{lm}\Gamma^{p}_{np}\right) \\ & + g^{kl}g^{mn}\left(\Gamma^{i}_{lp}\Gamma^{p}_{mn} + \Gamma^{i}_{mn}\Gamma^{p}_{lp} - \Gamma^{i}_{np}\Gamma^{p}_{lm} - \Gamma^{i}_{lm}\Gamma^{p}_{np}\right) \\ & + g^{lm}g^{np}\left(\Gamma^{i}_{ln}\Gamma^{k}_{mp} - \Gamma^{i}_{lm}\Gamma^{k}_{np}\right). \end{aligned} \tag{20.60}$$

The physical content of the result is read off from its form. It vanishes wherever the connection vanishes, which is at any chosen point in a suitably adapted frame: the field can always be transformed away locally, and with it any energy one had ascribed to it. Only the integral of (20.60) over a region, converted by (20.38) into a surface integral of the superpotential over the boundary, carries a meaning independent of the coordinates.

20.3.3 Symmetries of the Landau–Lifshitz Superpotential

The superpotential λ^{imkl} possesses the same algebraic symmetries as the Riemann curvature tensor:

$$\lambda^{imkl} = -\lambda^{mikl} \quad \text{(antisymmetric in first pair } i \leftrightarrow m\text{)}, \tag{20.61}$$

$$\lambda^{imkl} = -\lambda^{imlk} \quad \text{(antisymmetric in second pair } k \leftrightarrow l\text{)}, \tag{20.62}$$

$$\lambda^{imkl} = \lambda^{klim} \quad \text{(symmetric under pair interchange } (im) \leftrightarrow (kl)\text{)}. \tag{20.63}$$

Because of the antisymmetry in (m, l), the double divergence satisfies:

$$\partial_k\left(\partial_l\partial_m\lambda^{imkl}\right) \equiv 0, \tag{20.64}$$

which guarantees that the total energy-momentum complex is **identically conserved** in any coordinate system:

$$\partial_k\left[(-g)\left(T^{ik}_{\mathrm{matter}} + t^{ik}_{\mathrm{LL}}\right)\right] = 0. \tag{20.65}$$

Key Concept: Properties of the Landau–Lifshitz Pseudotensor t^{ik}_{LL}

The Landau–Lifshitz energy-momentum pseudotensor t^{ik}_{LL} satisfies several remarkable properties:

1. **Strict Symmetry:** $t^{ik}_{\mathrm{LL}} = t^{ki}_{\mathrm{LL}}$, ensuring that angular momentum conservation follows automatically without requiring additional spin-correction terms.

2. **Vanishing in LIF:** At any given spacetime event P, coordinates can be chosen such that $\Gamma^{a}_{bc}(P) = 0$, causing $t^{ik}_{\mathrm{LL}}(P) = 0$. This embodies the Equivalence Principle: gravitational energy cannot be localized.

3. **Quadratic in First Derivatives:** t^{ik}_{LL} is purely homogeneous of degree 2 in Christoffel symbols $\Gamma\Gamma$, with no second derivatives of the metric:

$$16\pi G\, t^{ik}_{\mathrm{LL}} = \left(2\Gamma^{n}_{lm}\Gamma^{p}_{np} - \Gamma^{n}_{lp}\Gamma^{p}_{mn} - \Gamma^{n}_{ln}\Gamma^{p}_{mp}\right)\left(g^{il}g^{km} - g^{ik}g^{lm}\right)$$

$$
\begin{aligned}
&+ g^{il} g^{mn} \left(\Gamma^k_{lp} \Gamma^p_{mn} + \Gamma^k_{mn} \Gamma^p_{lp} - \Gamma^k_{np} \Gamma^p_{lm} - \Gamma^k_{lm} \Gamma^p_{np} \right) \\
&+ g^{kl} g^{mn} \left(\Gamma^i_{lp} \Gamma^p_{mn} + \Gamma^i_{mn} \Gamma^p_{lp} - \Gamma^i_{np} \Gamma^p_{lm} - \Gamma^i_{lm} \Gamma^p_{np} \right) \\
&+ g^{lm} g^{np} \left(\Gamma^i_{ln} \Gamma^k_{mp} - \Gamma^i_{lm} \Gamma^k_{np} \right).
\end{aligned} \tag{20.66}
$$

20.4 Total Conserved Quantities in Landau–Lifshitz Theory

20.4.1 Conserved 4-Momentum Vector

The total 4-momentum P^i of an isolated system is obtained by integrating the component $(-g)(T^{i0} + t^{i0}_{\mathrm{LL}})$ over a 3-dimensional constant-time spatial volume V:

$$
P^i \equiv \int_V \mathrm{d}^3x \, (-g) \left(T^{i0} + t^{i0}_{\mathrm{LL}} \right) = \frac{1}{16\pi G} \int_V \mathrm{d}^3x \, \partial_\alpha \partial_m \lambda^{im0\alpha}. \tag{20.67}
$$

Because the $m = 0$ term vanishes by antisymmetry ($\lambda^{i00\alpha} = -\lambda^{i0\alpha0}$), the integral reduces to a spatial boundary flux over the 2-sphere at spatial infinity S_∞:

$$
P^i = \frac{1}{16\pi G} \oint_{S_\infty} \mathrm{d}^2 S_\alpha \, \partial_\beta \left[(-g) \left(g^{i0} g^{\alpha\beta} - g^{i\beta} g^{0\alpha} \right) \right]. \tag{20.68}
$$

20.4.2 Conserved Total Angular Momentum Tensor

The total conserved angular momentum tensor M^{ik} in the Landau–Lifshitz formalism is:

$$
M^{ik} \equiv \int_V \mathrm{d}^3x \left[x^i (-g) \mathcal{T}^{k0} - x^k (-g) \mathcal{T}^{i0} \right]. \tag{20.69}
$$

Evaluating the boundary integral as established in Section 20.2:

$$
M^{ik} = \oint_{S_\infty} \mathrm{d}^2 S_\alpha \left[x^i h^{k0\alpha} - x^k h^{i0\alpha} + \lambda^{ik0\alpha} \right], \tag{20.70}
$$

where $h^{k0\alpha} \equiv \partial_m \lambda^{km0\alpha}$.

20.5 Geometric Transport: Parallel Transport vs. Lie Transport

20.5.1 Connection-Dependent Parallel Transport

Let C be a smooth curve parameterized by λ with tangent vector $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$. A vector field v^b is said to be **parallel transported** along C if its directional covariant derivative vanishes:

$$
\frac{Dv^b}{\mathrm{d}\lambda} \equiv u^a \nabla_a v^b = 0. \tag{20.71}
$$

In local coordinates, this translates to the differential equation:

$$
\frac{\mathrm{d}v^a}{\mathrm{d}\lambda} + \Gamma^a_{bc} \, u^b v^c = 0. \tag{20.72}
$$

Integrating along the curve from point P to point Q:

$$
v^a(Q) = v^a(P) - \int_P^Q \Gamma^a_{bc}(x(\lambda)) \, v^b(\lambda) \, \frac{\mathrm{d}x^c}{\mathrm{d}\lambda} \, \mathrm{d}\lambda. \tag{20.73}
$$

Physical Insight & Motivation: Connection-Dependence of Parallel Transport

Notice from Eq. (20.73) that parallel transport **strictly requires knowledge of the connection Γ^a_{bc} (and thus the metric g_{ab})**. Without a metric or affine connection, the concept of parallel transport is undefined.

20.5.2 Connection-Independent Lie Transport

In contrast, consider transporting a geometric object along the integral curves (flow) of a vector field u^a via **Lie dragging**:

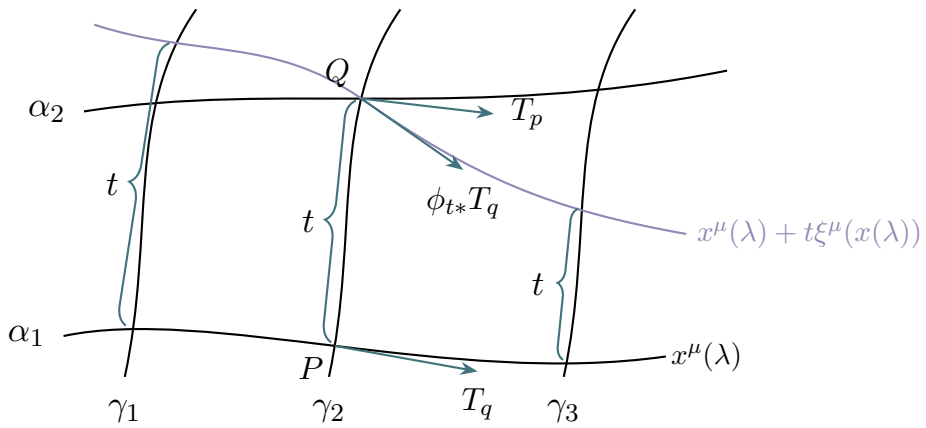
$$\mathcal{L}_u v^b \equiv [u, v]^b = u^a \partial_a v^b - v^a \partial_a u^b = 0. \tag{20.74}$$

Key Concept: Parallel Transport vs. Lie Transport

- **Underlying Geometric Structure:**
  - *Parallel Transport ($\nabla_u v = 0$):* Requires an affine connection Γ^a_{bc} or metric g_{ab} to define directional covariant differentiation.
  - *Lie Transport ($\mathcal{L}_u v = 0$):* Defined purely on a bare differentiable manifold $\mathcal{M}$ via the Lie bracket $[u, v]$, without any metric or connection structure!
- **Governing Equations:**
  - *Parallel Transport:* $u^a \nabla_a v^b = 0 \iff \frac{\mathrm{d}v^b}{\mathrm{d}\lambda} + \Gamma^b_{ac} u^a v^c = 0.$
  - *Lie Transport:* $\mathcal{L}_u v^b = 0 \iff [u, v]^b = u^a \partial_a v^b - v^a \partial_a u^b = 0.$
- **Geometric Meaning:**
  - *Parallel Transport:* Preserves inner products, lengths, and angles along the trajectory, but is path-dependent in the presence of curvature $R^a{}_{bcd}$.
  - *Lie Transport:* Drags the vector field v^b along the 1-parameter diffeomorphism flow generated by u^a, preserving the commutator algebra of vector fields.

20.5.3 Geometric Derivation of the Lie Derivative via Integral Curves and Flows

One intuitive way to derive the Lie derivative is with the notion of integral curves:



Let ξ be a vector field and let its integral curves be

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = \xi^\mu(x). \tag{20.75}$$

The integral curves generate a 1-parameter family of diffeomorphism maps $\phi_\epsilon : \mathcal{M} \to \mathcal{M}$, called the **flow** of ξ. For an infinitesimal parameter ϵ:

$$\phi_\epsilon(x)^\mu = x^\mu + \epsilon\, \xi^\mu(x) + \mathcal{O}(\epsilon^2). \tag{20.76}$$

Now consider another vector field $T = T^\mu \partial_\mu$ whose integral curves satisfy $\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} = T^\mu(x)$. We ask how T changes under an infinitesimal displacement generated by the flow ϕ_ϵ.

The flow ϕ_ϵ moves the points of the manifold and therefore maps the curve $x^\mu(\lambda)$ to a displaced curve:

$$\bar{x}^\mu(\lambda) = \phi_\epsilon(x(\lambda))^\mu = x^\mu(\lambda) + \epsilon\, \xi^\mu(x(\lambda)) + \mathcal{O}(\epsilon^2). \tag{20.77}$$

The tangent vector to this displaced curve is:

$$\bar{T} = \frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}\lambda}\bar{\partial}_\mu = \left[T^\mu + \epsilon\, T^\nu \partial_\nu \xi^\mu\right] \bar{\partial}_\mu + \mathcal{O}(\epsilon^2). \tag{20.78}$$

However, $\bar{T}$ is naturally expressed in the coordinate basis associated with the flowed coordinates $\bar{x}^\mu = \phi_\epsilon^\mu(x)$. To compare it with the original vector field T at the same point, we express the new basis in terms of the original basis via the inverse transformation $x^\mu = \bar{x}^\mu - \epsilon\, \xi^\mu(\bar{x}) + \mathcal{O}(\epsilon^2)$:

$$\bar{\partial}_\mu \equiv \frac{\partial}{\partial \bar{x}^\mu} = \frac{\partial x^\nu}{\partial \bar{x}^\mu}\partial_\nu = \left(\delta^\nu_\mu - \epsilon\, \partial_\mu \xi^\nu\right) \partial_\nu + \mathcal{O}(\epsilon^2). \tag{20.79}$$

Therefore, the pulled-back vector field evaluated back at the original base point is:

$$\begin{aligned} T^\mu_\epsilon &= \left[T^\nu + \epsilon\, \xi^\rho \partial_\rho T^\nu\right] \left[\delta^\mu_\nu - \epsilon\, \partial_\nu \xi^\mu\right] + \mathcal{O}(\epsilon^2) \\ &= T^\mu + \epsilon\, (\xi^\nu \partial_\nu T^\mu - T^\nu \partial_\nu \xi^\mu) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{20.80}$$

Thus, the infinitesimal change of T under the flow generated by ξ is:

$$\mathcal{L}_\xi T \equiv \lim_{\epsilon \to 0} \frac{T_\epsilon - T}{\epsilon} = (\xi^\nu \partial_\nu T^\mu - T^\nu \partial_\nu \xi^\mu)\, \partial_\mu. \tag{20.81}$$

Recognizing the Lie bracket $[\xi, T]^\mu \equiv \xi^\nu \partial_\nu T^\mu - T^\nu \partial_\nu \xi^\mu$, we obtain the fundamental identity:

$$\boxed{\mathcal{L}_\xi T = [\xi, T]} \tag{20.82}$$

Thus, the Lie derivative measures the intrinsic rate of change of the vector field T under the active flow generated by ξ, completely independent of any metric or affine connection.

20.5.4 Geodesic Motion and Conserved Momenta from Continuous Symmetries

For a particle of rest mass m moving along a geodesic $x^a(\lambda)$ with 4-velocity $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$, the geodesic equation $u^a \nabla_a u^b = 0$ can be expressed in canonical momentum form $p_l = m g_{lk} u^k$:

$$m\frac{\mathrm{d}u_l}{\mathrm{d}\lambda} = \frac{1}{2}\partial_l(g_{ik})\, u^i u^k. \tag{20.83}$$

From Eq. (20.83), we observe directly:

- If the metric components g_{ik} are **independent of a coordinate** x^l (i.e., $\partial_l g_{ik} = 0$), then the conjugate covariant momentum component is strictly conserved along the geodesic:

$$\frac{\mathrm{d}p_l}{\mathrm{d}\lambda} = 0 \implies p_l = \text{constant}. \tag{20.84}$$

- While coordinate-dependent descriptions require finding symmetry-adapted coordinate systems, the coordinate-free characterization is given by **Killing vector fields** ξ^a:

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0 \implies Q \equiv \xi_a u^a = \text{constant along geodesics.} \tag{20.85}$$

This connection between Lie transport, spacetime symmetries, and conserved geodesic constants will serve as the foundation for analyzing geodesic congruences and Raychaudhuri dynamics in subsequent chapters.

Chapter 21

Diffeomorphisms, Lie Derivatives, and Killing Vector Fields

In curved spacetime, the conventional notion of a conserved quantity arising from a cyclic coordinate (i.e., $\partial_l g_{ab} = 0$) is coordinate-dependent: an isometry may exist physically, but be obscured by an inconvenient choice of coordinates. To describe symmetries, conservation laws, and geometric deformations in a coordinate-independent manner, we require the machinery of **diffeomorphisms**, **Lie derivatives**, and **Killing vector fields**.

In this chapter, we develop the Lie derivative from first principles:

1. We introduce **congruences of curves** and the geometric definition of the Lie derivative for scalars, vectors, 1-forms, and arbitrary rank-(k, l) tensors.
2. We present the **geometric parallelogram loop construction**, demonstrating through rigorous second-order Taylor expansions that the non-closure of infinitesimal flows measures precisely the Lie bracket $[u, v]$.
3. We define **spacetime isometries** and derive the **Killing equation** $\nabla_{(a}\xi_{b)} = 0$, showing how Killing vector fields systematically generate conserved quantities along particle geodesics and divergence-free currents for relativistic matter.

21.1 From Coordinate Symmetries to Vector Field Generators

21.1.1 Conserved Momenta in Lagrangian Mechanics

Recall the relativistic point particle action parameterized by an affine parameter λ:

$$S = -m \int \mathrm{d}\lambda \sqrt{-g_{ab}(x)\, \dot{x}^a \dot{x}^b}, \qquad \dot{x}^a \equiv \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}. \tag{21.1}$$

Applying the Euler–Lagrange variational principle $\delta S = 0$, the equation of motion for the covariant 4-momentum $p_l \equiv m g_{lk} \dot{x}^k$ is:

$$\frac{\mathrm{d}p_l}{\mathrm{d}\lambda} = \frac{1}{2m} \partial_l (g_{ik})\, p^i p^k. \tag{21.2}$$

If the metric components happen to be independent of a particular coordinate x^l ($\partial_l g_{ab} = 0$), then the conjugate momentum component is strictly conserved:

$$\partial_l g_{ab} = 0 \implies \frac{\mathrm{d}p_l}{\mathrm{d}\lambda} = 0 \implies p_l = \text{constant}. \tag{21.3}$$

Physical Insight & Motivation: Limitation of Coordinate-Based Symmetries

The condition $\partial_l g_{ab} = 0$ is inherently coordinate-dependent. For example, flat Euclidean space in Cartesian coordinates satisfies $\partial_x \delta_{ij} = 0$, but in polar coordinates $g_{rr} = 1, g_{\theta\theta} = r^2$, the metric explicitly depends on r ($\partial_r g_{\theta\theta} \neq 0$), even though the underlying flat geometry is completely isotropic and homogeneous.
Therefore, we must reformulate the concept of a symmetry direction using **vector fields** $\xi^a(x)$ rather than coordinate axes.

21.2 Congruences of Curves and the Lie Derivative of Vector Fields

21.2.1 Congruence of Geodesics

A **congruence** in an open region $\mathcal{U} \subseteq \mathcal{M}$ is a family of smooth curves such that through each event $P \in \mathcal{U}$ there passes precisely one curve. Consider a 1-parameter family of curves forming a smooth 2-surface $x^a(\lambda, s)$, where λ is the parameter along each curve and s labels neighboring curves (Figure 21.1).

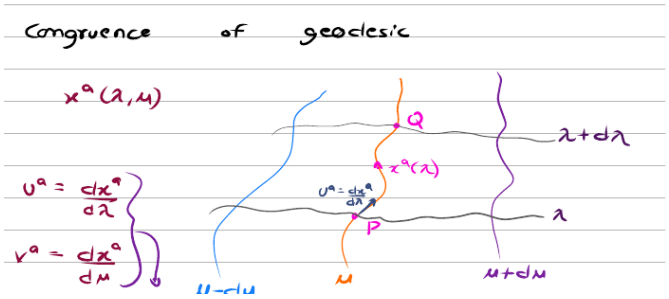


Figure 21.1: A congruence of curves $x^a(\lambda, s)$ with tangent velocity vector $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ and transverse separation vector $v^a = \frac{\mathrm{d}x^a}{\mathrm{d}s}$.

The tangent vector field u^a and transverse separation vector field v^a are defined as:

$$u^a \equiv \frac{\partial x^a}{\partial \lambda}, \qquad v^a \equiv \frac{\partial x^a}{\partial s}. \tag{21.4}$$

Because the curves in the congruence do not intersect, the parameters (λ, s) define a regular local coordinate system on the 2D sheet.

21.2.2 Active Diffeomorphisms and Lie Dragging

Let $u^a(x)$ be a smooth vector field generating an infinitesimal active coordinate shift (pushforward flow):

$$x'^a = x^a + \mathrm{d}x^a = x^a + u^a(x)\,\mathrm{d}\lambda. \tag{21.5}$$

Under this displacement, a vector field $v^a(x)$ at x transforms via the Jacobian matrix:

$$v'^a(x') = \frac{\partial x'^a}{\partial x^m} v^m(x) = (\delta^a_m + \partial_m u^a\,\mathrm{d}\lambda)\, v^m(x) = v^a(x) + v^m(x)\partial_m u^a\,\mathrm{d}\lambda. \tag{21.6}$$

To compare the transported vector with the original vector field evaluated at the new point x', we define the **Lie derivative** $\mathcal{L}_u v^a$:

$$\mathcal{L}_u v^a \equiv \lim_{\mathrm{d}\lambda \to 0} \frac{v^a(x + \mathrm{d}x) - v'^a(x + \mathrm{d}x)}{\mathrm{d}\lambda}. \tag{21.7}$$

Taylor-expanding $v^a(x+\mathrm{d}x) = v^a(x) + u^b\partial_b v^a\,\mathrm{d}\lambda$:

$$\begin{aligned}\mathcal{L}_u v^a &= \lim_{\mathrm{d}\lambda\to 0}\frac{\left[v^a(x)+u^b\partial_b v^a\,\mathrm{d}\lambda\right]-\left[v^a(x)+v^b\partial_b u^a\,\mathrm{d}\lambda\right]}{\mathrm{d}\lambda}\\ &= u^b\partial_b v^a - v^b\partial_b u^a. \end{aligned}\tag{21.8}$$

Theorem: Torsion-Free Covariant Form of the Lie Derivative

In any torsion-free connection ($\Gamma^a_{bc} = \Gamma^a_{cb}$), the partial derivatives in Eq. (21.8) can be replaced directly by covariant derivatives:

$$\begin{aligned}\mathcal{L}_u v^a &= u^b\left(\nabla_b v^a - \Gamma^a_{bc}v^c\right) - v^b\left(\nabla_b u^a - \Gamma^a_{bc}u^c\right)\\ &= u^b\nabla_b v^a - v^b\nabla_b u^a - \Gamma^a_{bc}u^b v^c + \Gamma^a_{cb}v^b u^c\\ &= u^b\nabla_b v^a - v^b\nabla_b u^a. \end{aligned}\tag{21.9}$$

Because the connection terms cancel identically, the Lie derivative $[u,v]^a \equiv \mathcal{L}_u v^a$ is a **genuine, connection-independent** $(1,0)$ **tensor**.

21.3 Lie Derivatives of Scalars, 1-Forms, and General Tensors

21.3.1 Lie Derivative of a Scalar Field

For a scalar field $f(x)$, under the infinitesimal diffeomorphism $x'^a = x^a + \xi^a\,\mathrm{d}\lambda$:

$$\begin{aligned}\mathcal{L}_\xi f &\equiv \lim_{\mathrm{d}\lambda\to 0}\frac{f(x+\xi\,\mathrm{d}\lambda)-f(x)}{\mathrm{d}\lambda}\\ &= \lim_{\mathrm{d}\lambda\to 0}\frac{f(x)+\xi^a\partial_a f\,\mathrm{d}\lambda - f(x)}{\mathrm{d}\lambda} = \xi^a\partial_a f = \xi^a\nabla_a f. \end{aligned}\tag{21.10}$$

21.3.2 Lie Derivative of a 1-Form (Dual Vector)

To determine the Lie derivative of a covariant 1-form A_a, we demand that the Lie derivative satisfies the **Leibniz product rule** on the scalar contraction A_aB^a:

$$\mathcal{L}_\xi\left(A_aB^a\right) = \xi^b\partial_b\left(A_aB^a\right) = \left(\xi^b\partial_b A_a\right)B^a + A_a\left(\xi^b\partial_b B^a\right).\tag{21.11}$$

Applying the Leibniz rule to the left-hand side:

$$\mathcal{L}_\xi\left(A_aB^a\right) = \left(\mathcal{L}_\xi A_a\right)B^a + A_a\left(\mathcal{L}_\xi B^a\right).\tag{21.12}$$

Substituting the known vector formula $\mathcal{L}_\xi B^a = \xi^b\partial_b B^a - B^b\partial_b\xi^a$:

$$\left(\mathcal{L}_\xi A_a\right)B^a + A_a\left(\xi^b\partial_b B^a - B^b\partial_b\xi^a\right) = \left(\xi^b\partial_b A_a\right)B^a + A_a\left(\xi^b\partial_b B^a\right).\tag{21.13}$$

The term $A_a\xi^b\partial_b B^a$ cancels from both sides:

$$\left(\mathcal{L}_\xi A_a\right)B^a - A_bB^a\partial_a\xi^b = \left(\xi^b\partial_b A_a\right)B^a \implies \left(\mathcal{L}_\xi A_a\right)B^a = \left(\xi^b\partial_b A_a + A_b\partial_a\xi^b\right)B^a.\tag{21.14}$$

Since this must hold for an arbitrary vector field B^a, we obtain:

$$\mathcal{L}_\xi A_a = \xi^b\partial_b A_a + A_b\partial_a\xi^b = \xi^b\nabla_b A_a + A_b\nabla_a\xi^b.\tag{21.15}$$

21.3.3 General Rank-(k,l) Tensor Formula

Key Concept: General Formula for the Lie Derivative of a Tensor

For an arbitrary tensor field $T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$, the Lie derivative along a vector field ξ^m is:

$$\mathcal{L}_\xi T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = \xi^m \nabla_m T^{a_1\ldots a_k}{}_{b_1\ldots b_l} - \sum_{i=1}^{k} T^{a_1\ldots m\ldots a_k}{}_{b_1\ldots b_l} \nabla_m \xi^{a_i} + \sum_{j=1}^{l} T^{a_1\ldots a_k}{}_{b_1\ldots m\ldots b_l} \nabla_{b_j} \xi^m. \tag{21.16}$$

Equivalently, all covariant derivatives ∇ can be replaced with partial derivatives ∂, as all Christoffel connection symbols Γ cancel identically.

21.4 Geometric Parallelogram Non-Closure and the Lie Bracket

21.4.1 The 4-Point Infinitesimal Loop Construction

Consider two vector fields $u = \frac{\partial}{\partial \lambda}$ and $v = \frac{\partial}{\partial s}$. We construct a closed quadrilateral path by flowing along their integral curves from a base point P_0 (Figure 21.2):

- **Path 1 ($P_0 \to P_1 \to P_4$):** Flow along u by parameter $\Delta\lambda$ to point P_1, then along v by parameter Δs to point P_4.
- **Path 2 ($P_0 \to P_2 \to P_3$):** Flow along v by parameter Δs to point P_2, then along u by parameter $\Delta\lambda$ to point P_3.

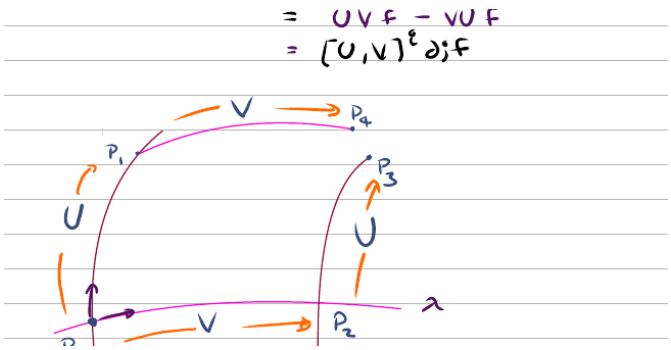


Figure 21.2: Geometric quadrilateral formed by the flows of vector fields u and v. The failure of the loop to close ($P_4 \neq P_3$) measures the Lie commutator $[u, v]$.

Let f be an arbitrary smooth test function on $\mathcal{M}$. We compute the difference $f(P_4) - f(P_3)$ by expanding around the intermediate vertices:

$$f(P_4) - f(P_3) = [f(P_4) - f(P_1)] + [f(P_1) - f(P_0)] - [f(P_2) - f(P_0)] - [f(P_3) - f(P_2)]\,. \tag{21.17}$$

21.4.2 Second-Order Taylor Expansions

Taylor-expanding each segment to second order:

$$f(P_4) - f(P_1) = \left(v^i \partial_i f\right)_{P_1} \Delta s + \frac{1}{2}\left(v^i v^j \partial_i \partial_j f\right)_{P_1} (\Delta s)^2 + \mathcal{O}(3), \tag{21.18}$$

$$f(P_1) - f(P_0) = \left(u^i \partial_i f\right)_{P_0} \Delta\lambda + \frac{1}{2}\left(u^i u^j \partial_i \partial_j f\right)_{P_0} (\Delta\lambda)^2 + \mathcal{O}(3), \tag{21.19}$$

$$f(P_2) - f(P_0) = \left(v^i \partial_i f\right)_{P_0} \Delta s + \frac{1}{2}\left(v^i v^j \partial_i \partial_j f\right)_{P_0} (\Delta s)^2 + \mathcal{O}(3), \tag{21.20}$$

$$f(P_3) - f(P_2) = \left(u^i \partial_i f\right)_{P_2} \Delta\lambda + \frac{1}{2}\left(u^i u^j \partial_i \partial_j f\right)_{P_2} (\Delta\lambda)^2 + \mathcal{O}(3). \tag{21.21}$$

Now expand the quantities evaluated at P_1 and P_2 around the origin P_0:

$$\left(v^i \partial_i f\right)_{P_1} = \left(v^i \partial_i f\right)_{P_0} + \partial_j \left(v^i \partial_i f\right)_{P_0} u^j \, \Delta\lambda + \mathcal{O}(2), \tag{21.22}$$

$$\left(u^i \partial_i f\right)_{P_2} = \left(u^i \partial_i f\right)_{P_0} + \partial_j \left(u^i \partial_i f\right)_{P_0} v^j \, \Delta s + \mathcal{O}(2). \tag{21.23}$$

Substituting these into Eq. (21.17):

- The zeroth-order terms cancel identically: $(u^i \partial_i f)_0 \Delta\lambda - (u^i \partial_i f)_0 \Delta\lambda = 0$ and $(v^i \partial_i f)_0 \Delta s - (v^i \partial_i f)_0 \Delta s = 0$.
- The pure second-order terms cancel: $\frac{1}{2}(u^i u^j \partial_i \partial_j f)_0 (\Delta\lambda)^2 - \frac{1}{2}(u^i u^j \partial_i \partial_j f)_0 (\Delta\lambda)^2 = 0$, and similarly for $(\Delta s)^2$.

The only non-vanishing terms are the mixed cross-terms of order $\Delta\lambda\,\Delta s$:

$$f(P_4) - f(P_3) = \left[u^j \partial_j \left(v^i \partial_i f\right) - v^j \partial_j \left(u^i \partial_i f\right)\right]_{P_0} \Delta\lambda\,\Delta s. \tag{21.24}$$

Applying the product rule:

$$\begin{aligned} u^j \partial_j \left(v^i \partial_i f\right) - v^j \partial_j \left(u^i \partial_i f\right) &= u^j \left(\partial_j v^i\right) \partial_i f + u^j v^i \partial_j \partial_i f - v^j \left(\partial_j u^i\right) \partial_i f - v^j u^i \partial_j \partial_i f \\ &= \left(u^j \partial_j v^i - v^j \partial_j u^i\right) \partial_i f + \underbrace{\left(u^j v^i - v^j u^i\right) \partial_j \partial_i f}_{=0 \text{ (symmetric } \partial_j \partial_i = \partial_i \partial_j)} \\ &= \left(\mathcal{L}_u v^i\right) \partial_i f. \end{aligned} \tag{21.25}$$

Theorem: Geometric Commutator of Vector Flows

The failure of the flow parallelogram to close is governed strictly by the Lie bracket:

$$f(P_4) - f(P_3) = (\mathcal{L}_u \mathcal{L}_v f - \mathcal{L}_v \mathcal{L}_u f)\,\Delta\lambda\,\Delta s = [u, v]^i \, \partial_i f \, \Delta\lambda\,\Delta s. \tag{21.26}$$

Thus, two flows commute and form closed coordinate meshes if and only if $[u, v] = 0$.

21.5 Isometries, the Killing Equation, and Conservation Laws

21.5.1 The Metric Lie Derivative

An **isometry** is a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ that leaves the metric tensor invariant:

$$\mathcal{L}_\xi g_{ab} = 0. \tag{21.27}$$

Using the general tensor formula (21.16) with the metric-compatibility property $\nabla_m g_{ab} = 0$:

$$\begin{aligned} \mathcal{L}_\xi g_{ab} &= \xi^m \nabla_m g_{ab} + g_{mb} \nabla_a \xi^m + g_{am} \nabla_b \xi^m \\ &= 0 + \nabla_a \xi_b + \nabla_b \xi_a = 2 \nabla_{(a} \xi_{b)}. \end{aligned} \tag{21.28}$$

Similarly, for the inverse metric g^{ab}:

$$\mathcal{L}_\xi g^{ab} = -\left(\nabla^a \xi^b + \nabla^b \xi^a\right) = -2 \nabla^{(a} \xi^{b)}. \tag{21.29}$$

Definition: Killing Vector Field and the Killing Equation

A vector field ξ^a is called a **Killing vector field** if it generates a 1-parameter group of isometries. The necessary and sufficient condition is the **Killing Equation**:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \iff \nabla_{(a}\xi_{b)} = 0. \tag{21.30}$$

21.5.2 Conserved Quantities along Geodesics

Theorem: First Integrals of Geodesic Motion

Let $x^a(\lambda)$ be an affinely parameterized geodesic with tangent 4-velocity $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ ($u^b \nabla_b u^a = 0$), and let ξ^a be a Killing vector field. Then the scalar quantity:

$$Q \equiv \xi_a u^a \tag{21.31}$$

is **strictly conserved** along the entire geodesic trajectory.

Proof. Differentiating Q along the worldline:

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}\lambda} &= \frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\partial_b \left(\xi_a u^a\right) = u^b \nabla_b \left(\xi_a u^a\right) \\ &= \left(u^b \nabla_b \xi_a\right) u^a + \xi_a \underbrace{\left(u^b \nabla_b u^a\right)}_{=0 \text{ (geodesic)}} \\ &= u^a u^b \nabla_b \xi_a = \frac{1}{2} u^a u^b \left(\nabla_a \xi_b + \nabla_b \xi_a\right) = 0, \end{aligned} \tag{21.32}$$

where we used the symmetry $u^a u^b = u^b u^a$ and the Killing equation $\nabla_{(a}\xi_{b)} = 0$. □

21.5.3 Conserved Killing Energy vs. Observer-Measured Energy

For a particle or photon with four-momentum p^μ, the physical energy measured by an observer with four-velocity U^μ (satisfying $U^\mu U_\mu = -1$, or $-c^2$) is given by the frame projection:

$$E_{\text{obs}} = -p_\mu U^\mu. \tag{21.33}$$

This measured energy is generally **not conserved** along the particle's geodesic worldline because U^μ varies across spacetime or from observer to observer.

In a **stationary spacetime** admitting a timelike Killing vector $\xi^\mu = (\partial_t)^\mu$, the contraction of p_μ with the Killing vector yields the **conserved Killing energy**:

$$E_\infty \equiv -p_\mu \xi^\mu = \text{constant}. \tag{21.34}$$

For a static observer (at rest with respect to the stationary coordinates, $x^i = \text{const}$), the observer's 4-velocity is aligned with ξ^μ. Normalizing $U^\mu U_\mu = -1$ gives:

$$U^\mu = \frac{\xi^\mu}{\sqrt{-\xi^\nu \xi_\nu}} = \frac{\xi^\mu}{\sqrt{-\xi^2}}. \tag{21.35}$$

Consequently, the locally measured energy by the static observer is directly related to the asymptotic Killing energy by:

$$\boxed{E_{\text{obs}} = -p_\mu \left(\frac{\xi^\mu}{\sqrt{-\xi^2}}\right) = \frac{-p_\mu \xi^\mu}{\sqrt{-\xi^2}} = \frac{E_\infty}{\sqrt{-\xi^2}}} \tag{21.36}$$

In an asymptotically flat spacetime where $\xi^2 = g_{00} \to -1$ as $r \to \infty$, $E_{\text{obs}} \to E_\infty$, identifying E_∞ as the conserved energy measured by an observer at spatial infinity. For an emitter at radius r_e and receiver at r_o in a static gravitational potential with metric $g_{00}(r) = -\left(1 - \frac{2GM}{c^2 r}\right)$, Eq. (21.36) provides the exact foundation for gravitational redshift:

$$\frac{E_{\text{obs}}(r_o)}{E_{\text{obs}}(r_e)} = \frac{\sqrt{-\xi^2(r_e)}}{\sqrt{-\xi^2(r_o)}} = \sqrt{\frac{1 - \frac{2GM}{c^2 r_e}}{1 - \frac{2GM}{c^2 r_o}}}. \tag{21.37}$$

21.5.4 Relativistic Matter Conservation Currents

Let T^{ab} be a symmetric, on-shell conserved stress-energy tensor ($\nabla_a T^{ab} = 0, T^{ab} = T^{ba}$). Given a Killing vector field ξ^b, define the 4-current vector:

$$J^a \equiv T^a{}_b\, \xi^b = T^{ab}\xi_b. \tag{21.38}$$

Taking the covariant divergence of J^a:

$$\begin{aligned} \nabla_a J^a &= \nabla_a \left(T^{ab}\xi_b\right) = \left(\nabla_a T^{ab}\right)\xi_b + T^{ab}\nabla_a \xi_b \\ &= 0 + \frac{1}{2} T^{ab} \left(\nabla_a \xi_b + \nabla_b \xi_a\right) = 0. \end{aligned} \tag{21.39}$$

Thus, J^a is a **strictly conserved matter 4-current**.

Applying Gauss's theorem on a 4-dimensional spacetime volume $\mathcal{M}$ bounded by spacelike hypersurfaces Σ_1, Σ_2 and spatial infinity:

$$\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \nabla_a J^a = 0 \iff \int_{\mathcal{M}} \mathrm{d}^4x\, \partial_a \left(\sqrt{-g}\, J^a\right) = 0. \tag{21.40}$$

This establishes that the total charge:

$$\mathcal{Q} \equiv \int_\Sigma \mathrm{d}^3x \sqrt{\gamma}\, n_a J^a = \int_\Sigma \mathrm{d}^3x \sqrt{\gamma}\, n_a T^{ab} \xi_b \tag{21.41}$$

is independent of the choice of Cauchy hypersurface Σ, providing the exact relativistic definition of energy (when $\xi = \partial_t$), linear momentum (when $\xi = \partial_x$), and angular momentum (when $\xi = \partial_\phi$).

Chapter 22

Geodesic Congruences, Kinematic Decomposition, and the Raychaudhuri Equation

The dynamics of extended relativistic media, gravitational collapse, and the global causal topology of spacetime are governed by the behavior of families of curves known as *congruences*. In this chapter, we develop the differential kinematics and non-linear dynamics of timelike congruences. We establish the first-principles Lie derivatives of tensor densities, examine the fundamental distinction between Lie transport and metric-affine parallel transport, and perform the complete irreducible decomposition of velocity gradients into expansion, shear, and vorticity. We then prove Frobenius' theorem of hypersurface orthogonality and derive the celebrated *Raychaudhuri equation*, demonstrating how spacetime curvature drives universal gravitational focusing, caustic formation, and geometric singularities.

22.1 Global Conservation Laws and Lie Derivatives of Densities

22.1.1 Conservation of Global Charges across Spacelike Slices

In Chapter 21, we proved that every Killing vector field ξ^a satisfying $\nabla_{(a}\xi_{b)} = 0$ yields a strictly conserved matter current $J^a = T^a{}_b\xi^b$ satisfying $\nabla_a J^a = 0$. We now analyze the global integral balance of this current across Cauchy hypersurfaces.

Theorem: Global Conservation of Hypersurface Flux

Let $\mathcal{M}$ be a 4-dimensional spacetime region bounded by two non-intersecting spacelike Cauchy hypersurfaces $\Sigma(t_1)$ and $\Sigma(t_2)$, and a timelike or null boundary $\mathcal{B}_\infty$ at spatial infinity where matter fields vanish. If $\nabla_a J^a = 0$, then the global charge:

$$\mathcal{Q}(t) \equiv \int_{\Sigma(t)} \mathrm{d}^3x\,\sqrt{-g}\,J^0 = \int_{\Sigma(t)} \mathrm{d}^3x\,\sqrt{\gamma}\,n_a J^a \tag{22.1}$$

is strictly independent of time, such that $\mathcal{Q}(t_1) = \mathcal{Q}(t_2)$.

Proof. Applying Gauss's divergence theorem to the 4-volume $\mathcal{M}$:

$$\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_a J^a = \int_{\partial\mathcal{M}} \mathrm{d}^3x\,\sqrt{-g}\,J^a\hat{n}_a = 0. \tag{22.2}$$

The closed boundary decomposes as $\partial\mathcal{M} = \Sigma(t_2) \cup (-\Sigma(t_1)) \cup \mathcal{B}_\infty$. Since fields vanish asymptotically on $\mathcal{B}_\infty$, we obtain:

$$\int_{\Sigma(t_2)} \mathrm{d}^3x\, \sqrt{-g}\, J^a \hat{n}_a - \int_{\Sigma(t_1)} \mathrm{d}^3x\, \sqrt{-g}\, J^a \hat{n}_a = 0. \tag{22.3}$$

For coordinate slices $t = \text{const}$, the unit normal is proportional to $\mathrm{d}t$, so the integrand evaluates to $\sqrt{-g}\, J^0 = \sqrt{-g}\, T^0{}_b \xi^b$. Thus:

$$\int_{\Sigma(t_2)} \mathrm{d}^3x\, \sqrt{-g}\, T^0{}_b \xi^b = \int_{\Sigma(t_1)} \mathrm{d}^3x\, \sqrt{-g}\, T^0{}_b \xi^b. \tag{22.4}$$

The integrated physical quantity $P^0 \equiv T^0{}_b \xi^b$ represents the conserved energy (if $\xi = \partial_t$), linear momentum (if $\xi = \partial_i$), or angular momentum (if $\xi = \partial_\phi$). □

22.1.2 First-Principles Lie Derivatives of Metric, Inverse, and Densities

We now establish the Lie transport of the metric determinant and scalar densities directly from first principles.

Key Concept: Lie Derivatives of Metric Inverses and Tensor Densities

For any vector field ξ^a, the Lie derivative of the metric tensor and its inverse are:

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 2\nabla_{(a}\xi_{b)}, \tag{22.5}$$

$$\mathcal{L}_\xi g^{ab} = -\nabla^a \xi^b - \nabla^b \xi^a = -2\nabla^{(a}\xi^{b)}. \tag{22.6}$$

For the metric volume density $\sqrt{-g}$ and any scalar field Q, the Lie derivatives satisfy:

$$\mathcal{L}_\xi \sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{ab} \mathcal{L}_\xi g_{ab} = \sqrt{-g}\, \nabla_a \xi^a = \partial_a \left(\sqrt{-g}\, \xi^a\right), \tag{22.7}$$

$$\mathcal{L}_\xi \left(Q\sqrt{-g}\right) = \sqrt{-g}\, \nabla_a \left(Q\xi^a\right) = \partial_a \left(\sqrt{-g}\, Q\xi^a\right). \tag{22.8}$$

Proof. To derive $\mathcal{L}_\xi g^{ab}$, differentiate the identity $g^{ac} g_{cb} = \delta^a_b$:

$$0 = \mathcal{L}_\xi \left(g^{ac} g_{cb}\right) = \left(\mathcal{L}_\xi g^{ac}\right) g_{cb} + g^{ac} \left(\mathcal{L}_\xi g_{cb}\right). \tag{22.9}$$

Contracting with g^{bd}:

$$\mathcal{L}_\xi g^{ad} = -g^{ac} g^{bd} \mathcal{L}_\xi g_{cb} = -g^{ac} g^{bd} \left(\nabla_c \xi_b + \nabla_b \xi_c\right) = -\nabla^a \xi^d - \nabla^d \xi^a. \tag{22.10}$$

Next, for the metric determinant $g \equiv \det(g_{ab})$, Jacobi's formula gives $\delta g = g g^{ab} \delta g_{ab}$, which implies:

$$\mathcal{L}_\xi \sqrt{-g} = \frac{1}{2\sqrt{-g}} \left(-\mathcal{L}_\xi g\right) = -\frac{1}{2\sqrt{-g}} \left(g g^{ab} \mathcal{L}_\xi g_{ab}\right) = \frac{1}{2}\sqrt{-g}\, g^{ab} \left(\nabla_a \xi_b + \nabla_b \xi_a\right). \tag{22.11}$$

Since g^{ab} is symmetric:

$$\mathcal{L}_\xi \sqrt{-g} = \sqrt{-g}\, g^{ab} \nabla_a \xi_b = \sqrt{-g}\, \nabla_a \xi^a = \partial_a \left(\sqrt{-g}\, \xi^a\right), \tag{22.12}$$

where we used the standard identity $\nabla_a \xi^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} \xi^a)$.

Finally, applying the product rule to $Q\sqrt{-g}$ where Q is a scalar ($\mathcal{L}_\xi Q = \xi^a \partial_a Q = \xi^a \nabla_a Q$):

$$\begin{aligned} \mathcal{L}_\xi \left(Q\sqrt{-g}\right) &= \left(\mathcal{L}_\xi Q\right) \sqrt{-g} + Q \left(\mathcal{L}_\xi \sqrt{-g}\right) \\ &= \sqrt{-g}\, \xi^a \nabla_a Q + Q\sqrt{-g}\, \nabla_a \xi^a \\ &= \sqrt{-g}\, \nabla_a \left(Q\xi^a\right) = \partial_a \left(\sqrt{-g}\, Q\xi^a\right). \end{aligned} \tag{22.13}$$

□

22.1.3 Killing Vector Fields of Flat Euclidean Space

In flat Euclidean 3-space $\mathbb{R}^3$ with metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$, the Killing equation $\partial_{(i}\xi_{j)} = 0$ admits six linearly independent solutions:

1. **Spatial Translations** (3 generators):

$$\xi_{(x)} = (1,0,0) = \partial_x, \quad \xi_{(y)} = (0,1,0) = \partial_y, \quad \xi_{(z)} = (0,0,1) = \partial_z. \tag{22.14}$$

2. **Spatial Rotations** (3 generators):

$$\xi_{(\phi)z} = -y\,\partial_x + x\,\partial_y = xP_y - yP_x = \partial_\phi \longleftrightarrow (-y, x, 0), \tag{22.15}$$
$$\xi_{(\phi)y} = z\,\partial_x - x\,\partial_z = xP_z - zP_x \longleftrightarrow (z, 0, -x), \tag{22.16}$$
$$\xi_{(\phi)x} = -z\,\partial_y + y\,\partial_z = yP_z - zP_y \longleftrightarrow (0, -z, y). \tag{22.17}$$

Under coordinate transformations $\xi^a_{(x)} = \frac{\partial x^a}{\partial y^b}\xi^b_{(\phi)}$, the rotational generators form the closed $\mathfrak{so}(3)$ Lie algebra:

$$\left[\xi_{(\phi)i}, \xi_{(\phi)j}\right] = \epsilon_{ijk}\,\xi_{(\phi)k}. \tag{22.18}$$

22.2 Can Curvature Be Quantified by the Lie Derivative?

Foundational Question: Can Spacetime Curvature Be Detected via Lie Derivatives?

Since the Lie bracket $[u, v]$ measures the failure of flow quadrilaterals to close in the tangent bundle, can the Lie derivative $\mathcal{L}_\xi$ be used to define or quantify intrinsic spacetime curvature?

Physical Insight & Motivation: Fundamental Distinction: Lie Transport vs. Parallel Transport

No. The Lie derivative cannot quantify intrinsic spacetime curvature.

- The Lie derivative $\mathcal{L}_\xi$ is a purely topological/differential structure on a differentiable manifold $\mathcal{M}$. It depends exclusively on active point mappings (diffeomorphisms generated by vector flows) and can be evaluated without any metric g_{ab} or affine connection Γ^a_{bc}.
- Intrinsic curvature (the Riemann tensor $R^a{}_{bcd}$) measures the path-dependence of **parallel transport** around an infinitesimal closed loop. Parallel transport requires an affine connection ∇_a (which establishes parallel direction along curves).
- In flat spacetime, vector fields generally fail to commute ($[u, v] \neq 0$) whenever the flow lines shear, rotate, or accelerate relative to one another. Hence, non-vanishing Lie derivatives reflect the deformation of the coordinate mesh/flows, not intrinsic curvature.

22.3 Kinematics of Timelike Congruences

22.3.1 Congruence Geometry and the Deviation Vector

Definition: Congruence of Timelike Worldlines

A **timelike congruence** in an open spacetime region $\mathcal{U} \subseteq \mathcal{M}$ is a family of smooth curves such that precisely one curve passes through every event $p \in \mathcal{U}$. The smooth tangent vector field to these curves is the 4-velocity $u^a(x)$, normalized such that:

$$u^a u_a = -1, \qquad u^a \nabla_b u_a = \frac{1}{2}\nabla_b \left(u^a u_a\right) = 0. \tag{22.19}$$

The 4-acceleration vector field a^a of the congruence is defined by:

$$a^a \equiv u^b \nabla_b u^a = \frac{\mathrm{D}u^a}{\mathrm{d}\tau}, \qquad u_a a^a = u_a u^b \nabla_b u^a = 0. \tag{22.20}$$

If $a^a = 0$, the curves are freely falling geodesics, and the family is termed a **geodesic congruence**.

Let $x^a(\tau, s)$ define a 2-parameter sheet of curves in the congruence, where τ is the proper time parameter along each individual worldline and s labels neighboring worldlines. The tangent velocity is $u^a = \frac{\partial x^a}{\partial \tau}$, and the transverse deviation vector field connecting neighboring particles is $\xi^a = \frac{\partial x^a}{\partial s}$.

Key Concept: Lie Transport of the Connecting Vector

Because u^a and ξ^a are coordinate basis vectors on the 2D sheet (τ, s), their Lie bracket vanishes identically:

$$\mathcal{L}_u \xi^a = [u, \xi]^a = 0. \tag{22.21}$$

In terms of the torsion-free covariant derivative:

$$u^b \nabla_b \xi^a - \xi^b \nabla_b u^a = 0 \implies \frac{\mathrm{D}\xi^a}{\mathrm{d}\tau} \equiv u^b \nabla_b \xi^a = \xi^b \nabla_b u^a. \tag{22.22}$$

We define the **velocity gradient tensor** $Q^a{}_b$ by:

$$Q^a{}_b \equiv \nabla_b u^a \implies \frac{\mathrm{D}\xi^a}{\mathrm{d}\tau} = \xi^b Q^a{}_b. \tag{22.23}$$

The tensor $Q^a{}_b$ completely governs how the separation between nearby particles evolves in proper time τ. Contraction of $Q^a{}_b$ with u_a and u^b yields:

$$u_a Q^a{}_b = u_a \nabla_b u^a = \frac{1}{2}\nabla_b (u_a u^a) = 0, \tag{22.24}$$

$$u^b Q^a{}_b = u^b \nabla_b u^a = a^a \quad (= 0 \text{ for geodesic congruences}). \tag{22.25}$$

Thus, in the local rest frame of an observer comoving with a geodesic congruence (where $u^a = (1, 0, 0, 0)$), $Q^a{}_b$ has only purely spatial components $Q^i{}_j$.

22.4 The Spatial Projection Tensor

To analyze physical deformations orthogonal to the 4-velocity u^a, we introduce the spatial projector onto the 3-dimensional rest space of comoving observers.

Definition: Spatial Projection Tensor

The projection tensor $P^a{}_b$ (or P_{ab}) onto the local 3-space orthogonal to u^a is defined as:

$$P^a{}_b \equiv \delta^a_b + u^a u_b, \qquad P_{ab} \equiv g_{ab} + u_a u_b. \tag{22.26}$$

Theorem: Algebraic Properties of P_{ab}

The projection tensor satisfies:

1. **Orthogonality / Transversality**:

$$P^a{}_b u^b = (\delta^a_b + u^a u_b)\, u^b = u^a + u^a(-1) = 0, \qquad u_a P^a{}_b = 0. \tag{22.27}$$

2. **Idempotence (Projector)**:

$$P^a{}_b P^b{}_c = (\delta^a_b + u^a u_b)\left(\delta^b_c + u^b u_c\right) = \delta^a_c + u^a u_c + u^a u_c + u^a(-1)u_c = \delta^a_c + u^a u_c = P^a{}_c. \tag{22.28}$$

3. **Spatial Dimension (Trace)**:

$$P^a{}_a = \delta^a_a + u^a u_a = 4 - 1 = 3. \tag{22.29}$$

4. **Contraction Norm**:

$$P_{ab}P^{ab} = (g_{ab} + u_a u_b)\left(g^{ab} + u^a u^b\right) = 4 + (-1) + (-1) + 1 = 3. \tag{22.30}$$

22.5 Irreducible Kinematic Decomposition: Expansion, Shear, and Vorticity

22.5.1 Projection and Splitting of the Velocity Gradient

We now decompose the velocity gradient tensor $\nabla_j u_i$ into its longitudinal (acceleration) component and its transverse spatial components.

Using the identity $\delta^m_j = P^m{}_j - u^m u_j$, we project $\nabla_j u_i$:

$$\begin{aligned}\nabla_j u_i &= \delta^m_j \nabla_m u_i = (P^m{}_j - u_j u^m)\,\nabla_m u_i \\ &= P^m{}_j \nabla_m u_i - u_j \underbrace{(u^m \nabla_m u_i)}_{a_i} \\ &= P^m{}_j \nabla_m u_i - a_i u_j.\end{aligned} \tag{22.31}$$

We now split the projected spatial tensor $B_{ji} \equiv P^m{}_j \nabla_m u_i$ into its symmetric part S_{ij} and antisymmetric part ω_{ij}:

$$P^m{}_j \nabla_m u_i = \frac{1}{2}\left(P^m{}_j \nabla_m u_i + P^m{}_i \nabla_m u_j\right) + \frac{1}{2}\left(P^m{}_j \nabla_m u_i - P^m{}_i \nabla_m u_j\right) \equiv S_{ij} + \omega_{ij}. \tag{22.32}$$

22.5.2 Trace, Expansion Scalar, and Traceless Shear

Let us evaluate the trace of the symmetric spatial tensor S_{ij}:

$$\mathrm{Tr}(S) \equiv S^i{}_i = P^{mi}\nabla_m u_i = \left(g^{mi} + u^m u^i\right)\nabla_m u_i$$

$$= \nabla_i u^i + u^m \underbrace{\left(u^i \nabla_m u_i\right)}_{=0} = \nabla_i u^i \equiv \theta. \tag{22.33}$$

Definition: Expansion Scalar θ

The scalar:

$$\theta \equiv \nabla_a u^a = P^{ab} \nabla_b u_a \tag{22.34}$$

is called the **expansion scalar**. It measures the fractional rate of change of the transverse 3-volume element $\delta\mathcal{V}$ enclosed by neighboring trajectories:

$$\theta = \frac{1}{\delta\mathcal{V}} \frac{\mathrm{d}(\delta\mathcal{V})}{\mathrm{d}\tau}. \tag{22.35}$$

- $\theta > 0$: The congruence is locally **diverging** (volume expanding).
- $\theta < 0$: The congruence is locally **converging** (volume contracting).

22.5.3 Volume Expansion via the Jacobian of Congruence Deformation

In a three-dimensional medium (the local rest frame orthogonal to u^a), the tensor $B_{ab} \equiv P^m{}_b \nabla_m u_a$ is decomposed as:

$$B_{ab} = \frac{1}{3}\theta\, \delta_{ab} + \sigma_{ab} + \omega_{ab}, \tag{22.36}$$

where $\theta = B^a{}_a$ is the expansion scalar, $\sigma_{ab} = B_{(ab)} - \frac{1}{3}\theta\, \delta_{ab}$ is the shear tensor, and $\omega_{ab} = B_{[ab]}$ is the rotation (vorticity) tensor.

In the three-dimensional case, the expansion is the fractional change of *volume* per unit time:

$$\theta = \frac{1}{V} \frac{\Delta V}{\Delta t}\,. \tag{22.37}$$

To see this, treat the three-dimensional relation:

$$\xi^a(t_1) = (\delta^a_b + B^a{}_b \Delta t)\xi^b(t_0) \tag{22.38}$$

as a coordinate transformation from $\xi^a(t_0)$ to $\xi^a(t_1)$. The Jacobian of this transformation is:

$$\begin{aligned} J &= \det\left[\delta^a_b + B^a{}_b \Delta t\right] \\ &= 1 + \mathrm{Tr}\left[B^a{}_b \Delta t\right] \\ &= 1 + \theta\, \Delta t. \end{aligned} \tag{22.39}$$

This implies that volumes at t_0 and t_1 are related by $V_1 = (1+\theta\, \Delta t)V_0$, so that $V_0\theta = (V_1 - V_0)/\Delta t$. This argument shows also that the volume is not affected by the shear and rotation tensors.

We isolate the trace part proportional to the induced spatial metric P_{ij} (where $P^i{}_i = 3$):

$$S_{ij} = \underbrace{\left(S_{ij} - \frac{1}{3}\theta P_{ij}\right)}_{\sigma_{ij}\ \text{(traceless)}} + \frac{1}{3}\theta P_{ij}. \tag{22.40}$$

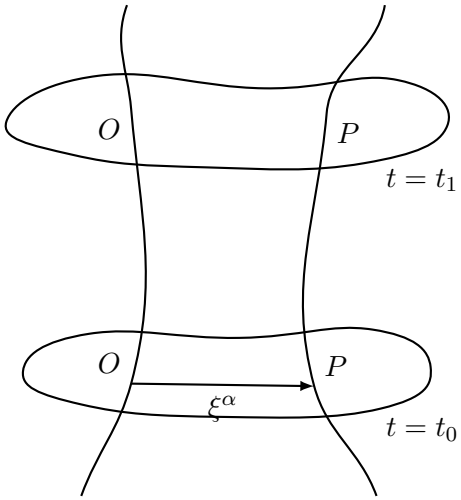


Figure 22.1: Two-dimensional deformable medium and connecting vector ξ^α evolving along worldlines between $t = t_0$ and $t = t_1$.

Definition: Shear Tensor σ_{ab} and Vorticity Tensor ω_{ab}

The **shear tensor** σ_{ab} and **vorticity (rotation) tensor** ω_{ab} are defined as:

$$\sigma_{ab} \equiv \frac{1}{2}\left(P^m{}_b \nabla_m u_a + P^m{}_a \nabla_m u_b\right) - \frac{1}{3}\theta P_{ab}, \tag{22.41}$$

$$\omega_{ab} \equiv \frac{1}{2}\left(P^m{}_b \nabla_m u_a - P^m{}_a \nabla_m u_b\right) = \frac{1}{2}\left(\nabla_b u_a - \nabla_a u_b + a_a u_b - a_b u_a\right). \tag{22.42}$$

Key Concept: Fundamental Properties of the Kinematic Tensors

The kinematic decomposition tensors satisfy the following strict algebraic properties:

1. **Shear Tensor σ_{ab}**:
   - Symmetric: $\sigma_{ab} = \sigma_{ba}$.
   - Traceless: $\sigma^a{}_a = P^{ab}\sigma_{ab} = 0$.
   - Purely spatial: $u^a \sigma_{ab} = 0$.
   - Shear scalar: $2\sigma^2 \equiv \sigma_{ab}\sigma^{ab} \geq 0$ (measures distortion of a sphere into an ellipsoid at constant volume).
2. **Vorticity Tensor ω_{ab}**:
   - Antisymmetric: $\omega_{ab} = -\omega_{ba} \implies \omega^a{}_a = 0$.
   - Purely spatial: $u^a \omega_{ab} = 0$.
   - Vorticity scalar: $2\omega^2 \equiv \omega_{ab}\omega^{ab} \geq 0$ (measures rigid spatial rotation of neighboring worldlines without shape change).
3. **Orthogonality Proof**:

$$u^i P^m{}_j \nabla_m u_i = P^m{}_j \left(u^i \nabla_m u_i\right) = 0, \qquad u^i P^m{}_i \nabla_m u_j = \underbrace{\left(u^i P^m{}_i\right)}_{=0} \nabla_m u_j = 0. \tag{22.43}$$

   Therefore:

$$u^i \sigma_{ij} = 0, \qquad u^i \omega_{ij} = 0, \qquad u^i P_{ij} = 0. \tag{22.44}$$

Theorem: Master Irreducible Kinematic Decomposition

For any timelike congruence with 4-velocity u^a and acceleration $a^a = u^b\nabla_b u^a$, the covariant velocity gradient decomposes uniquely as:

$$\nabla_b u_a = \sigma_{ab} + \frac{1}{3}\theta P_{ab} + \omega_{ab} - a_a u_b, \tag{22.45}$$

or with indices raised:

$$\nabla_b u^a = \sigma^a{}_b + \frac{1}{3}\theta P^a{}_b + \omega^a{}_b - a^a u_b. \tag{22.46}$$

22.6 Hypersurface Orthogonality and Frobenius' Theorem

A crucial property in general relativity is whether a timelike congruence is orthogonal to a family of global spacelike hypersurfaces (Cauchy surfaces).

Definition: Hypersurface-Orthogonal Congruence

A congruence is **hypersurface-orthogonal** if there exists a smooth scalar field foliation $\phi(x) = \text{const}$ such that the unit 4-velocity is proportional to the gradient $\partial_a\phi$:

$$u_a(x) = -\mu(x)\,\partial_a\phi(x) = -\mu(x)\,\nabla_a\phi(x), \tag{22.47}$$

where $\mu(x) = \left(-g^{ab}\partial_a\phi\partial_b\phi\right)^{-1/2} > 0$ is a non-vanishing lapse/normalization factor ensuring $u^a u_a = -1$.

Theorem: Frobenius' Integrability Theorem for Timelike Congruences

A unit timelike vector field u_a is hypersurface-orthogonal if and only if:

$$u_{[a}\nabla_b u_{c]} \equiv \frac{1}{6}\Big(u_a\nabla_b u_c + u_b\nabla_c u_a + u_c\nabla_a u_b - u_b\nabla_a u_c - u_c\nabla_b u_a - u_a\nabla_c u_b\Big) = 0. \tag{22.48}$$

Furthermore, this geometric condition holds if and only if the vorticity tensor vanishes identically:

$$u_{[a}\nabla_b u_{c]} = 0 \iff \omega_{ab} = 0. \tag{22.49}$$

Proof. We prove both directions explicitly:

Step 1: Direct proof that $u_a = -\mu\partial_a\phi \implies \omega_{ab} = 0$.

Differentiating $u_a = -\mu\partial_a\phi$:

$$\nabla_b u_a = -\nabla_b(\mu\partial_a\phi) = -(\partial_b\mu)(\partial_a\phi) - \mu\nabla_b\nabla_a\phi. \tag{22.50}$$

Because the connection is torsion-free, partial derivatives commute on scalars: $\nabla_b\nabla_a\phi = \nabla_a\nabla_b\phi$. Taking the antisymmetric curl:

$$\nabla_b u_a - \nabla_a u_b = -(\partial_b\mu)(\partial_a\phi) + (\partial_a\mu)(\partial_b\phi) = \frac{1}{\mu}\Big(u_a\partial_b\mu - u_b\partial_a\mu\Big). \tag{22.51}$$

We now project into the vorticity tensor:

$$\omega_{ab} = \frac{1}{2}P^m{}_b P^n{}_a\,(\nabla_m u_n - \nabla_n u_m)$$

$$= \frac{1}{2\mu} P^m{}_b P^n{}_a \left(u_n \partial_m \mu - u_m \partial_n \mu\right). \tag{22.52}$$

Since $P^n{}_a u_n = 0$ and $P^m{}_b u_m = 0$, both terms vanish identically:

$$\omega_{ab} = 0. \tag{22.53}$$

Step 2: Proof that $u_{[a}\nabla_b u_{c]} = 0 \iff \omega_{ab} = 0$.
Expand the fully antisymmetrized condition:

$$u_a \left(\nabla_b u_c - \nabla_c u_b\right) + u_b \left(\nabla_c u_a - \nabla_a u_c\right) + u_c \left(\nabla_a u_b - \nabla_b u_a\right) = 0. \tag{22.54}$$

Contract this equation with u^a. Using $u^a u_a = -1$, $u^a \nabla_b u_a = 0$, and $u^a \nabla_a u_b = a_b$:

$$\begin{aligned} u^a u_{[a}\nabla_b u_{c]} &= -1 \cdot \left(\nabla_b u_c - \nabla_c u_b\right) + u_b \left(u^a \nabla_c u_a - u^a \nabla_a u_c\right) + u_c \left(u^a \nabla_a u_b - u^a \nabla_b u_a\right) \\ &= -\left(\nabla_b u_c - \nabla_c u_b\right) - u_b a_c + u_c a_b = 0. \end{aligned} \tag{22.55}$$

Rearranging terms:

$$\nabla_b u_c - \nabla_c u_b + u_c a_b - u_b a_c = 0. \tag{22.56}$$

Comparing directly with the definition of vorticity in Eq. (22.42):

$$\omega_{bc} = \frac{1}{2}\Big(\nabla_c u_b - \nabla_b u_c + a_b u_c - a_c u_b\Big) = -\frac{1}{2}\Big(\nabla_b u_c - \nabla_c u_b + u_c a_b - u_b a_c\Big) = 0. \tag{22.57}$$

Hence, $u_{[a}\nabla_b u_{c]} = 0$ is mathematically equivalent to $\omega_{ab} = 0$. □

22.7 The Timelike Raychaudhuri Equation

We now derive the central dynamical equation governing the evolution of expansion θ along the worldlines of any timelike congruence.

Theorem: The Timelike Raychaudhuri Equation

Let u^a be the tangent velocity field of a timelike congruence with acceleration $a^a = u^b \nabla_b u^a$. The proper time rate of change of the expansion scalar $\theta \equiv \nabla_a u^a$ along the flow lines satisfies:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}u^a u^b + \nabla_a a^a, \tag{22.58}$$

or equivalently, in terms of the scalar magnitudes $2\sigma^2 \equiv \sigma_{ab}\sigma^{ab} \geq 0$ and $2\omega^2 \equiv \omega_{ab}\omega^{ab} \geq 0$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 - R_{ab}u^a u^b + \nabla_a a^a. \tag{22.59}$$

Proof. By definition, the directional derivative of $\theta = \nabla_a u^a$ along the congruence is:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{\mathrm{d}x^b}{\mathrm{d}\tau}\partial_b \theta = u^b \nabla_b \left(\nabla_a u^a\right) = u^b \nabla_b \nabla_a u^a. \tag{22.60}$$

Step 1: Commuting covariant derivatives using the Riemann tensor.
Using the definition of the Riemann tensor on a vector field:

$$\left[\nabla_b, \nabla_a\right] u^a = \nabla_b \nabla_a u^a - \nabla_a \nabla_b u^a = R^a{}_{cba} u^c = -R^a{}_{cab} u^c = -R_{cb} u^c. \tag{22.61}$$

Therefore:

$$u^b \nabla_b \nabla_a u^a = u^b \nabla_a \nabla_b u^a - R_{cb} u^c u^b = u^b \nabla_a \nabla_b u^a - R_{ab} u^a u^b. \tag{22.62}$$

Step 2: Applying the product rule to the first term.
We rewrite $u^b\nabla_a\nabla_b u^a$ as:

$$\begin{aligned} u^b\nabla_a\nabla_b u^a &= \nabla_a\left(u^b\nabla_b u^a\right) - \left(\nabla_a u^b\right)\left(\nabla_b u^a\right) \\ &= \nabla_a a^a - \left(\nabla_a u_b\right)\left(\nabla^b u^a\right). \end{aligned} \tag{22.63}$$

Step 3: Algebraic evaluation of the quadratic contraction $(\nabla_a u_b)(\nabla^b u^a)$.
We substitute the kinematic decomposition from Eq. (22.45):

$$\nabla_a u_b = \sigma_{ba} + \frac{1}{3}\theta P_{ba} + \omega_{ba} - a_b u_a = \sigma_{ab} + \frac{1}{3}\theta P_{ab} - \omega_{ab} - a_b u_a, \tag{22.64}$$

$$\nabla^b u^a = \sigma^{ab} + \frac{1}{3}\theta P^{ab} + \omega^{ab} - a^a u^b. \tag{22.65}$$

We compute the full tensor contraction term by term:

$$\begin{aligned} (\nabla_a u_b)(\nabla^b u^a) &= \left(\sigma_{ab} + \frac{1}{3}\theta P_{ab} - \omega_{ab} - a_b u_a\right)\left(\sigma^{ab} + \frac{1}{3}\theta P^{ab} + \omega^{ab} - a^a u^b\right) \\ &= \sigma_{ab}\sigma^{ab} + \frac{1}{3}\theta\sigma_{ab}P^{ab} + \sigma_{ab}\omega^{ab} - \sigma_{ab}a^a u^b \\ &\quad + \frac{1}{3}\theta P_{ab}\sigma^{ab} + \frac{1}{9}\theta^2 P_{ab}P^{ab} + \frac{1}{3}\theta P_{ab}\omega^{ab} - \frac{1}{3}\theta P_{ab}a^a u^b \\ &\quad - \omega_{ab}\sigma^{ab} - \frac{1}{3}\theta\omega_{ab}P^{ab} - \omega_{ab}\omega^{ab} + \omega_{ab}a^a u^b \\ &\quad - a_b u_a\sigma^{ab} - \frac{1}{3}\theta a_b u_a P^{ab} - a_b u_a\omega^{ab} + a_b u_a a^a u^b. \end{aligned} \tag{22.66}$$

We now evaluate each algebraic piece using the orthogonality and symmetry properties:

1. $\sigma_{ab}P^{ab} = \sigma^a{}_a = 0$ (tracelessness).
2. $\sigma_{ab}\omega^{ab} = 0$ (contraction of symmetric with antisymmetric tensor).
3. $P_{ab}\omega^{ab} = \omega^a{}_a = 0$ (antisymmetry).
4. $P_{ab}P^{ab} = 3$ (spatial dimension).
5. All terms contracted with u^a or u^b vanish identically because $\sigma_{ab}u^b = 0$, $\omega_{ab}u^b = 0$, $P_{ab}u^b = 0$, and $a_a u^a = 0$. In particular, $a_b u_a a^a u^b = (a_b u^b)(u_a a^a) = 0 \cdot 0 = 0$.

Thus, all cross-terms vanish, leaving only the non-zero quadratic terms:

$$\begin{aligned} (\nabla_a u_b)(\nabla^b u^a) &= \sigma_{ab}\sigma^{ab} + \frac{1}{9}\theta^2(3) - \omega_{ab}\omega^{ab} \\ &= \sigma_{ab}\sigma^{ab} + \frac{1}{3}\theta^2 - \omega_{ab}\omega^{ab} \\ &= 2\sigma^2 + \frac{1}{3}\theta^2 - 2\omega^2. \end{aligned} \tag{22.67}$$

Step 4: Assembling the master equation.
Substituting Eq. (22.67) and Eq. (22.63) into Eq. (22.62):

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= \nabla_a a^a - \left(\sigma_{ab}\sigma^{ab} + \frac{1}{3}\theta^2 - \omega_{ab}\omega^{ab}\right) - R_{ab}u^a u^b \\ &= -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}u^a u^b + \nabla_a a^a. \end{aligned} \tag{22.68}$$

This completes the rigorous derivation of the Timelike Raychaudhuri equation. □

22.8 Gravitational Focusing, Energy Conditions, and Caustic Singularities

22.8.1 The Strong Energy Condition and Ricci Curvature

We examine the physical implications of Raychaudhuri's equation for geodesic ($a^a = 0$) and hypersurface-orthogonal ($\omega_{ab} = 0$) congruences. Under these conditions, Eq. (22.59) simplifies to:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - 2\sigma^2 - R_{ab}u^a u^b. \tag{22.69}$$

Postulate / Principle: Einstein Field Equations and the Strong Energy Condition (SEC)

Via Einstein's field equations $R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}$, taking the trace yields $R - 2R = 8\pi G T \implies R = -8\pi G T$. Substituting this back gives:

$$R_{ab} = 8\pi G\left(T_{ab} - \frac{1}{2}T g_{ab}\right). \tag{22.70}$$

Contracting with the timelike velocity u^a ($u^a u_a = -1$):

$$R_{ab}u^a u^b = 8\pi G\left(T_{ab}u^a u^b - \frac{1}{2}T(-1)\right) = 8\pi G\left(T_{ab}u^a u^b + \frac{1}{2}T\right). \tag{22.71}$$

The **Strong Energy Condition (SEC)** states that for any timelike unit vector u^a:

$$T_{ab}u^a u^b \geq -\frac{1}{2}T u^a u_a = \frac{1}{2}T \iff R_{ab}u^a u^b \geq 0. \tag{22.72}$$

For a perfect fluid with energy density ρ and isotropic pressure p, $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, which yields $T = -\rho + 3p$ and:

$$R_{ab}u^a u^b = 4\pi G(\rho + 3p) \geq 0 \iff \rho + 3p \geq 0 \quad \text{and} \quad \rho + p \geq 0. \tag{22.73}$$

22.8.2 The Gravitational Focusing Theorem

Theorem: Focusing Theorem for Timelike Geodesic Congruences

In any spacetime where Einstein's equations and the Strong Energy Condition hold ($R_{ab}u^a u^b \geq 0$), every hypersurface-orthogonal timelike geodesic congruence ($\omega_{ab} = 0, a^a = 0$) satisfies:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \leq -\frac{1}{3}\theta^2. \tag{22.74}$$

Consequently, gravitation acts as an unconditionally attractive force, causing the expansion θ to monotonically decrease along every geodesic trajectory.

Proof. Because shear is purely spatial, $\sigma_{ab}\sigma^{ab} = 2\sigma^2 \geq 0$. Under the SEC, $R_{ab}u^a u^b \geq 0$. Thus, every term on the right-hand side of Eq. (22.69) is strictly non-positive:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \underbrace{2\sigma^2}_{\geq 0} - \underbrace{R_{ab}u^a u^b}_{\geq 0} \leq -\frac{1}{3}\theta^2 \leq 0. \tag{22.75}$$

□

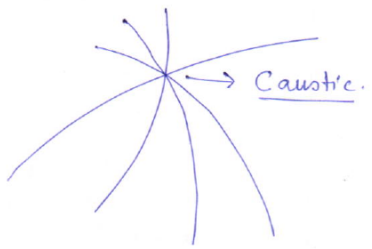


Figure 22.2: Gravitational focusing and the formation of a caustic (conjugate point). Due to the universal non-linear attraction $\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \leq -\frac{1}{3}\theta^2$, neighboring geodesics in an initially converging congruence are focused to a mutual intersection in a finite proper time $\tau \leq \frac{3}{|\theta_0|}$, at which point the expansion diverges $\theta \to -\infty$ and the congruence develops an envelope singularity.

22.8.3 Riccati Integration and Finite Proper Time Caustic Singularities

Theorem: Formation of Caustics in Finite Proper Time

If a hypersurface-orthogonal timelike geodesic congruence is initially converging at $\tau = 0$, such that:

$$\theta(0) \equiv \theta_0 < 0, \tag{22.76}$$

then the expansion scalar must diverge to negative infinity:

$$\theta(\tau) \longrightarrow -\infty \tag{22.77}$$

within a finite proper time τ_{foc} bounded by:

$$\tau_{\text{foc}} \leq \frac{3}{|\theta_0|}. \tag{22.78}$$

Proof. The differential inequality $\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \leq -\frac{1}{3}\theta^2$ can be integrated analytically. Dividing by θ^2:

$$\frac{1}{\theta^2}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \leq -\frac{1}{3} \implies -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\theta}\right) \leq -\frac{1}{3} \implies \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\theta}\right) \geq \frac{1}{3}. \tag{22.79}$$

Integrating from $\tau = 0$ to τ:

$$\frac{1}{\theta(\tau)} - \frac{1}{\theta(0)} \geq \frac{\tau}{3} \implies \frac{1}{\theta(\tau)} \geq \frac{1}{\theta_0} + \frac{\tau}{3}. \tag{22.80}$$

Since $\theta_0 < 0$, we write $\frac{1}{\theta_0} = -\frac{1}{|\theta_0|}$:

$$\frac{1}{\theta(\tau)} \geq -\frac{1}{|\theta_0|} + \frac{\tau}{3} = \frac{\tau - 3/|\theta_0|}{3}. \tag{22.81}$$

As $\tau \to \tau_* \equiv \frac{3}{|\theta_0|}$, the upper bound on $\theta^{-1}(\tau)$ reaches 0 from below:

$$\frac{1}{\theta(\tau)} \leq -\left(\frac{1}{|\theta_0|} - \frac{\tau}{3}\right) \xrightarrow{\tau \to 3/|\theta_0|} 0^-. \tag{22.82}$$

Because $\theta^{-1}(\tau) \to 0^-$, the expansion scalar itself diverges to negative infinity:

$$\theta(\tau) \longrightarrow -\infty \quad \text{at some proper time} \quad \tau \leq \frac{3}{|\theta_0|}. \tag{22.83}$$

□

Physical Insight & Motivation: Physical Meaning of Caustics and Conjugate Points

The divergence $\theta \to -\infty$ signifies that the cross-sectional 3-volume $\delta\mathcal{V} \to 0$. Nearby geodesics in the congruence cross one another, creating an envelope known as a **caustic** or **conjugate point** (Figure 22.2).

- At a caustic, the congruence loses its coordinate regularity (the mapping from congruence coordinates to spacetime ceases to be a diffeomorphism).
- This does not necessarily imply that the underlying spacetime manifold has a curvature singularity; rather, it proves that geodesics inevitably cross.
- However, when combined with global topological theorems (such as global hyperbolicity and the absence of trapped surface boundaries), this universal focusing property forms the indispensable core of the **Penrose–Hawking Singularity Theorems**, proving that spacetime singularities are inevitable in general relativity whenever gravitational collapse or cosmic expansion occurs.

22.9 Null Geodesic Congruences and the Transverse Screen Projector

22.9.1 Geometry of Null Congruences

Let $\mathcal{M}$ be a 4-dimensional spacetime manifold. A **null geodesic congruence** is a family of smooth null curves such that precisely one curve passes through each point of a region $\mathcal{U} \subseteq \mathcal{M}$. The tangent vector field k^a satisfies:

$$k^a k_a = 0, \qquad k^b \nabla_b k^a = 0, \tag{22.84}$$

where λ is an affine parameter along each null curve.

Because k^a is orthogonal to itself ($k^a k_a = 0$), k^a lies simultaneously in its own tangent space and orthogonal subspace. To isolate physical deformations transverse to the light rays, we must project onto a 2-dimensional spatial screen orthogonal to both k^a and an auxiliary null direction.

22.9.2 The Auxiliary Null Vector l^a and 2D Screen Metric h_{ab}

Definition: Transverse Screen Projector h_{ab}

We introduce an auxiliary null vector field l^a satisfying:

$$l^a l_a = 0, \qquad k^a l_a = -1. \tag{22.85}$$

The **transverse screen metric projector** h_{ab} is defined by:

$$h_{ab} \equiv g_{ab} + k_a l_b + l_a k_b, \qquad h^a{}_b \equiv \delta^a_b + k^a l_b + l^a k_b. \tag{22.86}$$

Theorem: Algebraic Properties of the 2D Screen Projector

The tensor h_{ab} satisfies:

1. **Transversality**:

$$h^a{}_b k^b = (\delta^a_b + k^a l_b + l^a k_b)\, k^b = k^a + k^a (l_b k^b) + l^a (k_b k^b) = k^a - k^a + 0 = 0, \quad (22.87)$$

$$h^a{}_b l^b = (\delta^a_b + k^a l_b + l^a k_b)\, l^b = l^a + k^a (l_b l^b) + l^a (k_b l^b) = l^a + 0 - l^a = 0. \quad (22.88)$$

2. **Idempotence (Projector onto 2D Space)**:

$$\begin{aligned} h^a{}_b h^b{}_c &= (\delta^a_b + k^a l_b + l^a k_b)\left(\delta^b_c + k^b l_c + l^b k_c\right) \\ &= \delta^a_c + k^a l_c + l^a k_c + k^a (l_b k^b) l_c + l^a (k_b l^b) k_c \\ &= \delta^a_c + k^a l_c + l^a k_c - k^a l_c - l^a k_c + k^a l_c + l^a k_c = h^a{}_c. \end{aligned} \quad (22.89)$$

3. **Trace (Dimension of Screen)**:

$$h^a{}_a = \delta^a_a + k^a l_a + l^a k_a = 4 - 1 - 1 = 2. \quad (22.90)$$

4. **Contraction**:

$$h_{ab} h^{ab} = 2. \quad (22.91)$$

22.10 The Null Raychaudhuri Equation and Sachs Optical Scalars

22.10.1 Irreducible Kinematic Decomposition of Null Congruences

We project the null velocity gradient tensor $\nabla_b k_a$ onto the 2-dimensional transverse screen:

$$\hat{B}_{ab} \equiv h^c{}_a h^d{}_b \nabla_d k_c. \quad (22.92)$$

The 2D tensor $\hat{B}_{ab}$ decomposes irreducibly into trace, symmetric traceless, and antisymmetric parts:

$$\hat{B}_{ab} = \frac{1}{2}\theta\, h_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab}, \quad (22.93)$$

where the **Sachs optical scalars** and tensors are:

- **Expansion Scalar θ**:

$$\begin{aligned} \theta &\equiv h^{ab}\hat{B}_{ab} = h^{ab}\nabla_b k_a \\ &= \left(g^{ab} + k^a l^b + l^a k^b\right)\nabla_b k_a = \nabla_a k^a + l^b \underbrace{(k^a \nabla_b k_a)}_{=0} + k^b l^a \nabla_b k_a = \nabla_a k^a. \end{aligned} \quad (22.94)$$

- **Shear Tensor $\hat{\sigma}_{ab}$ (Shape Distortion)**:

$$\hat{\sigma}_{ab} \equiv \hat{B}_{(ab)} - \frac{1}{2}\theta\, h_{ab}, \qquad h^{ab}\hat{\sigma}_{ab} = 0, \qquad 2\hat{\sigma}^2 \equiv \hat{\sigma}_{ab}\hat{\sigma}^{ab} \geq 0. \quad (22.95)$$

- **Rotation / Twist Tensor $\hat{\omega}_{ab}$**:

$$\hat{\omega}_{ab} \equiv \hat{B}_{[ab]}, \qquad 2\hat{\omega}^2 \equiv \hat{\omega}_{ab}\hat{\omega}^{ab} \geq 0. \quad (22.96)$$

22.10.2 Quadratic Contraction Identity

Theorem: Tensor Contraction Identity for Null Congruences

Using the transversality of k^a and the geodesic condition $k^a\nabla_a k^b = 0$, the contraction of the projected tensor equals the unprojected contraction:

$$\hat{B}_{ba}\hat{B}^{ab} = (\nabla_a k_b)\left(\nabla^b k^a\right) = \frac{1}{2}\theta^2 + \hat{\sigma}_{ab}\hat{\sigma}^{ab} - \hat{\omega}_{ab}\hat{\omega}^{ab}. \tag{22.97}$$

Proof. Substituting $h^c{}_a = \delta^c_a + k^c l_a + l^c k_a$ into $\hat{B}_{ba}\hat{B}^{ab}$:

$$\begin{aligned}\hat{B}_{ba}\hat{B}^{ab} &= (h^m{}_b h^n{}_a \nabla_n k_m)\left(h^a{}_j h^b{}_i \nabla^j k^i\right) = h^m{}_i h^n{}_j (\nabla_n k_m)(\nabla^j k^i)\\ &= (\delta^m_i + k^m l_i + l^m k_i)\left(\delta^n_j + k^n l_j + l^n k_j\right)(\nabla_n k_m)(\nabla^j k^i)\\ &= (\nabla_j k_i)(\nabla^j k^i) + l_i \underbrace{(k^m \nabla_n k_m)}_{=0} + l_j \underbrace{(k^n \nabla_n k_m)}_{=0} + \cdots \equiv (\nabla_a k_b)(\nabla^b k^a).\end{aligned} \tag{22.98}$$

Squaring the decomposition (22.93):

$$\begin{aligned}\hat{B}_{ba}\hat{B}^{ab} &= \left(\frac{1}{2}\theta h_{ba} + \hat{\sigma}_{ba} + \hat{\omega}_{ba}\right)\left(\frac{1}{2}\theta h^{ab} + \hat{\sigma}^{ab} + \hat{\omega}^{ab}\right)\\ &= \frac{1}{4}\theta^2 (h_{ba}h^{ab}) + \hat{\sigma}_{ba}\hat{\sigma}^{ab} + \hat{\omega}_{ba}\hat{\omega}^{ab} + \theta\underbrace{(h_{ab}\hat{\sigma}^{ab})}_{=0} + \theta\underbrace{(h_{ab}\hat{\omega}^{ab})}_{=0} + 2\underbrace{\hat{\sigma}_{ab}\hat{\omega}^{ab}}_{=0}\\ &= \frac{1}{4}\theta^2(2) + \hat{\sigma}_{ab}\hat{\sigma}^{ab} - \hat{\omega}_{ab}\hat{\omega}^{ab} = \frac{1}{2}\theta^2 + \hat{\sigma}_{ab}\hat{\sigma}^{ab} - \hat{\omega}_{ab}\hat{\omega}^{ab},\end{aligned} \tag{22.99}$$

where we used $\hat{\omega}_{ba}\hat{\omega}^{ab} = -\hat{\omega}_{ab}\hat{\omega}^{ab}$ due to antisymmetry. □

22.10.3 Derivation of the Null Raychaudhuri Equation

We compute the directional derivative of the expansion scalar along the affine generator:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \equiv k^a\nabla_a\theta = k^a\nabla_a(\nabla_b k^b). \tag{22.100}$$

Commuting the covariant derivatives using the Ricci identity $\nabla_a\nabla_b k^b - \nabla_b\nabla_a k^b = R^b{}_{cab}k^c = R_{ca}k^c$:

$$\begin{aligned}k^a\nabla_a(\nabla_b k^b) &= k^a\left(\nabla_b\nabla_a k^b - R_{ba}k^b\right)\\ &= k^a\nabla_b\nabla_a k^b - R_{ab}k^a k^b\\ &= \nabla_b\left(k^a\nabla_a k^b\right) - (\nabla_b k^a)\left(\nabla_a k^b\right) - R_{ab}k^a k^b.\end{aligned} \tag{22.101}$$

Since the congruence is geodesic ($k^a\nabla_a k^b = 0$), the first divergence term vanishes identically. Substituting the contraction identity (22.97):

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -(\nabla_b k^a)\left(\nabla_a k^b\right) - R_{ab}k^a k^b = -\left(\frac{1}{2}\theta^2 + \hat{\sigma}_{ab}\hat{\sigma}^{ab} - \hat{\omega}_{ab}\hat{\omega}^{ab}\right) - R_{ab}k^a k^b. \tag{22.102}$$

Theorem: The Null Raychaudhuri Equation

The exact evolution of the expansion scalar θ along an affinely parametrized null geodesic congruence is governed by:

$$\boxed{\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}k^a k^b} \tag{22.103}$$

Key Concept: Timelike vs. Null Raychaudhuri Equations

- **Timelike Case (3D Rest Space):**

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}u^a u^b. \tag{22.104}$$

- **Null Case (2D Screen Space):**

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}k^a k^b. \tag{22.105}$$

  The numerical factor $-\frac{1}{d}\theta^2$ reflects the spatial dimension d orthogonal to the congruence ($d = 3$ for timelike, $d = 2$ for null).

- **Null Focusing Theorem:** For a twist-free ($\hat{\omega}_{ab} = 0$, hypersurface-orthogonal) null congruence in a spacetime satisfying the **Null Energy Condition (NEC)**:

$$T_{ab}k^a k^b \geq 0 \implies R_{ab}k^a k^b = 8\pi G\, T_{ab}k^a k^b \geq 0, \tag{22.106}$$

  all terms on the right-hand side of Eq. (22.103) are strictly non-positive:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \leq -\frac{1}{2}\theta^2. \tag{22.107}$$

  If the congruence is initially converging ($\theta_0 < 0$), integration yields:

$$\frac{1}{\theta(\lambda)} \geq \frac{1}{\theta_0} + \frac{1}{2}\lambda \implies \theta(\lambda) \to -\infty \quad \text{at or before } \lambda \leq \frac{2}{|\theta_0|}. \tag{22.108}$$

  This inevitable focus to a **caustic** ($\theta \to -\infty$) is the foundational engine behind the Penrose Singularity Theorem and the Hawking Area Theorem.

22.11 Transverse Area Evolution and the Physical Meaning of θ

22.11.1 Adapted Coordinates on the 2D Transverse Screen

We now prove rigorously that θ represents the fractional rate of change of the cross-sectional area of the null beam.

Let the spacetime region be spanned by adapted coordinates $(\lambda, \mu, \theta^1, \theta^2)$, where λ is the affine parameter along the null rays, μ labels distinct null hypersurfaces, and θ^A ($A = 1, 2$) are intrinsic coordinates on the 2D transverse cross-sections.

The tangent vector to the null rays and the basis vectors tangent to the transverse 2-surface

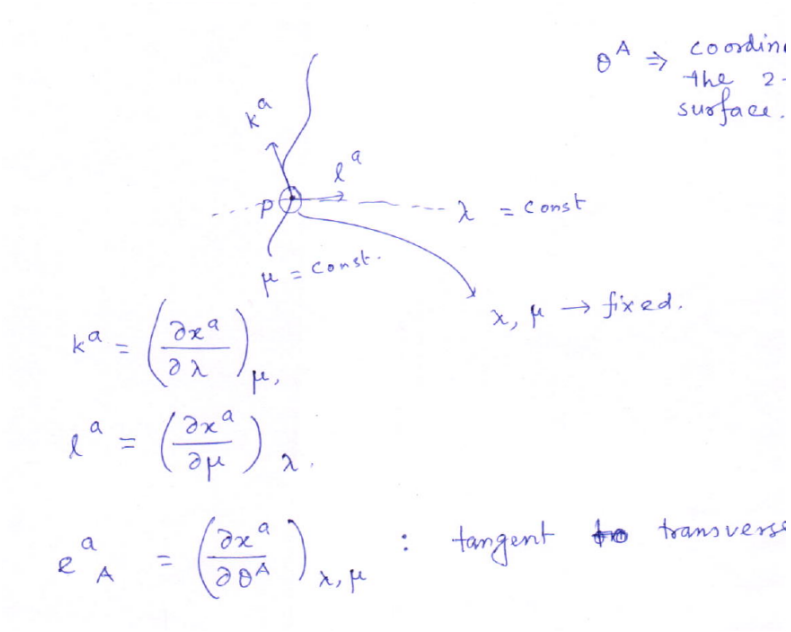


Figure 22.3: Two-dimensional transverse cross-section spanned by coordinates θ^A, showing the null generators $k^a = (\partial x^a/\partial\lambda)$ and transverse connecting vectors $e^a_A = (\partial x^a/\partial\theta^A)$.

are:

$$k^a = \left(\frac{\partial x^a}{\partial \lambda}\right)_{\mu,\theta^A}, \qquad e^a_A = \left(\frac{\partial x^a}{\partial \theta^A}\right)_{\lambda,\mu}. \tag{22.109}$$

The induced metric on the 2-dimensional transverse surface is:

$$h_{AB} = g_{ab} e^a_A e^b_B, \qquad (A, B = 1, 2). \tag{22.110}$$

22.11.2 Lie Commutator Identity

Because k^a and e^a_A are coordinate basis vectors, their Lie bracket vanishes identically:

$$\mathcal{L}_k e^a_A = [k, e_A]^a = k^b \partial_b e^a_A - e^b_A \partial_b k^a = \frac{\partial^2 x^a}{\partial\lambda \partial\theta^A} - \frac{\partial^2 x^a}{\partial\theta^A \partial\lambda} = 0. \tag{22.111}$$

In terms of the torsion-free covariant derivative:

$$k^b \nabla_b e^a_A - e^b_A \nabla_b k^a = 0 \implies k^b \nabla_b e^a_A = e^b_A \nabla_b k^a. \tag{22.112}$$

22.11.3 Metric Evolution and the Area Rate

Differentiating the induced 2D metric h_{AB} with respect to λ:

$$\begin{aligned} \frac{\mathrm{d}h_{AB}}{\mathrm{d}\lambda} &= k^m \nabla_m \left(g_{ab} e^a_A e^b_B\right) = g_{ab}\left[\left(k^m \nabla_m e^a_A\right) e^b_B + e^a_A \left(k^m \nabla_m e^b_B\right)\right] \\ &= g_{ab}\left[\left(e^m_A \nabla_m k^a\right) e^b_B + e^a_A \left(e^m_B \nabla_m k^b\right)\right] \\ &= e^a_A e^b_B \nabla_a k_b + e^a_A e^b_B \nabla_b k_a = e^a_A e^b_B \left(\nabla_a k_b + \nabla_b k_a\right). \end{aligned} \tag{22.113}$$

The physical cross-sectional area element of the transverse bundle is:

$$\delta A(\lambda) = \sqrt{\det h_{AB}}\, \mathrm{d}^2\theta. \tag{22.114}$$

Using Jacobi's determinant formula $\frac{\mathrm{d}}{\mathrm{d}\lambda}\det(h) = \det(h)\, h^{AB}\frac{\mathrm{d}h_{AB}}{\mathrm{d}\lambda}$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\delta A) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sqrt{h}\right)\mathrm{d}^2\theta = \frac{1}{2\sqrt{h}}\left(h\, h^{AB}\frac{\mathrm{d}h_{AB}}{\mathrm{d}\lambda}\right)\mathrm{d}^2\theta = \frac{1}{2}\sqrt{h}\, h^{AB}\frac{\mathrm{d}h_{AB}}{\mathrm{d}\lambda}\,\mathrm{d}^2\theta. \tag{22.115}$$

Dividing by $\delta A = \sqrt{h}\,\mathrm{d}^2\theta$:

$$\frac{1}{\delta A}\frac{\mathrm{d}(\delta A)}{\mathrm{d}\lambda} = \frac{1}{2}h^{AB}\frac{\mathrm{d}h_{AB}}{\mathrm{d}\lambda}. \tag{22.116}$$

Substituting Eq. (22.113) and using $h^{AB}e^a_A e^b_B = h^{ab}$:

$$\begin{aligned}\frac{1}{\delta A}\frac{\mathrm{d}(\delta A)}{\mathrm{d}\lambda} &= \frac{1}{2}\left(h^{AB}e^a_A e^b_B\right)\left(\nabla_a k_b + \nabla_b k_a\right)\\ &= \frac{1}{2}h^{ab}\left(\nabla_a k_b + \nabla_b k_a\right) = h^{ab}\nabla_a k_b = \theta.\end{aligned} \tag{22.117}$$

Theorem: Transverse Area Rate Theorem

The null expansion scalar θ is identically the fractional rate of change of the cross-sectional beam area δA along the congruence:

$$\boxed{\theta = \frac{1}{\delta A}\frac{\mathrm{d}(\delta A)}{\mathrm{d}\lambda}} \tag{22.118}$$

Chapter 23

Spherically Symmetric Spacetimes, Exact Schwarzschild Derivation, Horizons, and Kerr Geometry

The exterior gravitational field of any static, spherically symmetric distribution of mass in vacuum is uniquely described by the **Schwarzschild metric**. In this chapter, we develop spherically symmetric geometry from first principles. We establish the coordinate reduction of the most general spherically symmetric line element to standard diagonal form, compute the complete set of Christoffel symbols and Ricci curvature components, and solve Einstein's vacuum field equations to derive the exact exterior Schwarzschild metric. We then analyze the global geometry and causal structure of the spacetime, resolving the coordinate singularity at the event horizon using Regge–Wheeler tortoise, ingoing Eddington–Finkelstein, and Kruskal–Szekeres coordinates. Finally, we formulate the Penrose conformal diagram, examine geodesic orbits, derive the Kerr solution via the Newman–Janis algorithm, and evaluate black hole surface gravity.

23.1 Reduction of General Spherically Symmetric Metrics

23.1.1 The Most General Spherically Symmetric Line Element

The most general time-dependent, spherically symmetric spacetime metric in coordinates (t, r, θ, ϕ) is given by:

$$\mathrm{d}s^2 = -A(t,r)\,\mathrm{d}t^2 + B(t,r)\,\mathrm{d}t\,\mathrm{d}r + C(t,r)\,\mathrm{d}r^2 + D(t,r)\,\mathrm{d}\Omega^2, \tag{23.1}$$

where $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$, and A, B, C, D are smooth arbitrary positive functions of (t, r).

23.1.2 Areal Radius Coordinate Transformation

Because spherical symmetry guarantees that surfaces of constant (t, r) are geometric 2-spheres of invariant proper area $4\pi D(t, r)$, we define the **areal radius coordinate** $\bar{r}$:

$$\bar{r}^2 \equiv D(t,r) \implies \mathrm{d}\bar{r} = \frac{1}{2\sqrt{D}}\left(\frac{\partial D}{\partial t}\mathrm{d}t + \frac{\partial D}{\partial r}\mathrm{d}r\right). \tag{23.2}$$

Assuming $\frac{\partial D}{\partial r} \neq 0$, we invert for $r(t, \bar{r})$, casting the line element into the form:

$$\mathrm{d}s^2 = -A(t,\bar{r})\,\mathrm{d}t^2 + B(t,\bar{r})\,\mathrm{d}t\,\mathrm{d}\bar{r} + C(t,\bar{r})\,\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2. \tag{23.3}$$

23.1.3 Elimination of Off-Diagonal Terms via Integrating Factors

To eliminate the off-diagonal cross term $\mathrm{d}t\,\mathrm{d}\bar{r}$, we seek a new coordinate time $\bar{t}(t,\bar{r})$ such that $\mathrm{d}\bar{t}$ is proportional to the 1-form $A(t,\bar{r})\mathrm{d}t - \frac{1}{2}B(t,\bar{r})\mathrm{d}\bar{r}$:

$$\mathrm{d}\bar{t} = \psi(t,\bar{r})\left[A(t,\bar{r})\,\mathrm{d}t - \frac{1}{2}B(t,\bar{r})\,\mathrm{d}\bar{r}\right], \tag{23.4}$$

where $\psi(t,\bar{r})$ is an **integrating factor** chosen to make the 1-form an exact differential:

$$\frac{\partial}{\partial\bar{r}}\left(\psi A\right) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\psi B\right). \tag{23.5}$$

Squaring Eq. (23.4):

$$\frac{\mathrm{d}\bar{t}^2}{\psi^2} = A^2\,\mathrm{d}t^2 - AB\,\mathrm{d}t\,\mathrm{d}\bar{r} + \frac{B^2}{4}\,\mathrm{d}\bar{r}^2. \tag{23.6}$$

Dividing by A:

$$-A\,\mathrm{d}t^2 + B\,\mathrm{d}t\,\mathrm{d}\bar{r} = -\frac{\mathrm{d}\bar{t}^2}{\psi^2 A} + \frac{B^2}{4A}\,\mathrm{d}\bar{r}^2. \tag{23.7}$$

Substituting this into Eq. (23.3):

$$\begin{aligned}\mathrm{d}s^2 &= -\frac{\mathrm{d}\bar{t}^2}{\psi^2 A} + \frac{B^2}{4A}\,\mathrm{d}\bar{r}^2 + C\,\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2 \\ &= -\frac{\mathrm{d}\bar{t}^2}{\psi^2 A} + \left(C + \frac{B^2}{4A}\right)\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2.\end{aligned} \tag{23.8}$$

Theorem: Standard Diagonal Form of Spherically Symmetric Metrics

Defining the positive metric potentials:

$$e^{2\Phi(\bar{t},\bar{r})} \equiv \frac{1}{\psi^2 A}, \qquad e^{2\Lambda(\bar{t},\bar{r})} \equiv C(t,\bar{r}) + \frac{B^2(t,\bar{r})}{4A(t,\bar{r})}, \tag{23.9}$$

any spherically symmetric spacetime metric can be reduced to the standard diagonal form:

$$\boxed{\mathrm{d}s^2 = -e^{2\Phi(\bar{t},\bar{r})}\,\mathrm{d}\bar{t}^2 + e^{2\Lambda(\bar{t},\bar{r})}\,\mathrm{d}\bar{r}^2 + \bar{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)} \tag{23.10}$$

For static spacetimes, the metric functions depend solely on radial coordinate r, yielding $\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$.

23.2 First-Principles Derivation of the Exterior Schwarzschild Metric

23.2.1 Static Spherically Symmetric Metric Ansatz

We consider the static spherically symmetric line element:

$$\mathrm{d}s^2 = -A(r)c^2\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{23.11}$$

where $A(r) > 0, B(r) > 0$, and primes denote radial derivatives ($A' \equiv \mathrm{d}A/\mathrm{d}r, B' \equiv \mathrm{d}B/\mathrm{d}r$).

23.2.2 Euler–Lagrange Derivation of Connection Coefficients

Using the geodesic kinetic energy Lagrangian $2\mathcal{L} = -A\dot{t}^2 + B\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$, the Euler–Lagrange equations yield all non-zero Christoffel symbols:

1. **Time component:** $\frac{\mathrm{d}}{\mathrm{d}\tau}(-A\dot{t}) = 0 \implies \ddot{t} + \frac{A'}{A}\dot{t}\dot{r} = 0 \implies \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{A'}{2A}$.
2. **Radial component:** $\frac{\mathrm{d}}{\mathrm{d}\tau}(B\dot{r}) - \frac{1}{2}(-A'\dot{t}^2 + B'\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2) = 0$:

$$\Gamma^r_{tt} = \frac{A'}{2B}, \qquad \Gamma^r_{rr} = \frac{B'}{2B}, \qquad \Gamma^r_{\theta\theta} = -\frac{r}{B}, \qquad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B}. \tag{23.12}$$

3. **Angular components:**

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{23.13}$$

23.2.3 Symmetries and Calculation of the Ricci Tensor

By time-reversal invariance ($t \to -t$) and parity reflections ($\theta \to -\theta, \phi \to -\phi$), all off-diagonal Ricci components vanish: $R_{tr} = R_{t\theta} = R_{t\phi} = R_{r\theta} = R_{r\phi} = R_{\theta\phi} = 0$. Rotational isometry on S^2 requires $R_{\phi\phi} = \sin^2\theta\,R_{\theta\theta}$.

Evaluating the diagonal components:

$$R_{tt} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB}, \tag{23.14}$$

$$R_{rr} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB}, \tag{23.15}$$

$$R_{\theta\theta} = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right). \tag{23.16}$$

23.2.4 Solving the Vacuum Equations and Weak-Field Matching

In vacuum ($R_{\mu\nu} = 0$), the linear combination $BR_{tt} + AR_{rr} = 0$ gives:

$$BR_{tt} + AR_{rr} = \frac{1}{rB}\left(A'B + AB'\right) = \frac{1}{rB}\frac{\mathrm{d}}{\mathrm{d}r}(AB) = 0 \implies A(r)B(r) = \text{const} = 1, \tag{23.17}$$

where asymptotic flatness ($A, B \to 1$ as $r \to \infty$) sets the constant to 1, so $B(r) = 1/A(r)$.

Substituting $B = 1/A$ and $B'/B = -A'/A$ into $R_{\theta\theta} = 0$:

$$R_{\theta\theta} = 1 - A - rA' = 1 - \frac{\mathrm{d}}{\mathrm{d}r}(rA) = 0 \implies rA(r) = r + C \implies A(r) = 1 + \frac{C}{r}. \tag{23.18}$$

Matching $g_{00} = -(1 + C/r)$ to the Newtonian limit $g_{00} \approx -(1 + 2\Phi/c^2) = -(1 - 2GM/c^2 r)$ via the Poisson equation $\nabla^2 h_{00} = -16\pi GM/c^2\,\delta^3(\vec{r})$ uniquely fixes $C = -\frac{2GM}{c^2}$.

Theorem: The Schwarzschild Metric

The unique static, spherically symmetric, asymptotically flat vacuum solution is:

$$\boxed{\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)} \tag{23.19}$$

23.3 Horizon Singularity and Curvature Invariants

where $f(r) \equiv 1 - \frac{2GM}{r}$, $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$, and $r_g \equiv 2GM$ is the Schwarzschild radius (gravitational radius in units with $c = 1$).

Key Concept: Coordinate vs. Physical Curvature Singularities

Examining the metric components in Eq. (23.19):

- As $r \to 2GM$, $g_{tt} = -f(r) \to 0$ and $g_{rr} = f(r)^{-1} \to \infty$.
- As $r \to 0$, $g_{tt} \to \infty$ and $g_{rr} \to 0$.

To distinguish between coordinate artifacts and genuine physical singularities, we compute diffeomorphism-invariant curvature scalars. In vacuum ($R_{\mu\nu} = 0, R = 0$), the lowest-order non-trivial scalar is the **Kretschmann scalar**:

$$K \equiv R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{48\, G^2 M^2}{r^6}. \tag{23.20}$$

- At $r = 2GM$: $K = \frac{48G^2M^2}{(2GM)^6} = \frac{3}{4G^4M^4} < \infty$. The curvature invariants are smooth and finite; $r = 2GM$ is a **removable coordinate singularity** marking a null event horizon.
- At $r = 0$: $K \to \infty$ and tidal forces diverge unboundedly. The origin $r = 0$ is a **genuine physical curvature singularity**.

23.3.1 Radial Null Geodesics in Schwarzschild Coordinates

For radial light rays ($\mathrm{d}\theta = \mathrm{d}\phi = 0$) traversing the spacetime, the null condition $\mathrm{d}s^2 = 0$ yields:

$$-f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} = 0 \implies \left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = \frac{1}{f(r)^2} = \left(1 - \frac{2GM}{r}\right)^{-2}. \tag{23.21}$$

Thus, the coordinate speed of radial light is:

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{23.22}$$

As $r \to 2GM$, $\left|\frac{\mathrm{d}t}{\mathrm{d}r}\right| \to \infty$, meaning the coordinate light cones $\Delta t/\Delta r$ close up and become infinitely steep. In static Schwarzschild coordinates, an infalling photon or particle appears to require an infinite coordinate time $t \to \infty$ to reach the horizon, even though its proper time τ remains finite.

23.4 Regge–Wheeler Tortoise Coordinates

Definition: Regge–Wheeler Tortoise Coordinate r_*

To eliminate the coordinate singularity in the radial null slopes, we define the **tortoise coordinate** r_* by:

$$\mathrm{d}r_* \equiv \frac{\mathrm{d}r}{f(r)} = \frac{\mathrm{d}r}{1 - \frac{2GM}{r}} = \left(1 + \frac{2GM}{r - 2GM}\right)\mathrm{d}r. \tag{23.23}$$

Integrating for $r > 2GM$:

$$r_*(r) = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|. \tag{23.24}$$

The tortoise coordinate maps the exterior domain $r \in (2GM, \infty)$ monotonically onto the entire real line:

$$r \to 2GM \implies r_* \to -\infty, \qquad r \to \infty \implies r_* \to +\infty. \tag{23.25}$$

In (t, r_*, θ, ϕ) coordinates, the line element takes the conformally flat $(1+1)$-dimensional radial form:

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r_*^2\right) + r^2(r_*)\,\mathrm{d}\Omega^2. \tag{23.26}$$

Radial null geodesics satisfy:

$$-\mathrm{d}t^2 + \mathrm{d}r_*^2 = 0 \implies \frac{\mathrm{d}t}{\mathrm{d}r_*} = \pm 1. \tag{23.27}$$

In the (t, r_*) plane, light cones maintain a constant 45° opening angle everywhere.

Defining the standard retarded and advanced null coordinates:

$$u \equiv t - r_*, \qquad v \equiv t + r_*, \qquad (-\infty < u, v < \infty), \tag{23.28}$$

the line element becomes:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}u\,\mathrm{d}v + r^2(u,v)\,\mathrm{d}\Omega^2. \tag{23.29}$$

23.5 Ingoing Eddington–Finkelstein Coordinates

23.5.1 Coordinate Construction and Horizon Regularity

While tortoise coordinates rectify the slope of null geodesics, the horizon is pushed to $r_* = -\infty$. To construct a coordinate system that is regular and non-singular across the future event horizon, we replace the static time coordinate t with the advanced null coordinate $v = t + r_*$.

Key Concept: Ingoing Eddington–Finkelstein Transformation

Using $t = v - r_*$, the differential is:

$$\mathrm{d}t = \mathrm{d}v - \mathrm{d}r_* = \mathrm{d}v - \frac{\mathrm{d}r}{1 - \frac{2GM}{r}}. \tag{23.30}$$

Substituting this into the standard Schwarzschild line element:

$$\begin{aligned}
\mathrm{d}s^2 &= -\left(1 - \frac{2GM}{r}\right)\left(\mathrm{d}v - \frac{\mathrm{d}r}{1 - \frac{2GM}{r}}\right)^2 + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{r}} + r^2\,\mathrm{d}\Omega^2 \\
&= -\left(1 - \frac{2GM}{r}\right)\left[\mathrm{d}v^2 - \frac{2\,\mathrm{d}v\,\mathrm{d}r}{1 - \frac{2GM}{r}} + \frac{\mathrm{d}r^2}{\left(1 - \frac{2GM}{r}\right)^2}\right] + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{r}} + r^2\,\mathrm{d}\Omega^2 \\
&= -\left(1 - \frac{2GM}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2.
\end{aligned} \tag{23.31}$$

Theorem: Metric Regularity at the Horizon

In the coordinates (v, r, θ, ϕ), the metric tensor components are:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{23.32}$$

The metric determinant is:

$$g \equiv \det(g_{\mu\nu}) = \left(0 - 1^2\right) \cdot r^4 \sin^2\theta = -r^4 \sin^2\theta. \tag{23.33}$$

Because $\det(g) = -r^4 \sin^2\theta \neq 0$ and all metric components are smooth for all $r > 0$, the metric is **manifestly non-singular and analytic across the entire event horizon** $r = 2GM$.

23.5.2 Lightcone Tilting and the Trapped Horizon

For radial null geodesics ($\mathrm{d}s^2 = 0, \mathrm{d}\Omega = 0$) in ingoing Eddington–Finkelstein coordinates:

$$-\left(1 - \frac{2GM}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r = \mathrm{d}v\left[-\left(1 - \frac{2GM}{r}\right)\mathrm{d}v + 2\,\mathrm{d}r\right] = 0. \tag{23.34}$$

This factors into two distinct branches of radial light rays:

1. **Ingoing Null Rays**:
$$\mathrm{d}v = 0 \implies v = \text{const} \implies \frac{\mathrm{d}v}{\mathrm{d}r} = 0. \tag{23.35}$$
Ingoing light rays are straight horizontal lines at 45° in (t, r) or straight lines along $v = \text{const}$, freely propagating across $r = 2GM$ into the singularity $r = 0$ in finite proper time.

2. **Outgoing Null Rays**:
$$-\left(1 - \frac{2GM}{r}\right)\mathrm{d}v + 2\,\mathrm{d}r = 0 \implies \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2}{1 - \frac{2GM}{r}}. \tag{23.36}$$

Analyzing the outgoing light ray slope across regions (Figure 23.1):

- For $r > 2GM$: $1 - \frac{2GM}{r} > 0 \implies \frac{\mathrm{d}v}{\mathrm{d}r} > 0 \iff \frac{\mathrm{d}r}{\mathrm{d}v} > 0$. Outgoing light rays move outward to larger radii.
- At $r = 2GM$: $\frac{\mathrm{d}v}{\mathrm{d}r} = \infty \iff \frac{\mathrm{d}r}{\mathrm{d}v} = 0$. Outgoing photons stay frozen at $r = 2GM$; the event horizon is a **null hypersurface generated by outgoing null geodesics**.
- For $r < 2GM$: $1 - \frac{2GM}{r} < 0 \implies \frac{\mathrm{d}r}{\mathrm{d}v} < 0$. The outgoing light cone tilts inward: **both** null ray solutions have $\mathrm{d}r < 0$. The radial coordinate r becomes a timelike coordinate, making future motion toward $r = 0$ as inevitable as future motion in time.

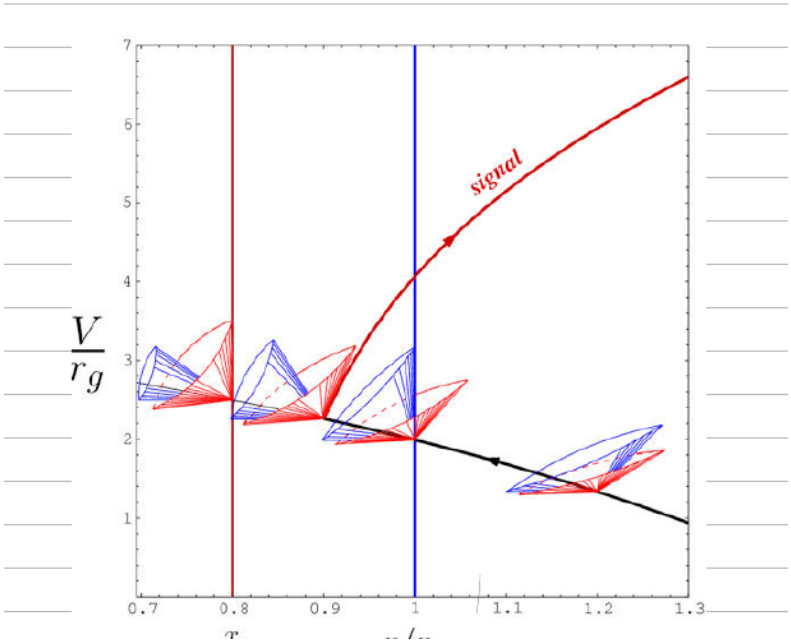


Figure 23.1: Lightcone tilting across the event horizon in ingoing Eddington–Finkelstein coordinates $(v/r_g, r/r_g)$. For $r > r_g = 2GM$, outgoing rays escape to infinity ($\mathrm{d}r/\mathrm{d}v > 0$). At the horizon $r = r_g$, outgoing light rays are trapped at $\mathrm{d}r/\mathrm{d}v = 0$, making the horizon a null hypersurface. For $r < r_g$, both the ingoing and outgoing null boundaries of the lightcone point strictly inward toward decreasing r, forcing all physical trajectories into the central curvature singularity at $r = 0$.

23.6 Painlevé–Gullstrand Coordinates and the River Model of Space

While Eddington–Finkelstein coordinates utilize a null foliation ($\mathrm{d}v = 0$ corresponds to incoming light rays), **Painlevé–Gullstrand coordinates** provide an alternative, deeply intuitive spacetime slicing based on the proper time of **freely falling observers dropped from rest at infinity**.

23.6.1 Physical Construction: The Freely Falling "Rain" Frame

Consider a family of test particles dropped radially from rest at spatial infinity ($r \to \infty, \dot{r} \to 0$). The conserved energy per unit mass along each radial geodesic is:

$$E \equiv -\xi_\mu u^\mu = \left(1 - \frac{2GM}{c^2 r}\right) c \frac{\mathrm{d}t}{\mathrm{d}\tau} = c \implies \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \tag{23.37}$$

where τ is the proper time registered by the freely falling clock. From the 4-velocity normalization $u^\mu u_\mu = -c^2$:

$$-\left(1 - \frac{2GM}{c^2 r}\right) c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -c^2. \tag{23.38}$$

Substituting $\frac{\mathrm{d}t}{\mathrm{d}\tau}$ yields the radial free-fall velocity:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -c\sqrt{\frac{2GM}{c^2 r}} \equiv -v_{\mathrm{ff}}(r). \tag{23.39}$$

Remarkably, the relativistic free-fall velocity $v_{\mathrm{ff}}(r) = \sqrt{\frac{2GM}{r}}$ coincides identically with the Newtonian escape velocity from radius r.

23.6.2 Coordinate Transformation to Painlevé Time

We define the new time coordinate t_{PG} to coincide with the proper time τ recorded by these freely falling rain observers, synchronized at $r \to \infty$. Along any radial displacement:

$$c\,\mathrm{d}t_{\mathrm{PG}} \equiv c\,\mathrm{d}\tau = c\,\mathrm{d}t - \frac{\sqrt{\frac{2GM}{c^2 r}}}{1-\frac{2GM}{c^2 r}}\,\mathrm{d}r \iff c\,\mathrm{d}t = c\,\mathrm{d}t_{\mathrm{PG}} + \frac{\sqrt{\frac{2GM}{c^2 r}}}{1-\frac{2GM}{c^2 r}}\,\mathrm{d}r. \tag{23.40}$$

Substituting Eq. (23.40) into the standard Schwarzschild line element (setting $c = 1$ for clarity):

$$\begin{aligned}
\mathrm{d}s^2 &= -\left(1-\frac{2GM}{r}\right)\left(\mathrm{d}t_{\mathrm{PG}} + \frac{\sqrt{\frac{2GM}{r}}}{1-\frac{2GM}{r}}\,\mathrm{d}r\right)^2 + \frac{\mathrm{d}r^2}{1-\frac{2GM}{r}} + r^2\,\mathrm{d}\Omega^2 \\
&= -\left(1-\frac{2GM}{r}\right)\mathrm{d}t_{\mathrm{PG}}^2 - 2\sqrt{\frac{2GM}{r}}\,\mathrm{d}t_{\mathrm{PG}}\,\mathrm{d}r - \frac{2GM/r}{1-\frac{2GM}{r}}\,\mathrm{d}r^2 + \frac{\mathrm{d}r^2}{1-\frac{2GM}{r}} + r^2\,\mathrm{d}\Omega^2 \\
&= -\left(1-\frac{2GM}{r}\right)\mathrm{d}t_{\mathrm{PG}}^2 - 2\sqrt{\frac{2GM}{r}}\,\mathrm{d}t_{\mathrm{PG}}\,\mathrm{d}r + \left(\frac{1-\frac{2GM}{r}}{1-\frac{2GM}{r}}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2.
\end{aligned} \tag{23.41}$$

For infalling observers (absorbing the sign into the cross-term), the line element is:

Theorem: The Painlevé–Gullstrand Metric

In coordinates $(t_{\mathrm{PG}}, r, \theta, \phi)$, the Schwarzschild line element is:

$$\boxed{\mathrm{d}s^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2\mathrm{d}t_{\mathrm{PG}}^2 + 2c\sqrt{\frac{2GM}{c^2 r}}\,\mathrm{d}t_{\mathrm{PG}}\,\mathrm{d}r + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)} \tag{23.42}$$

23.6.3 Remarkable Geometric Properties

The Painlevé–Gullstrand form possesses three extraordinary properties:

1. **Spatially Flat Hypersurfaces:** On any constant time slice $t_{\mathrm{PG}} = \text{constant}$ ($\mathrm{d}t_{\mathrm{PG}} = 0$), the induced 3-metric is:

$$\mathrm{d}\sigma_{(3)}^2 = \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{23.43}$$

   The spatial 3-slices are **strictly flat Euclidean space** $\mathbb{R}^3$! Spacetime curvature is encoded entirely in the non-zero off-diagonal shift term $g_{t_{\mathrm{PG}}r}$ and lapse function.

2. **Manifest Horizon Regularity:** The metric tensor components are:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{r}\right) & \sqrt{\frac{2GM}{r}} & 0 & 0 \\ \sqrt{\frac{2GM}{r}} & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad \det(g_{\mu\nu}) = -r^4\sin^2\theta. \tag{23.44}$$

   Because $\det(g_{\mu\nu}) \neq 0$ and all components are completely smooth for all $r > 0$, the metric is analytic across the event horizon $r = 2GM$.

3. **Continuous Universal Free-Fall Time:** The coordinate t_{PG} is the exact proper time measured by freely falling rain observers, meaning clocks carried by infalling astronauts smoothly tick across the horizon into the singularity at $r = 0$ without any coordinate freezing.

23.6.4 The River Model of Spacetime

Completing the square in the Painlevé–Gullstrand line element (23.42):

$$\boxed{\mathrm{d}s^2 = -c^2\mathrm{d}t_{\mathrm{PG}}^2 + \left(\mathrm{d}r + v_{\mathrm{river}}(r)\,\mathrm{d}t_{\mathrm{PG}}\right)^2 + r^2\,\mathrm{d}\Omega^2, \qquad v_{\mathrm{river}}(r) \equiv \sqrt{\frac{2GM}{r}}} \tag{23.45}$$

Key Concept: Physical Interpretation: The River Model

Equation (23.45) presents the celebrated **River Model of Black Holes** (Hamilton and Lisle):

- Space itself behaves like a flat Euclidean medium (a "river") flowing radially inward with velocity $v_{\mathrm{river}}(r) = \sqrt{\frac{2GM}{r}}$ toward the central singularity.
- Physical objects and light move through the locally flowing river at speed $\leq c$ relative to the fluid.
- **Subsonic/Subluminal Upstream ($r > 2GM$):** $v_{\mathrm{river}} < c$. Photons swimming radially outward against the river move faster than the inflow and successfully escape to infinity.
- **The Event Horizon ($r = 2GM$):** $v_{\mathrm{river}} = c$. The inflow velocity exactly equals the speed of light. Outgoing photons swimming outward at speed c stand motionless at $r = 2GM$ relative to asymptotic space ($\mathrm{d}r/\mathrm{d}t_{\mathrm{PG}} = 0$).
- **Supercritical/Superluminal Inflow ($r < 2GM$):** $v_{\mathrm{river}} > c$. The river cascades faster than light. Even outward-directed photons are inexorably dragged backward toward the singularity at $r = 0$.

23.6.5 Lightcone Tilting and Radial Null Geodesics

Setting $\mathrm{d}s^2 = 0$ and $\mathrm{d}\Omega = 0$ in Eq. (23.45):

$$-\mathrm{d}t_{\mathrm{PG}}^2 + \left(\mathrm{d}r + \sqrt{\frac{2GM}{r}}\,\mathrm{d}t_{\mathrm{PG}}\right)^2 = 0 \implies \frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{PG}}} = -\sqrt{\frac{2GM}{r}} \pm 1. \tag{23.46}$$

- **Ingoing Light Rays:** $\frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{PG}}} = -\left(1 + \sqrt{\frac{2GM}{r}}\right) < -1$. Photons fall inward faster than standard speed of light because their motion adds to the river current.
- **Outgoing Light Rays:** $\frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{PG}}} = 1 - \sqrt{\frac{2GM}{r}}$.
  - For $r > 2GM$: $\frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{PG}}} > 0$ (escapes to infinity).
  - At $r = 2GM$: $\frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{PG}}} = 0$ (held stationary at the horizon).
  - For $r < 2GM$: $\frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{PG}}} < 0$ (swept inward into the singularity).

23.7 Kruskal–Szekeres Coordinates and Maximal Extension

23.7.1 Construction of Kruskal Coordinates

To construct a coordinate system that is regular across both future and past horizons simultaneously, we introduce the Kruskal–Szekeres coordinates.

Starting from the double null coordinates (u, v) with $u = t - r_*$ and $v = t + r_*$, we define the null Kruskal coordinates (u', v'):

$$v' \equiv e^{v/(4GM)} = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r+t)/(4GM)}, \tag{23.47}$$

$$u' \equiv -e^{-u/(4GM)} = -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r-t)/(4GM)}. \tag{23.48}$$

Taking differentials and computing the line element:

$$\mathrm{d}u'\,\mathrm{d}v' = \frac{1}{(4GM)^2}\left(\frac{r}{2GM} - 1\right) e^{r/(2GM)}\,\mathrm{d}u\,\mathrm{d}v = \frac{r - 2GM}{32G^3M^3} e^{r/(2GM)}\,\mathrm{d}u\,\mathrm{d}v. \tag{23.49}$$

Substituting into Eq. (23.29):

$$\mathrm{d}s^2 = -\frac{32G^3M^3}{r} e^{-r/(2GM)}\,\mathrm{d}u'\,\mathrm{d}v' + r^2(u', v')\,\mathrm{d}\Omega^2. \tag{23.50}$$

We now define the timelike coordinate T and spacelike coordinate R:

$$T \equiv \frac{1}{2}\left(v' + u'\right) = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/(4GM)} \sinh\left(\frac{t}{4GM}\right), \tag{23.51}$$

$$R \equiv \frac{1}{2}\left(v' - u'\right) = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/(4GM)} \cosh\left(\frac{t}{4GM}\right). \tag{23.52}$$

Theorem: The Kruskal–Szekeres Metric

In coordinates (T, R, θ, ϕ), the Schwarzschild line element becomes:

$$\mathrm{d}s^2 = \frac{32G^3M^3}{r} e^{-r/(2GM)}\left(-\mathrm{d}T^2 + \mathrm{d}R^2\right) + r^2(T, R)\,\mathrm{d}\Omega^2, \tag{23.53}$$

where the radial coordinate $r(T, R)$ is determined implicitly by the algebraic relation:

$$R^2 - T^2 = \left(\frac{r}{2GM} - 1\right) e^{r/(2GM)}. \tag{23.54}$$

The static time t is given by:

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right). \tag{23.55}$$

23.7.2 Spacetime Structure of the Kruskal Diagram

The Kruskal coordinates $(-\infty < T < \infty, -\infty < R < \infty)$ reveal the full global analytic extension of the geometry (Figure 23.2):

1. **Surfaces of Constant** r: Hyperbolae $R^2 - T^2 = \text{const}$.
   - For $r > 2GM$: Timelike hyperbolae in the right (Region I) and left (Region III) quadrants.
   - At $r = 2GM$: The null asymptotes $T = \pm R$.
   - At $r = 0$: Spacelike hyperbolae $T^2 - R^2 = 1$ in Region II (future singularity) and Region IV (past singularity).

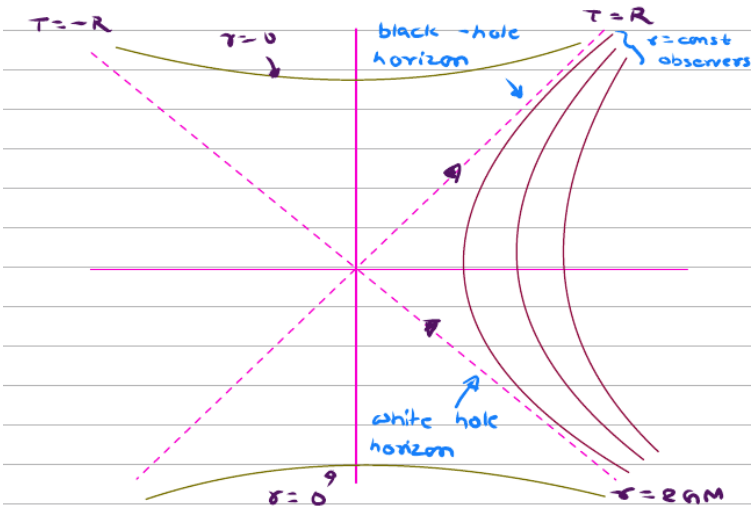


Figure 23.2: The Kruskal–Szekeres diagram representing the maximal analytic extension of Schwarzschild spacetime. Radial null rays travel at strictly $\pm 45°$. The four distinct spacetime regions are: Region I (our exterior universe, $R > |T|$), Region II (future black hole interior, $T > |R|$), Region III (parallel mirror universe, $R < -|T|$), and Region IV (past white hole interior, $T < -|R|$). The physical singularities are hyperbolae at $T^2 - R^2 = 1$ ($r = 0$), while future/past horizons are the null lines $T = \pm R$ ($r = 2GM$).

2. **Surfaces of Constant t**: Straight radial lines through the origin with slope $T/R = \tanh(t/4GM)$.
   - $t \to +\infty$: Approaches the future horizon $T = R$ ($u' = 0$).
   - $t \to -\infty$: Approaches the past horizon $T = -R$ ($v' = 0$).
   - $t = 0$: The horizontal axis $T = 0$.
3. **The Four Spacetime Quadrants**:
   - **Region I (Exterior Universe)**: $r > 2GM, R > |T|$.
   - **Region II (Black Hole)**: $r < 2GM, T > |R|$. Everything entering crosses $T = R$ and must hit the spacelike singularity at $r = 0$.
   - **Region III (Mirror Universe)**: $r > 2GM, R < -|T|$. An identical asymptotically flat universe.
   - **Region IV (White Hole)**: $r < 2GM, T < -|R|$. Time-reversed black hole from which matter and light can emerge across $T = -R$, but nothing can enter.
4. **The Einstein–Rosen Bridge (Wormhole)**: On the spatial slice $T = 0$, the metric is $\mathrm{d}s^2 = \frac{32G^3M^3}{r} e^{-r/(2GM)} \mathrm{d}R^2 + r^2 \, \mathrm{d}\Omega^2$. As R goes from $-\infty \to 0 \to +\infty$, the area radius r decreases from ∞ to a minimum throat at $r = 2GM$ ($R = 0$) and expands back to ∞. Because the throat pinches off faster than the speed of light, the wormhole is non-traversable.

23.7.3 Global Time Orientation: Kruskal Time T vs. Schwarzschild Coordinate t

A fundamental question in the global geometry of the maximally extended Schwarzschild spacetime is: *What determines the true physical direction of time across all four quadrants, and why does the static coordinate t behave differently in the left and right wedges?*

Key Concept: Global Time Orientation and the Arrow of Time in Kruskal Coordinates

In the Kruskal–Szekeres line element (23.53):

$$\mathrm{d}s^2 = \frac{32G^3M^3}{r} e^{-r/(2GM)} \left(-\mathrm{d}T^2 + \mathrm{d}R^2\right) + r^2(T,R)\,\mathrm{d}\Omega^2, \tag{23.56}$$

the conformal factor $\frac{32G^3M^3}{r} e^{-r/(2GM)}$ is strictly positive and non-singular everywhere outside the physical curvature singularity ($0 < r < \infty$).
Consequently, the light cones in the (T, R) plane are **everywhere oriented at strictly** $\pm 45°$, opening vertically along the $+T$ axis. This provides a globally consistent, smooth **time orientation** across the entire manifold:

- The physical future direction is universally defined by the vector field ∂_T (the direction of **increasing Kruskal time** T).
- Any physical particle (timelike worldline parameterized by proper time τ) or photon (null worldline parameterized by an affine parameter λ) must move forward in global time:
$$\frac{\mathrm{d}T}{\mathrm{d}\tau} > 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}\lambda} > 0. \tag{23.57}$$
The affine parameter λ and proper time τ always increase in the direction of increasing T, establishing a universal causal arrow of time throughout all four quadrants.

The Reversal of Schwarzschild Time t in the Left Wedge (Region III)

Confusion often arises because the static Schwarzschild time coordinate t behaves very differently in the right (Region I) and left (Region III) exterior wedges:

1. **Right Wedge (Region I, $R > |T|$):** Here $R > 0$, and the coordinate relations (23.51)–(23.52) yield:
$$\frac{\partial T}{\partial t} = \frac{R}{4GM} > 0. \tag{23.58}$$
The static timelike Killing vector $\xi^a = (\partial_t)^a$ is **future-directed** (∂_t points upwards in the diagram). Thus, in our exterior universe, increasing Schwarzschild time t coincides with increasing Kruskal time T and increasing proper time τ.

2. **Left Wedge / Mirror Universe (Region III, $R < -|T|$):** In the left quadrant, $R < 0$. The coordinate transformation from Schwarzschild coordinates (t, r) to Kruskal coordinates is:
$$\begin{aligned} T &= -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/(4GM)} \sinh\left(\frac{t}{4GM}\right), \\ R &= -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/(4GM)} \cosh\left(\frac{t}{4GM}\right). \end{aligned} \tag{23.59}$$
Differentiating with respect to t:
$$\frac{\partial T}{\partial t} = \frac{R}{4GM} < 0 \quad (\text{since } R < 0). \tag{23.60}$$
The Killing Vector Points Downwards: In Region III, the static Killing vector $\xi^a = (\partial_t)^a$ points in the direction of **decreasing Kruskal time** T (towards the global past)!

Physical Insight & Motivation: Resolution: Coordinate Time t vs. Physical Time τ and Affine Parameter λ

Does time run backwards for an observer living in Region III?

- **No!** Physical observers in Region III are governed by local physics and the global light-cone structure. Their worldlines must have $\mathrm{d}T/\mathrm{d}\tau > 0$, meaning they travel **upward** in the Kruskal diagram toward the future black hole horizon ($T > 0$).
- Because $\partial T/\partial t < 0$ in Region III, a physical observer moving into the future experiences **decreasing coordinate time** t ($\mathrm{d}t/\mathrm{d}\tau < 0$).
- The coordinate t is merely a label associated with the static isometry ∂_t. The true physical arrow of time is determined by the global Kruskal time T (along which proper time τ and affine parameter λ increase), not by the static coordinate t.

Causal Flow in the Interior Regions (II and IV)

- **Region II (Future Black Hole Interior, $T > |R|$):** Since $T > 0$, all future-directed timelike curves ($\mathrm{d}T/\mathrm{d}\tau > 0$) are inexorably directed toward increasing T. From $T^2 - R^2 = (1 - r/2GM)e^{r/2GM}$, increasing T forces the areal radius r to decrease monotonically toward $r = 0$. The spacelike singularity at $r = 0$ ($T = \sqrt{1 + R^2}$) lies strictly in the **future** of all events in Region II.
- **Region IV (Past White Hole Interior, $T < -|R|$):** Here $T < 0$. The spacelike singularity at $r = 0$ ($T = -\sqrt{1 + R^2}$) lies in the global **past**. Because all physical trajectories must move toward increasing T ($\mathrm{d}T/\mathrm{d}\tau > 0$), matter and radiation can only emerge *out* of the singularity and cross the past horizon $T = -R$ into Regions I or III; nothing can enter Region IV from the outside.

23.8 Conformal Penrose–Carter Diagrams

To map the infinite Kruskal spacetime onto a compact, finite diagram while preserving causal lightcone angles (45°), we perform a conformal compactification.

Definition: Conformal Transformation to Penrose Coordinates

Define coordinates (ψ, ξ) via:

$$T + R = \tan\left[\frac{1}{2}(\psi + \xi)\right], \tag{23.61}$$

$$T - R = \tan\left[\frac{1}{2}(\psi - \xi)\right], \tag{23.62}$$

with range $-\pi < \psi < \pi$ and $-\pi < \xi < \pi$.

In (ψ, ξ), the line element becomes:

$$\mathrm{d}s^2 = \frac{32G^3M^3}{r}\frac{e^{-r/(2GM)}}{4\cos^2\left[\frac{1}{2}(\psi+\xi)\right]\cos^2\left[\frac{1}{2}(\psi-\xi)\right]}\left(-\mathrm{d}\psi^2 + \mathrm{d}\xi^2\right) + r^2\,\mathrm{d}\Omega^2 = \Omega^{-2}(\psi,\xi)\,\mathrm{d}\tilde{s}^2. \tag{23.63}$$

The conformal boundary structure shown in Figure 23.3 consists of:

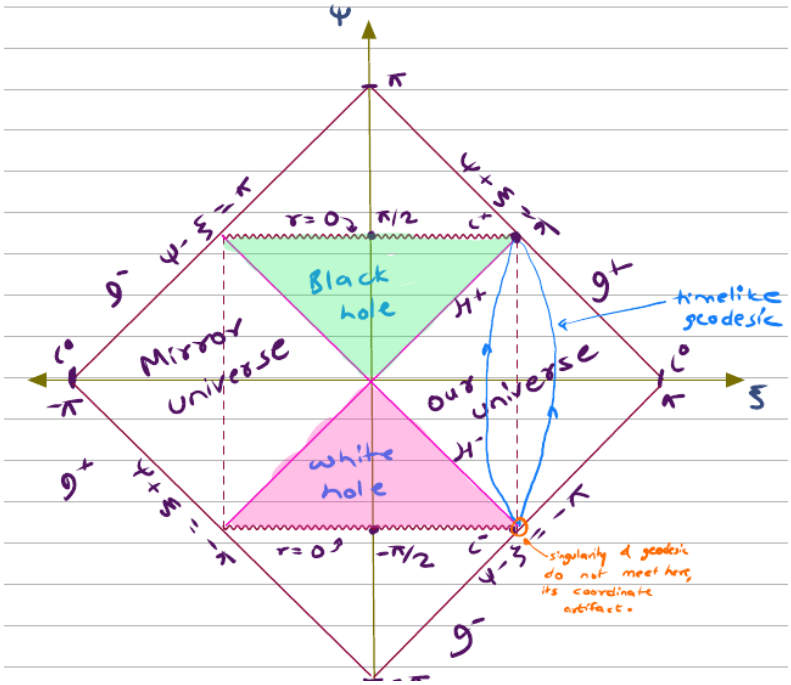


Figure 23.3: Penrose–Carter conformal diagram of the maximally extended Schwarzschild geometry. Light cones are oriented at precisely $\pm 45°$. Boundaries include: future/past null infinity $\mathcal{I}^+, \mathcal{I}^-$; future/past timelike infinity i^+, i^-; spatial infinity i^0; future/past event horizons $\mathcal{H}^+, \mathcal{H}^-$; and the physical spacelike singularities $r = 0$ at $\psi = \pm\pi/2$.

- $\mathcal{I}^+$ (**Future Null Infinity**): Where outgoing light rays terminate ($r \to \infty, t \to \infty$).
- $\mathcal{I}^-$ (**Past Null Infinity**): Where ingoing light rays originate ($r \to \infty, t \to -\infty$).
- i^+ (**Future Timelike Infinity**): Where massive particles at rest or bounded orbits end ($t \to +\infty$).
- i^- (**Past Timelike Infinity**): Where massive particles originate ($t \to -\infty$).
- i^0 (**Spatial Infinity**): Spatial asymptotic limit ($r \to \infty$ at finite t).
- $\mathcal{H}^+$ and $\mathcal{H}^-$: The future and past event horizons at $r = 2GM$.
- $r = 0$: The spacelike curvature singularities represented by jagged horizontal boundaries at $\psi = +\pi/2$ (black hole) and $\psi = -\pi/2$ (white hole).

23.9 Marginally Bound Geodesic Congruences in Schwarzschild

To illustrate the kinematic decomposition and Raychaudhuri equation developed in Chapter 22, we consider a congruence of freely falling, radial, marginally bound timelike particles in Schwarzschild spacetime.

Theorem: Kinematics of Marginally Bound Geodesics

A particle of unit mass ($m = 1$) dropped from rest at infinity has conserved energy $E \equiv -u_a \xi^a_{(t)} = 1$. Its 4-velocity field $u^a = (\dot{t}, \dot{r}, 0, 0)$ satisfies:

$$u_t = -1 \implies u^t = g^{tt} u_t = \frac{1}{f(r)}, \tag{23.64}$$

$$u_r = \pm \frac{1}{f(r)} \sqrt{\frac{2GM}{r}} \implies u^r = g^{rr} u_r = \pm \sqrt{\frac{2GM}{r}}, \tag{23.65}$$

where $(+)$ denotes outgoing geodesics and $(-)$ denotes ingoing geodesics.

Proof. From the normalization condition $g_{ab} u^a u^b = -1$:

$$-f(r) \left(u^t\right)^2 + \frac{1}{f(r)} (u^r)^2 = -1 \implies -\frac{1}{f(r)} + \frac{1}{f(r)} (u^r)^2 = -1. \tag{23.66}$$

Multiplying by $f(r) = 1 - \frac{2GM}{r}$:

$$-1 + (u^r)^2 = -1 + \frac{2GM}{r} \implies u^r = \pm \sqrt{\frac{2GM}{r}}. \tag{23.67}$$

□

23.9.1 Expansion Scalar and Explicit Verification of the Focusing Theorem

We compute the expansion scalar $\theta \equiv \nabla_a u^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g}\, u^a)$. For the Schwarzschild metric, $\sqrt{-g} = r^2 \sin\theta$:

$$\begin{aligned} \theta &= \frac{1}{r^2 \sin\theta} \partial_r \left(r^2 \sin\theta\, u^r \right) = \frac{1}{r^2} \partial_r \left(\pm r^2 \sqrt{\frac{2GM}{r}} \right) \\ &= \pm \frac{\sqrt{2GM}}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^{3/2} \right) = \pm \frac{3}{2} \frac{\sqrt{2GM}}{r^2} r^{1/2} = \pm \frac{3}{2} \sqrt{\frac{2GM}{r^3}}. \end{aligned} \tag{23.68}$$

Thus:

- For outgoing geodesics $(+)$: $\theta > 0$, the congruence is **diverging**.
- For ingoing geodesics $(-)$: $\theta < 0$, the congruence is **converging**.

Now we evaluate the rate of change of θ along the proper time τ of the congruence:

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= u^r \frac{\mathrm{d}\theta}{\mathrm{d}r} = \left(\pm \sqrt{\frac{2GM}{r}} \right) \frac{\mathrm{d}}{\mathrm{d}r} \left(\pm \frac{3}{2} \sqrt{2GM}\, r^{-3/2} \right) \\ &= \sqrt{\frac{2GM}{r}} \left(\frac{3}{2} \sqrt{2GM} \right) \left(-\frac{3}{2} r^{-5/2} \right) \\ &= -\frac{9GM}{2r^3} < 0. \end{aligned} \tag{23.69}$$

Because $\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{9GM}{2r^3}$ is strictly negative everywhere, this provides an exact analytical realization of the **Raychaudhuri focusing theorem** in vacuum ($R_{\mu\nu} = 0$).

23.10 Hamilton–Jacobi Theory and Orbital Mechanics

23.10.1 The Relativistic Hamilton–Jacobi Equation

For a test particle of rest mass m moving in an external metric $g_{\mu\nu}$, let $S(x^\mu)$ denote Hamilton's principal function (action). The 4-momentum is given by the gradient:

$$p_a = \partial_a S, \qquad p^a = g^{ab}\partial_b S. \tag{23.70}$$

The relativistic mass-shell condition $g^{ab}p_a p_b = -m^2$ yields the **Hamilton–Jacobi equation**:

$$g^{ab}\,\partial_a S\,\partial_b S + m^2 = 0. \tag{23.71}$$

In Schwarzschild coordinates:

$$-\frac{1}{f(r)}\left(\frac{\partial S}{\partial t}\right)^2 + f(r)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 + m^2 = 0. \tag{23.72}$$

23.10.2 Separation of Variables in Equatorial Motion

Due to spherical symmetry, any orbit is confined to a 2D plane; without loss of generality, we choose the equatorial plane $\theta = \pi/2$ ($\partial_\theta S = 0, \sin\theta = 1$).

Since t and ϕ are cyclic coordinates associated with the Killing vectors $\xi_{(t)} = \partial_t$ and $\xi_{(\phi)} = \partial_\phi$, the conserved energy E and angular momentum L are:

$$p_0 = \frac{\partial S}{\partial t} = -E, \qquad p_\phi = \frac{\partial S}{\partial \phi} = L. \tag{23.73}$$

We separate the action:

$$S(t, r, \phi) = -E\,t + L\,\phi + S_r(r). \tag{23.74}$$

Substituting this ansatz into Eq. (23.72):

$$-\frac{E^2}{f(r)} + f(r)\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \frac{L^2}{r^2} + m^2 = 0. \tag{23.75}$$

Solving algebraically for $\frac{\mathrm{d}S_r}{\mathrm{d}r}$:

$$\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 = \frac{1}{f(r)^2}\left[E^2 - f(r)\left(m^2 + \frac{L^2}{r^2}\right)\right], \tag{23.76}$$

which integrates to:

$$S_r(r) = \pm\int \mathrm{d}r\,\frac{1}{f(r)}\sqrt{E^2 - \left(1 - \frac{2GM}{r}\right)\left(m^2 + \frac{L^2}{r^2}\right)}. \tag{23.77}$$

Theorem: Complete Action Integral

The complete separated Hamilton–Jacobi action for equatorial motion in Schwarzschild spacetime is:

$$S(t, r, \phi; E, L) = -E\,t + L\,\phi \pm \int \frac{\mathrm{d}r}{1 - \frac{2GM}{r}}\sqrt{E^2 - \left(1 - \frac{2GM}{r}\right)\left(m^2 + \frac{L^2}{r^2}\right)}. \tag{23.78}$$

23.10.3 Equations of Motion via Jacobi's Variation

According to Jacobi's theorem of analytical mechanics, the trajectory equations are obtained by differentiating the action S with respect to the constants of motion $\alpha_i = (E, L)$ and setting the results equal to new constants $\beta_i = (-t_0, \phi_0)$:

1. **Coordinate Time Evolution** $t(r)$:

$$\frac{\partial S}{\partial E} = -t_0 \implies -t \pm \int \mathrm{d}r \, \frac{E}{f(r)\sqrt{E^2 - f(r)\left(m^2 + \frac{L^2}{r^2}\right)}} = -t_0, \tag{23.79}$$

which yields:

$$t - t_0 = \pm \int \mathrm{d}r \, \frac{E}{\left(1 - \frac{2GM}{r}\right)\sqrt{E^2 - \left(1 - \frac{2GM}{r}\right)\left(m^2 + \frac{L^2}{r^2}\right)}}. \tag{23.80}$$

2. **Orbital Path** $\phi(r)$:

$$\frac{\partial S}{\partial L} = \phi_0 \implies \phi \mp \int \mathrm{d}r \, \frac{L}{r^2\sqrt{E^2 - f(r)\left(m^2 + \frac{L^2}{r^2}\right)}} = \phi_0, \tag{23.81}$$

which yields the exact trajectory relation:

$$\phi - \phi_0 = \pm \int \mathrm{d}r \, \frac{L/r^2}{\sqrt{E^2 - \left(1 - \frac{2GM}{r}\right)\left(m^2 + \frac{L^2}{r^2}\right)}}. \tag{23.82}$$

23.10.4 Effective Potential and Characteristic Orbits

Defining the specific energy $\mathcal{E} \equiv E/m$ and specific angular momentum $\ell \equiv L/m$, the radial equation of motion $\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}(r) = \mathcal{E}^2$ is governed by the **effective potential**:

$$V_{\text{eff}}(r) \equiv \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) = 1 - \frac{2GM}{r} + \frac{\ell^2}{r^2} - \frac{2GM\ell^2}{r^3}. \tag{23.83}$$

Key Concept: Relativistic Orbital Regimes

The relativistic $-2GM\ell^2/r^3$ correction modifies the Newtonian effective potential fundamentally:

1. **Innermost Stable Circular Orbit (ISCO)**: Setting $V'_{\text{eff}}(r) = 0$ and $V''_{\text{eff}}(r) = 0$ yields:

$$r_{\text{ISCO}} = 6GM, \qquad \ell_{\text{ISCO}} = \sqrt{12}\, GM. \tag{23.84}$$

For $r < 6GM$, no stable circular orbits exist; test particles inevitably plunge into the event horizon.

2. **Photon Sphere (Null Geodesics** $m = 0$**)**: Setting $m = 0$ gives $V_{\text{eff,null}}(r) = \frac{L^2}{r^2}\left(1 - \frac{2GM}{r}\right)$. Extrema occur at:

$$r_{\text{ph}} = 3GM. \tag{23.85}$$

At $r = 3GM$, photons execute an unstable circular orbit around the black hole.

23.11 The Kerr Metric and the Newman–Janis Algorithm

The **Kerr metric** represents the unique stationary, axisymmetric, asymptotically flat vacuum solution to Einstein's field equations describing a rotating black hole of mass M and specific angular momentum $a \equiv J/M$. In standard **Boyer–Lindquist coordinates** (t, r, θ, ϕ), the Kerr line element is given by:

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\rho^2}\mathrm{d}t\,\mathrm{d}\phi + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\,\mathrm{d}\theta^2 + \frac{\Sigma}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2, \tag{23.86}$$

where the auxiliary metric functions are defined by:

$$\rho^2 \equiv r^2 + a^2\cos^2\theta, \qquad \Delta \equiv r^2 - 2Mr + a^2, \qquad \Sigma \equiv (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \tag{23.87}$$

In matrix form, the covariant metric components $g_{\mu\nu}$ in Boyer–Lindquist coordinates (t, r, θ, ϕ) are:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2Mr}{\rho^2}\right) & 0 & 0 & -\frac{2Mar\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2Mar\sin^2\theta}{\rho^2} & 0 & 0 & \frac{\Sigma}{\rho^2}\sin^2\theta \end{pmatrix}. \tag{23.88}$$

23.11.1 Newman–Penrose Tetrads for Schwarzschild Spacetime

To derive the rotating solution from the static Schwarzschild solution, we employ the **Newman–Janis algorithm** using the Newman–Penrose null tetrad formalism.

We begin with the Schwarzschild metric in static coordinates:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{23.89}$$

Transforming to advanced Eddington–Finkelstein coordinates with $v \equiv t + r_*$ (where $\mathrm{d}v = \mathrm{d}t + \frac{\mathrm{d}r}{1-2M/r}$), the line element becomes:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{23.90}$$

In the basis (v, r, θ, ϕ), the metric and its inverse are:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{23.91}$$

In terms of a complex null tetrad basis $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ satisfying the normalization conditions:

$$l\cdot n = 1, \qquad m\cdot\bar{m} = -1, \qquad l\cdot l = n\cdot n = m\cdot m = \bar{m}\cdot\bar{m} = 0, \qquad l\cdot m = n\cdot m = 0, \tag{23.92}$$

the contravariant metric tensor is decomposed as:

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu\bar{m}^\nu - \bar{m}^\mu m^\nu. \tag{23.93}$$

In advanced Eddington–Finkelstein coordinates, the principal null congruence is tangent to ∂_v; taking l^μ along the null geodesic and setting $m_v = m_r = 0$, the tetrad vectors are:

$$l^\mu = (0, 1, 0, 0) = \delta^\mu_r, \tag{23.94}$$

$$n^\mu = \left(1, -\frac{1}{2}\left(1 - \frac{2M}{r}\right), 0, 0\right) = \delta^\mu_v - \frac{f(r)}{2}\delta^\mu_r, \tag{23.95}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{\mathrm{i}}{\sin\theta}\right) = \frac{1}{\sqrt{2}r}\left(\delta^\mu_\theta + \frac{\mathrm{i}}{\sin\theta}\delta^\mu_\phi\right), \tag{23.96}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, -\frac{\mathrm{i}}{\sin\theta}\right) = \frac{1}{\sqrt{2}r}\left(\delta^\mu_\theta - \frac{\mathrm{i}}{\sin\theta}\delta^\mu_\phi\right), \tag{23.97}$$

where $f(r) \equiv 1 - \frac{2M}{r}$.

23.11.2 The Complex Newman–Janis Transformation

Theorem: Newman–Janis Complex Coordinate Shift

To generate rotation, we perform a complex coordinate transformation on the coordinates (r, v):

$$r' = r + \mathrm{i}a\cos\theta, \qquad v' = v - \mathrm{i}a\cos\theta, \tag{23.98}$$

while keeping $\theta' = \theta$ and $\phi' = \phi$. The complex conjugate of the radial coordinate is $\bar{r}' = r - \mathrm{i}a\cos\theta$, so that:

$$r'\bar{r}' = r^2 + a^2\cos^2\theta = \rho^2. \tag{23.99}$$

The function $\frac{1}{r}$ in the mass potential is analytically continued as:

$$\frac{1}{r} \longrightarrow \frac{1}{2}\left(\frac{1}{r'} + \frac{1}{\bar{r}'}\right) = \frac{r}{\rho^2} = \frac{r}{r^2 + a^2\cos^2\theta}. \tag{23.100}$$

Under this transformation, the differentials of the complex coordinates with respect to θ evaluate to:

$$\frac{\partial r'}{\partial\theta} = \frac{\partial}{\partial\theta}(r + \mathrm{i}a\cos\theta) = -\mathrm{i}a\sin\theta, \tag{23.101}$$

$$\frac{\partial v'}{\partial\theta} = \frac{\partial}{\partial\theta}(v - \mathrm{i}a\cos\theta) = \mathrm{i}a\sin\theta. \tag{23.102}$$

The vector components transform according to $V'^a = \frac{\partial x'^a}{\partial x^\mu}V^\mu$. For the real null vectors:

$$l'^a = \frac{\partial x'^a}{\partial x^\mu}l^\mu = \frac{\partial x'^a}{\partial r}l^r = \delta^a_r, \tag{23.103}$$

$$\begin{aligned} n'^a &= \frac{\partial x'^a}{\partial x^\mu}n^\mu = \frac{\partial x'^a}{\partial v}n^v + \frac{\partial x'^a}{\partial r}n^r \\ &= \delta^a_v - \frac{1}{2}\left(1 - \frac{M}{r'} - \frac{M}{\bar{r}'}\right)\delta^a_r = \delta^a_v - \frac{1}{2}\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)\delta^a_r. \end{aligned} \tag{23.104}$$

For the complex null vector m^a and its conjugate $\bar{m}^a$:

$$\begin{aligned} m'^a &= \frac{\partial x'^a}{\partial x^\mu}m^\mu = \frac{\partial x'^a}{\partial\theta}m^\theta + \frac{\partial x'^a}{\partial\phi}m^\phi \\ &= \left(\frac{\partial x'^a}{\partial r'}\frac{\partial r'}{\partial\theta} + \frac{\partial x'^a}{\partial v'}\frac{\partial v'}{\partial\theta} + \frac{\partial x'^a}{\partial\theta'}\frac{\partial\theta'}{\partial\theta}\right)m^\theta + \frac{\partial x'^a}{\partial\phi}m^\phi \\ &= (-\mathrm{i}a\sin\theta\,\delta^a_r + \mathrm{i}a\sin\theta\,\delta^a_v + \delta^a_\theta)\,m^\theta + \frac{\partial x'^a}{\partial\phi}m^\phi \\ &= \frac{1}{\sqrt{2}(r - \mathrm{i}a\cos\theta)}\left[\delta^a_\theta + \mathrm{i}a\sin\theta(\delta^a_v - \delta^a_r) + \frac{\mathrm{i}}{\sin\theta}\delta^a_\phi\right]. \end{aligned} \tag{23.105}$$

Similarly, for $\bar{m}'^a$:

$$\begin{aligned}\bar{m}'^a &= \frac{\partial x'^a}{\partial x^\mu}\bar{m}^\mu = \frac{\partial x'^a}{\partial \theta}\bar{m}^\theta + \frac{\partial x'^a}{\partial \phi}\bar{m}^\phi \\ &= \left(\frac{\partial x'^a}{\partial \bar{r}'}\frac{\partial \bar{r}'}{\partial \theta} + \frac{\partial x'^a}{\partial \bar{v}'}\frac{\partial \bar{v}'}{\partial \theta} + \frac{\partial x'^a}{\partial \theta'}\frac{\partial \theta'}{\partial \theta}\right)\bar{m}^\theta + \frac{\partial x'^a}{\partial \phi}\bar{m}^\phi \\ &= (\mathrm{i}a\sin\theta\,\delta^a_r - \mathrm{i}a\sin\theta\,\delta^a_v + \delta^a_\theta)\,\bar{m}^\theta + \frac{\partial x'^a}{\partial \phi}\bar{m}^\phi \\ &= \frac{1}{\sqrt{2}(r+\mathrm{i}a\cos\theta)}\left[\delta^a_\theta - \mathrm{i}a\sin\theta(\delta^a_v - \delta^a_r) - \frac{\mathrm{i}}{\sin\theta}\delta^a_\phi\right]. \end{aligned} \tag{23.106}$$

23.11.3 Reconstruction of the Contravariant and Covariant Kerr Metric

We reconstruct the inverse metric tensor $g^{\mu\nu}$ via Eq. (23.93):

$$\begin{aligned} g^{vv} &= l^v n^v + n^v l^v - m^v \bar{m}^v - \bar{m}^v m^v \\ &= -2\left(\frac{\mathrm{i}a\sin\theta}{\sqrt{2}(r-\mathrm{i}a\cos\theta)}\right)\left(\frac{-\mathrm{i}a\sin\theta}{\sqrt{2}(r+\mathrm{i}a\cos\theta)}\right) = -\frac{a^2\sin^2\theta}{\rho^2}, \end{aligned} \tag{23.107}$$

$$\begin{aligned} g^{rr} &= l^r n^r + n^r l^r - m^r \bar{m}^r - \bar{m}^r m^r \\ &= 2\left(-\frac{1}{2}\left(1 - \frac{2Mr}{\rho^2}\right)\right) - 2\left(\frac{-\mathrm{i}a\sin\theta}{\sqrt{2}(r-\mathrm{i}a\cos\theta)}\right)\left(\frac{\mathrm{i}a\sin\theta}{\sqrt{2}(r+\mathrm{i}a\cos\theta)}\right) \\ &= -\left(1 - \frac{2Mr}{\rho^2}\right) - \frac{a^2\sin^2\theta}{\rho^2} = -\frac{\Delta}{\rho^2}, \end{aligned} \tag{23.108}$$

$$g^{\theta\theta} = -m^\theta \bar{m}^\theta - \bar{m}^\theta m^\theta = -2\left(\frac{1}{\sqrt{2}(r-\mathrm{i}a\cos\theta)}\right)\left(\frac{1}{\sqrt{2}(r+\mathrm{i}a\cos\theta)}\right) = -\frac{1}{\rho^2}, \tag{23.109}$$

$$g^{\phi\phi} = -m^\phi \bar{m}^\phi - \bar{m}^\phi m^\phi = -2\left(\frac{\mathrm{i}}{\sqrt{2}(r-\mathrm{i}a\cos\theta)\sin\theta}\right)\left(\frac{-\mathrm{i}}{\sqrt{2}(r+\mathrm{i}a\cos\theta)\sin\theta}\right) = -\frac{1}{\rho^2\sin^2\theta}, \tag{23.110}$$

$$\begin{aligned} g^{vr} &= l^v n^r + n^v l^r - m^v \bar{m}^r - \bar{m}^v m^r \\ &= 1 - \left(\frac{\mathrm{i}a\sin\theta}{\sqrt{2}(r-\mathrm{i}a\cos\theta)}\right)\left(\frac{\mathrm{i}a\sin\theta}{\sqrt{2}(r+\mathrm{i}a\cos\theta)}\right) - \text{c.c.} = 1 + \frac{a^2\sin^2\theta}{\rho^2}, \end{aligned} \tag{23.111}$$

$$g^{v\theta} = -m^v \bar{m}^\theta - \bar{m}^v m^\theta = 0, \tag{23.112}$$

$$g^{v\phi} = -m^v \bar{m}^\phi - \bar{m}^v m^\phi = -\left(\frac{\mathrm{i}a\sin\theta}{\sqrt{2}(r-\mathrm{i}a\cos\theta)}\right)\left(\frac{-\mathrm{i}}{\sqrt{2}(r+\mathrm{i}a\cos\theta)\sin\theta}\right) + \text{c.c.} = -\frac{a}{\rho^2}, \tag{23.113}$$

$$g^{r\phi} = -m^r \bar{m}^\phi - \bar{m}^r m^\phi = -\left(\frac{-\mathrm{i}a\sin\theta}{\sqrt{2}(r-\mathrm{i}a\cos\theta)}\right)\left(\frac{-\mathrm{i}}{\sqrt{2}(r+\mathrm{i}a\cos\theta)\sin\theta}\right) + \text{c.c.} = \frac{a}{\rho^2}. \tag{23.114}$$

Assembling these components, the contravariant metric tensor in Kerr–Schild / advanced Eddington–Finkelstein coordinates (v, r, θ, ϕ) is:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{a^2\sin^2\theta}{\rho^2} & 1 + \frac{a^2\sin^2\theta}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 1 + \frac{a^2\sin^2\theta}{\rho^2} & -\frac{\Delta}{\rho^2} & 0 & \frac{a}{\rho^2} \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ -\frac{a}{\rho^2} & \frac{a}{\rho^2} & 0 & -\frac{1}{\rho^2\sin^2\theta} \end{pmatrix}. \tag{23.115}$$

Inverting the matrix $g^{\mu\nu}$ yields the covariant metric tensor in Kerr Eddington–Finkelstein coordinates:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2Mr}{\rho^2}\right) & 1 & 0 & \frac{2Mar\sin^2\theta}{\rho^2} \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ \frac{2Mar\sin^2\theta}{\rho^2} & -a\sin^2\theta & 0 & \left[(r^2+a^2) + \frac{2Mar\sin^2\theta}{\rho^2}\right]\sin^2\theta \end{pmatrix}. \tag{23.116}$$

Performing the final coordinate transformation to standard Boyer–Lindquist coordinates (t, r, θ, ϕ) via:

$$\mathrm{d}v = \mathrm{d}t + \frac{r^2+a^2}{\Delta}\,\mathrm{d}r, \qquad \mathrm{d}\phi_{\mathrm{EF}} = \mathrm{d}\phi + \frac{a}{\Delta}\,\mathrm{d}r, \tag{23.117}$$

eliminates the off-diagonal g_{vr} and $g_{r\phi}$ terms, producing the standard Boyer–Lindquist line element in Eq. (23.86).

23.12 Killing Horizons and Surface Gravity: Exact Calculations for Kerr and Kerr–Newman Black Holes

23.12.1 Geometric Foundations of Surface Gravity

Let $\mathcal{H}$ be a **Killing horizon** of a stationary spacetime, defined as a null hypersurface to which a Killing vector field χ^μ is normal and tangent. Since χ^μ is null on $\mathcal{H}$ ($\chi^\mu\chi_\mu = 0$), its integral curves are null geodesics. The failure of the affine parametrization along these generators defines the **surface gravity** κ:

$$\chi^\nu\nabla_\nu\chi^\mu = \kappa\chi^\mu. \tag{23.118}$$

Because χ^μ is a Killing vector field, it satisfies the **Killing equation**:

$$\nabla_\nu\chi_\sigma + \nabla_\sigma\chi_\nu = 0. \tag{23.119}$$

By the Frobenius theorem, because the null vector field χ^μ is hypersurface-orthogonal on the horizon, the differential 3-form $\chi_{[\mu}\nabla_\nu\chi_{\sigma]}$ vanishes identically ($\chi_{[\mu}\nabla_\nu\chi_{\sigma]} = 0$). Expanding the total antisymmetrization explicitly:

$$\chi_\mu\nabla_\nu\chi_\sigma + \chi_\nu\nabla_\sigma\chi_\mu + \chi_\sigma\nabla_\mu\chi_\nu - \chi_\sigma\nabla_\nu\chi_\mu - \chi_\mu\nabla_\sigma\chi_\nu - \chi_\nu\nabla_\mu\chi_\sigma = 0. \tag{23.120}$$

Using Killing's equation (23.119) to interchange indices ($\nabla_\sigma\chi_\mu = -\nabla_\mu\chi_\sigma$), the negative terms duplicate the positive terms, reducing Eq. (23.120) to:

$$\chi_\mu\nabla_\nu\chi_\sigma + \chi_\nu\nabla_\sigma\chi_\mu + \chi_\sigma\nabla_\mu\chi_\nu = 0. \tag{23.121}$$

Multiplying Eq. (23.121) by the gradient $\nabla^\nu\chi^\sigma$:

$$\begin{aligned} \chi_\mu(\nabla^\nu\chi^\sigma)\nabla_\nu\chi_\sigma &= -\underbrace{\chi_\nu(\nabla^\nu\chi^\sigma)}_{\kappa\chi^\sigma}\nabla_\sigma\chi_\mu - \chi_\sigma\underbrace{(\nabla^\nu\chi^\sigma)}_{-\nabla^\sigma\chi^\nu}\underbrace{\nabla_\mu\chi_\nu}_{-\nabla_\nu\chi_\mu} \\ &= -\kappa\chi^\sigma\nabla_\sigma\chi_\mu - \underbrace{\chi_\sigma\nabla^\sigma\chi^\nu}_{\kappa\chi^\nu}\nabla_\nu\chi_\mu \\ &= -\kappa\chi^\sigma\nabla_\sigma\chi_\mu - \kappa\chi^\nu\nabla_\nu\chi_\mu \\ &= -2\kappa\underbrace{\chi^\sigma\nabla_\sigma\chi_\mu}_{\kappa\chi_\mu} \\ &= -2\kappa^2\chi_\mu. \end{aligned} \tag{23.122}$$

Contracting both sides with χ^μ:

$$\chi^\mu \chi_\mu (\nabla^\nu \chi^\sigma) \nabla_\nu \chi_\sigma = -2\kappa^2 \chi^\mu \chi_\mu. \tag{23.123}$$

Cancelling the non-zero prefactor along the horizon approach yields the invariant formula:

$$\boxed{\kappa^2 = -\frac{1}{2} \left(\nabla^\nu \chi^\sigma\right) \left(\nabla_\nu \chi_\sigma\right)} \tag{23.124}$$

A second, immensely practical form for calculating surface gravity is obtained by rewriting the acceleration relation using Killing's equation:

$$\begin{aligned} \chi^\nu \underbrace{\nabla_\nu \chi^\mu}_{-\nabla^\mu \chi_\nu} &= \kappa \chi^\mu \\ -\chi^\nu \nabla^\mu \chi_\nu &= \kappa \chi^\mu \\ \frac{1}{2} \nabla^\mu \left(\chi^\nu \chi_\nu\right) &= -\kappa \chi^\mu. \end{aligned} \tag{23.125}$$

Thus, the gradient of the norm of the horizon Killing field directly determines κ (Wald, Eq. 12.5.2):

$$\boxed{\nabla^\mu \left(\chi^\nu \chi_\nu\right) = -2\kappa \chi^\mu} \tag{23.126}$$

23.12.2 Exact Surface Gravity Calculation for the Kerr Black Hole

To compute κ for a rotating Kerr black hole, we use coordinates that remain regular and non-singular at the horizon $\Delta = 0$. In advanced Kerr–Schild / Eddington–Finkelstein coordinates (v, r, θ, ϕ), the Kerr metric is:

$$\begin{aligned} \mathrm{d}s^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\rho^2}\right) \mathrm{d}v^2 + \rho^2 \, \mathrm{d}\theta^2 + \frac{\Sigma \sin^2\theta}{\rho^2} \mathrm{d}\phi^2 + 2 \, \mathrm{d}v \, \mathrm{d}r \\ - 2a \sin^2\theta \, \mathrm{d}r \, \mathrm{d}\phi - \frac{4GMar \sin^2\theta}{\rho^2} \mathrm{d}v \, \mathrm{d}\phi, \end{aligned} \tag{23.127}$$

or in covariant matrix representation:

$$g_{ij} = \begin{pmatrix} -\left(\frac{\Delta - a^2 \sin^2\theta}{\rho^2}\right) & 1 & 0 & -\frac{2GMar \sin^2\theta}{\rho^2} \\ 1 & 0 & 0 & -a \sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2GMar \sin^2\theta}{\rho^2} & -a \sin^2\theta & 0 & \frac{\Sigma \sin^2\theta}{\rho^2} \end{pmatrix}, \tag{23.128}$$

where:

$$\Delta \equiv r^2 - 2GMr + a^2, \qquad \rho^2 \equiv r^2 + a^2 \cos^2\theta = r^2 + a^2 - a^2 \sin^2\theta, \qquad \Sigma \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta. \tag{23.129}$$

Because the metric components are independent of v and ϕ, the spacetime possesses two fundamental Killing vectors: the stationary Killing vector $K^\mu = \delta^\mu_v = \partial_v$ and the axisymmetric Killing vector $R^\mu = \delta^\mu_\phi = \partial_\phi$. Neither K^μ nor R^μ is null at the outer horizon $r_+ \equiv GM + \sqrt{G^2M^2 - a^2}$ (where $\Delta(r_+) = 0$). Instead, the null generator of the event horizon is a linear combination:

$$\chi^\mu = \delta^\mu_v + \Omega_H \delta^\mu_\phi, \tag{23.130}$$

where Ω_H is the angular velocity of the horizon:

$$\Omega_H \equiv \omega \pm \sqrt{\omega^2 - \frac{g_{vv}}{g_{\phi\phi}}}, \qquad \omega \equiv -\frac{g_{v\phi}}{g_{\phi\phi}} = \frac{2GMar}{\Sigma}. \tag{23.131}$$

We evaluate the norm $\chi^\mu \chi_\mu$:

$$\begin{aligned}
\chi^\mu \chi_\mu &= g_{\mu\nu}\chi^\mu \chi^\nu \\
&= g_{\mu\nu}(\delta^\mu_v + \Omega_H \delta^\mu_\phi)(\delta^\nu_v + \Omega_H \delta^\nu_\phi) \\
&= g_{\mu\nu}\delta^\mu_v \delta^\nu_v + g_{\mu\nu}\delta^\mu_v \Omega_H \delta^\nu_\phi + g_{\mu\nu}\Omega_H \delta^\mu_\phi \delta^\nu_v + g_{\mu\nu}\Omega_H^2 \delta^\mu_\phi \delta^\nu_\phi \\
&= g_{vv} + 2\Omega_H g_{v\phi} + g_{\phi\phi}\Omega_H^2 \\
&= g_{vv} + g_{\phi\phi}(\Omega_H - \omega)^2 - \left(\frac{g_{v\phi}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} \\
&= g_{\phi\phi}(\Omega_H - \omega)^2 + g_{vv} - \left(\frac{g_{v\phi}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} \\
&= \frac{\Sigma \sin^2\theta}{\rho^2}(\Omega_H - \omega)^2 - \frac{\rho^2 \Delta}{\Sigma}.
\end{aligned} \tag{23.132}$$

Theorem: Algebraic Identity for the Horizon Norm

To establish Eq. (23.132), we prove the identity $g_{vv} - \left(\frac{g_{v\phi}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} = -\frac{\rho^2\Delta}{\Sigma}$:

$$\begin{aligned}
g_{vv} - \left(\frac{g_{v\phi}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} &= -\frac{\Delta - a^2\sin^2\theta}{\rho^2} - \frac{(2GM)^2 a^2 r^2 \sin^2\theta}{\Sigma\rho^2} \\
&= -\frac{\Delta}{\rho^2} + \frac{a^2\sin^2\theta}{\Sigma\rho^2}\left[\Sigma - (a^2 + r^2 - \Delta)^2\right] \\
&= -\frac{\Delta}{\rho^2} + \frac{a^2\sin^2\theta}{\Sigma\rho^2}\Big[(a^2+r^2)^2 - a^2\Delta\sin^2\theta - (a^2+r^2)^2 \\
&\qquad -\Delta^2 + 2\Delta(a^2+r^2)\Big] \\
&= \frac{1}{\Sigma\rho^2}\left[-\Delta\Sigma - a^2\sin^2\theta\left\{\Delta a^2\sin^2\theta + \Delta^2 - 2\Delta(a^2+r^2)\right\}\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\Big[(a^2+r^2)^2 - a^2\Delta\sin^2\theta + a^4\sin^4\theta \\
&\qquad +a^2\Delta\sin^2\theta - 2a^2\sin^2\theta(a^2+r^2)\Big] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[a^4 + 2a^2r^2 + (r^2 - a^2\sin^2\theta)^2 - 2a^4\sin^2\theta\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[a^4 + 2a^2r^2 + (\rho^2 - a^2)^2 - 2a^4\sin^2\theta\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[a^4 + 2a^2r^2 + \rho^4 + a^4 - 2\rho^2a^2 - 2a^4\sin^2\theta\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[2a^4 + 2a^2r^2 + \rho^4 - 2(a^2 + r^2 - a^2\sin^2\theta)a^2 - 2a^4\sin^2\theta\right] \\
&= -\frac{\Delta\rho^2}{\Sigma}.
\end{aligned} \tag{23.133}$$

At the horizon $r = r_+$, $\Delta(r_+) = 0$ and $\Omega_H = \omega(r_+)$, ensuring that $\chi^\mu\chi_\mu = 0$ vanishes identically. We now apply the master gradient equation (23.126):

$$\nabla_\nu(\chi^\mu\chi_\mu) = -2\kappa\chi_\nu$$

$$\partial_\nu\left(\frac{\Sigma\sin^2\theta}{\rho^2}(\Omega_H - \omega)^2 - \frac{\rho^2\Delta}{\Sigma}\right) = -2\kappa(g_{\alpha\nu}\chi^\alpha)$$

$$\partial_\nu\left\{\frac{\Sigma\sin^2\theta}{\rho^2}\right\}(\Omega_H-\omega)^2+\frac{\Sigma\sin^2\theta}{\rho^2}\partial_\nu(\Omega_H-\omega)^2-\partial_\nu(\rho^2)\frac{\Delta}{\Sigma}$$
$$-\partial_\nu\left(\frac{1}{\Sigma}\right)\rho^2\Delta-\frac{\rho^2}{\Sigma}\partial_\nu(\Delta)=-2\kappa(g_{v\nu}+\Omega_H g_{\phi\nu}). \quad (23.134)$$

Evaluating this equation on the horizon $r=r_+$ where $\Delta=0$ and $\Omega_H=\omega$:

$$\frac{\rho^2}{\Sigma}\partial_\nu(r^2-2GMr+a^2)\Big|_{r=r_+}=2\kappa(g_{v\nu}+\Omega_H g_{\phi\nu})\Big|_{r=r_+}. \quad (23.135)$$

Setting the index $\nu=r$:

$$\frac{\rho^2}{\Sigma}\partial_r(r^2-2GMr+a^2)\Big|_{r=r_+}=2\kappa(g_{vr}+\Omega_H g_{\phi r})\Big|_{r=r_+}$$
$$\frac{\rho^2}{\Sigma}(2r_+-2GM)=2\kappa(1-\Omega_H a\sin^2\theta)$$
$$\kappa=\frac{(r_+-GM)\rho^2}{\Sigma(1-\Omega_H a\sin^2\theta)}. \quad (23.136)$$

Using $\Omega_H=\frac{2GMar_+}{\Sigma(r_+)}=\frac{a}{r_+^2+a^2}$ and $\Sigma(r_+)=(r_+^2+a^2)^2$:

$$\kappa=\frac{(r_+-GM)\rho^2}{(r_+^2+a^2)(r_+^2+a^2-a^2\sin^2\theta)}=\frac{(r_+-GM)\rho^2}{(r_+^2+a^2)\rho^2}. \quad (23.137)$$

Cancelling the angular factor ρ^2, we obtain the exact surface gravity of the Kerr black hole:

$$\boxed{\kappa_{\rm Kerr}=\frac{r_+-GM}{r_+^2+a^2}} \quad (23.138)$$

23.12.3 Exact Surface Gravity Calculation for the Kerr–Newman Black Hole

We extend this analysis to the charged, rotating **Kerr–Newman black hole**. In Kerr–Schild coordinates:

$$\mathrm{d}s^2=-\left(\frac{\Delta-a^2\sin^2\theta}{\rho^2}\right)\mathrm{d}v^2+\rho^2\,\mathrm{d}\theta^2+\frac{\Sigma\sin^2\theta}{\rho^2}\mathrm{d}\phi^2+2\,\mathrm{d}v\,\mathrm{d}r-2a\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\phi$$
$$-\frac{2a\sin^2\theta}{\rho^2}(2GMr-Q^2)\mathrm{d}v\,\mathrm{d}\phi, \quad (23.139)$$

or in matrix form:

$$g_{ij}=\begin{pmatrix}-\left(\frac{\Delta-a^2\sin^2\theta}{\rho^2}\right) & 1 & 0 & -\frac{(2GMr-Q^2)a\sin^2\theta}{\rho^2}\\ 1 & 0 & 0 & -a\sin^2\theta\\ 0 & 0 & \rho^2 & 0\\ -\frac{(2GMr-Q^2)a\sin^2\theta}{\rho^2} & -a\sin^2\theta & 0 & \frac{\Sigma\sin^2\theta}{\rho^2}\end{pmatrix}, \quad (23.140)$$

where the horizon metric function includes the electric charge Q:

$$\Delta=r^2-2GMr+a^2+Q^2,\qquad \rho^2=r^2+a^2\cos^2\theta,\qquad \Sigma=(r^2+a^2)^2-a^2\Delta\sin^2\theta. \quad (23.141)$$

The outer event horizon is located at $r_+=GM+\sqrt{G^2M^2-a^2-Q^2}$.

When $Q\to 0$, this metric continuously reduces to the uncharged Kerr solution. The horizon Killing vector is defined by the same linear combination:

$$\chi^\mu=\delta^\mu_v+\Omega_H\delta^\mu_\phi, \quad (23.142)$$

where:

$$\Omega_H = \omega \pm \sqrt{\omega^2 - \frac{g_{vv}}{g_{\phi\phi}}}, \qquad \omega = -\frac{g_{v\phi}}{g_{\phi\phi}} = \frac{a}{\Sigma}(2GMr - Q^2). \tag{23.143}$$

Evaluating the norm $\chi^\mu \chi_\mu$:

$$\begin{aligned}
\chi^\mu \chi_\mu &= g_{\mu\nu}\chi^\mu \chi^\nu \\
&= g_{\mu\nu}(\delta^\mu_v + \Omega_H \delta^\mu_\phi)(\delta^\nu_v + \Omega_H \delta^\nu_\phi) \\
&= g_{\mu\nu}\delta^\mu_v \delta^\nu_v + g_{\mu\nu}\delta^\mu_v \Omega_H \delta^\nu_\phi + g_{\mu\nu}\Omega_H \delta^\nu_\phi \delta^\mu_v + g_{\mu\nu}\Omega_H^2 \delta^\nu_\phi \delta^\nu_\phi \\
&= g_{vv} + 2\Omega_H g_{v\phi} + g_{\phi\phi}\Omega_H^2 \\
&= g_{vv} + g_{\phi\phi}(\Omega_H - \omega)^2 - \left(\frac{g_{v\phi}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} \\
&= g_{\phi\phi}(\Omega_H - \omega)^2 + g_{vv} - \left(\frac{g_{v\phi}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} \\
&= \frac{\Sigma \sin^2\theta}{\rho^2}(\Omega_H - \omega)^2 - \frac{\rho^2 \Delta}{\Sigma}.
\end{aligned} \tag{23.144}$$

Following the exact algebraic steps of the Kerr case with $2GMr \to 2GMr - Q^2$:

$$\begin{aligned}
g_{vv} - \left(\frac{g_{v\phi}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} &= -\frac{\Delta - a^2\sin^2\theta}{\rho^2} - \frac{(2GMr - Q^2)^2 a^2 \sin^2\theta}{\Sigma\rho^2} \\
&= -\frac{\Delta}{\rho^2} + \frac{a^2\sin^2\theta}{\Sigma\rho^2}\left[\Sigma - (a^2 + r^2 - \Delta)^2\right] \\
&= -\frac{\Delta}{\rho^2} + \frac{a^2\sin^2\theta}{\Sigma\rho^2}\Big[(a^2+r^2)^2 - a^2\Delta\sin^2\theta - (a^2+r^2)^2 \\
&\qquad\qquad -\Delta^2 + 2\Delta(a^2+r^2)\Big] \\
&= \frac{1}{\Sigma\rho^2}\left[-\Delta\Sigma - a^2\sin^2\theta\left\{\Delta a^2\sin^2\theta + \Delta^2 - 2\Delta(a^2+r^2)\right\}\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\Big[(a^2+r^2)^2 - a^2\Delta\sin^2\theta + a^4\sin^4\theta \\
&\qquad\qquad +a^2\Delta\sin^2\theta - 2a^2\sin^2\theta(a^2+r^2)\Big] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[a^4 + 2a^2r^2 + (r^2 - a^2\sin^2\theta)^2 - 2a^4\sin^2\theta\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[a^4 + 2a^2r^2 + (\rho^2 - a^2)^2 - 2a^4\sin^2\theta\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[a^4 + 2a^2r^2 + \rho^4 + a^4 - 2\rho^2a^2 - 2a^4\sin^2\theta\right] \\
&= \frac{-\Delta}{\Sigma\rho^2}\left[2a^4 + 2a^2r^2 + \rho^4 - 2(a^2 + r^2 - a^2\sin^2\theta)a^2 - 2a^4\sin^2\theta\right] \\
&= -\frac{\Delta\rho^2}{\Sigma}.
\end{aligned} \tag{23.145}$$

Evaluating $\nabla_\nu(\chi^\mu\chi_\mu) = -2\kappa\chi_\nu$ at $r = r_+$ for $\nu = r$:

$$\begin{aligned}
\nabla_\nu(\chi^\mu\chi_\mu) &= -2\kappa\chi_\nu \\
\partial_\nu\left(\frac{\Sigma\sin^2\theta}{\rho^2}(\Omega_H - \omega)^2 - \frac{\rho^2\Delta}{\Sigma}\right) &= -2\kappa(g_{v\nu} + \Omega_H g_{\phi\nu})
\end{aligned}$$

$$\frac{\rho^2}{\Sigma}\partial_r\Delta = 2\kappa(1-\Omega_H a\sin^2\theta)$$
$$\kappa = \frac{(a^2+r_+^2-a^2\sin^2\theta)(r_+-GM)}{(r_+^2+a^2)(1-\Omega_H a\sin^2\theta)}. \tag{23.146}$$

Using $\Omega_H = \frac{a(2GMr_+-Q^2)}{(r_+^2+a^2)^2}$ and the horizon condition $2GMr_+ - Q^2 = r_+^2 + a^2$:

$$\boxed{\kappa_{\text{Kerr-Newman}} = \frac{r_+ - GM}{r_+^2+a^2}} \tag{23.147}$$

While this formula appears functionally identical to the uncharged Kerr expression, the radius r_+ explicitly depends on the electric charge Q via $r_+ = GM + \sqrt{G^2M^2 - a^2 - Q^2}$.

23.12.4 Evaluation Along the Symmetry Axis and General Metric Formula

The calculation of surface gravity simplifies dramatically along the **axis of symmetry** ($\sin\theta = 0$), providing an elegant invariant shortcut.

On the symmetry axis ($\sin\theta = 0$), the metric line element reduces to:

$$\mathrm{d}s^2 = -\frac{\Delta}{r^2+a^2}\mathrm{d}t^2 + \frac{r^2+a^2}{\Delta}\mathrm{d}r^2 \qquad (\text{for } \sin\theta = 0). \tag{23.148}$$

Because the rotation terms vanish identically along the axis, the horizon-generating Killing vector is simply:

$$\chi^\mu = \delta^\mu_t, \qquad \chi^\nu\chi_\nu = g_{tt}, \qquad g_{tt}g^{tt} = 1. \tag{23.149}$$

Starting from the master gradient equation $\nabla^\mu(\chi^\nu\chi_\nu) = -2\kappa\chi^\mu$:

$$(-2\kappa\chi^\mu)(-2\kappa\chi_\mu) = \nabla^\mu(\chi^\nu\chi_\nu)\nabla_\mu(\chi^\nu\chi_\nu)$$
$$(\chi^\nu\chi_\nu)\kappa^2 = \frac{1}{4}\left|g^{ij}\nabla_i(\chi^\nu\chi_\nu)\nabla_j(\chi^\nu\chi_\nu)\right|$$
$$\kappa^2 = \frac{1}{4}\left|g^{ij}g^{tt}\partial_i g_{tt}\partial_j g_{tt}\right|. \tag{23.150}$$

In Boyer–Lindquist coordinates on the rotation axis:

$$\kappa^2 = \frac{1}{4}\left|g^{rr}g^{tt}\partial_r g_{tt}\partial_r g_{tt}\right|$$
$$= \frac{1}{4}\left|\partial_r g_{tt}\partial_r g_{tt}\right|$$
$$\kappa = \frac{1}{2}\left|\partial_r g_{tt}\right|$$
$$= \frac{1}{2}\left|\frac{1}{r^2+a^2}\partial_r\Delta\right| + \mathcal{O}(\Delta)$$
$$= \frac{r_+ - GM}{r_+^2+a^2} \quad (\text{evaluated at } r = r_+). \tag{23.151}$$

Evaluating surface gravity along the axis of symmetry is therefore computationally the most efficient and direct method, yielding the exact universal result across the entire Killing horizon due to the zeroth law of black hole mechanics ($\kappa = \text{const}$ on $\mathcal{H}$).

Chapter 24

Relativistic Stellar Structure, the TOV Equation, and the Interior Schwarzschild Solution

Having formulated the exterior vacuum Schwarzschild spacetime and its global causal geometry, we now investigate the interior geometry of static, spherically symmetric matter distributions. When gravity is sufficiently strong, Newtonian hydrostatic equilibrium fails, and general-relativistic pressure gradients become necessary to support a massive body against gravitational collapse. In this chapter, we derive the fundamental **Tolman–Oppenheimer–Volkoff (TOV) equation** of relativistic hydrostatic equilibrium from Einstein's field equations. We then integrate these equations analytically for an incompressible (constant-density) fluid star to obtain the exact **Interior Schwarzschild Solution**. Finally, we analyze the central pressure divergence, deduce **Buchdahl's theorem** ($R > \frac{9}{8}r_s$), and examine the Israel junction conditions matching the interior to the exterior vacuum metric across the stellar surface.

24.1 From the Non-Relativistic Euler Equation to Relativistic Conservation

24.1.1 Classical Hydrostatic Equilibrium from the Euler Equation

In Newtonian fluid mechanics, the dynamics of an ideal fluid with mass density ρ, isotropic pressure p, and velocity field $\mathbf{v}$ under an external body force density $\mathbf{f}$ is governed by Euler's equation of motion:

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p + \mathbf{f}. \tag{24.1}$$

For a self-gravitating mass distribution, the body force is Newtonian gravity, $\mathbf{f} = -\rho\nabla\Phi_{\mathrm{N}}$, where Φ_{N} is the Newtonian gravitational potential satisfying Poisson's equation $\nabla^2\Phi_{\mathrm{N}} = 4\pi G\rho$.

For a static configuration ($\mathbf{v} = 0$, $\partial/\partial t = 0$), the kinematic acceleration terms vanish identically, yielding the condition of hydrostatic balance:

$$\nabla p = -\rho\nabla\Phi_{\mathrm{N}}. \tag{24.2}$$

Under spherical symmetry, the potential gradient is radially directed. Applying Gauss's law over a sphere of radius r, the gravitational acceleration is:

$$\frac{\mathrm{d}\Phi_{\mathrm{N}}}{\mathrm{d}r} = \frac{Gm(r)}{r^2}, \qquad m(r) = 4\pi\int_0^r \rho(r')\,r'^2\,\mathrm{d}r'. \tag{24.3}$$

Substituting Eq. (24.3) into Eq. (24.2) produces the classical Newtonian equation of hydrostatic equilibrium for a star:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}. \tag{24.4}$$

Here, the inward gravitational attraction pulling on a spherical shell of density ρ is counterbalanced solely by the outward radial pressure gradient $-\mathrm{d}p/\mathrm{d}r$.

24.1.2 Relativistic Generalization: The Relativistic Euler Equation

In General Relativity, the fluid equation of motion is embodied by covariant conservation of the stress-energy tensor:

$$\nabla_\mu T^{\mu\nu} = 0, \tag{24.5}$$

which holds on-shell when the matter fields satisfy their equations of motion (and is identically enforced by the contracted Bianchi identities $\nabla_\mu G^{\mu\nu} \equiv 0$ through Einstein's field equations).

For an isotropic perfect fluid with 4-velocity u^μ (normalized as $u^\mu u_\mu = -c^2$), the stress-energy tensor is:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p\, g^{\mu\nu}. \tag{24.6}$$

Projecting $\nabla_\mu T^{\mu\nu} = 0$ orthogonal to u^ν using the spatial projector $h^\alpha{}_\nu \equiv \delta^\alpha{}_\nu + \frac{1}{c^2} u^\alpha u_\nu$ yields the relativistic Euler equation:

$$\left(\rho + \frac{p}{c^2}\right) u^\mu \nabla_\mu u^\alpha = -h^{\alpha\mu}\nabla_\mu p. \tag{24.7}$$

The left-hand side represents the fluid 4-acceleration $a^\alpha = u^\mu \nabla_\mu u^\alpha$ weighted by the relativistic inertial enthalpy density $\rho + p/c^2$, while the right-hand side is the spatial pressure gradient. In a static spacetime, this balance directly yields the relativistic hydrostatic equilibrium condition.

24.2 Static Spherically Symmetric Matter Distributions

24.2.1 Interior Metric Ansatz and Perfect Fluid Stress-Energy

We model a static, spherically symmetric star of total radius R and total mass M. In standard static coordinates (ct, r, θ, ϕ), the interior spacetime metric is given by:

$$\mathrm{d}s^2 = -e^{2\Phi(r)}c^2\mathrm{d}t^2 + e^{2\Lambda(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{24.8}$$

where $\Phi(r)$ is the gravitational potential metric function and $e^{2\Lambda(r)}$ is the radial metric potential.

The stellar matter is modeled as an isotropic perfect fluid with 4-velocity $u^\mu = e^{-\Phi}(1, 0, 0, 0)$:

$$T^\mu{}_\nu = \Big(\rho(r)c^2 + p(r)\Big)\, u^\mu u_\nu + p(r)\,\delta^\mu_\nu = \mathrm{diag}\Big(-\rho(r)c^2,\, p(r),\, p(r),\, p(r)\Big), \tag{24.9}$$

where $\rho(r)$ is the proper energy mass density and $p(r)$ is the isotropic fluid pressure.

24.2.2 Einstein Field Equations with Matter

The mixed Einstein tensor $G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu_\nu R$ for the diagonal metric (24.8) has non-zero components:

$$G^0{}_0 = -\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(1 - e^{-2\Lambda}\right)\right], \tag{24.10}$$

$$G^1{}_1 = e^{-2\Lambda}\left(\frac{2\Phi'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{24.11}$$

$$G^2{}_2 = G^3{}_3 = e^{-2\Lambda}\left[\Phi'' + (\Phi')^2 - \Phi'\Lambda' + \frac{\Phi' - \Lambda'}{r}\right]. \tag{24.12}$$

Einstein's field equations $G^\mu{}_\nu = \frac{8\pi G}{c^4} T^\mu{}_\nu$ yield the governing equations for the stellar interior:

1. **Time-Time Component ($G^0{}_0 = \frac{8\pi G}{c^4} T^0{}_0$):**

$$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(1 - e^{-2\Lambda(r)}\right)\right] = -\frac{8\pi G}{c^2}\rho(r). \tag{24.13}$$

2. **Radial Component ($G^1{}_1 = \frac{8\pi G}{c^4} T^1{}_1$):**

$$e^{-2\Lambda(r)}\left(\frac{2\Phi'(r)}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^4} p(r). \tag{24.14}$$

24.2.3 The Enclosed Mass Function and Radial Metric Potential

Integrating the time-time equation (24.13) from the stellar center $r = 0$ to radius r:

$$r\left(1 - e^{-2\Lambda(r)}\right) = \frac{8\pi G}{c^2}\int_0^r \rho(r')\, r'^2\, \mathrm{d}r' \equiv \frac{2Gm(r)}{c^2}, \tag{24.15}$$

where we define the **enclosed mass function** $m(r)$:

$$m(r) \equiv 4\pi \int_0^r \rho(r')\, r'^2\, \mathrm{d}r'. \tag{24.16}$$

Solving for the radial metric component:

$$e^{-2\Lambda(r)} = 1 - \frac{2Gm(r)}{c^2 r} \implies e^{2\Lambda(r)} = \left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}. \tag{24.17}$$

At the stellar surface $r = R$, the total mass is $M \equiv m(R)$, matching the exterior Schwarzschild metric function.

24.3 The Tolman–Oppenheimer–Volkoff (TOV) Equation

24.3.1 The Gravitational Potential Gradient

We solve the radial Einstein equation (24.14) for the potential derivative $\Phi'(r)$:

$$e^{-2\Lambda}\left(\frac{2\Phi'}{r}\right) = \frac{8\pi G}{c^4} p + \frac{1 - e^{-2\Lambda}}{r^2}. \tag{24.18}$$

Substituting $e^{-2\Lambda} = 1 - \frac{2Gm(r)}{c^2 r}$ and $1 - e^{-2\Lambda} = \frac{2Gm(r)}{c^2 r}$:

$$\left(1 - \frac{2Gm(r)}{c^2 r}\right)\left(\frac{2\Phi'}{r}\right) = \frac{8\pi G}{c^4} p + \frac{2Gm(r)}{c^2 r^3} = \frac{2G}{c^2 r^2}\left[\frac{m(r)}{r} + \frac{4\pi r^2 p(r)}{c^2}\right]. \tag{24.19}$$

Dividing by $2/r$ yields the exact general-relativistic potential gradient:

$$\Phi'(r) = \frac{G\left(m(r) + \frac{4\pi r^3 p(r)}{c^2}\right)}{c^2 r^2\left(1 - \frac{2Gm(r)}{c^2 r}\right)}. \tag{24.20}$$

24.3.2 Covariant Hydrostatic Equilibrium

The covariant conservation of energy-momentum $\nabla_\mu T^\mu{}_\nu = 0$ for the radial component $\nu = 1$ gives:

$$\nabla_\mu T^\mu{}_1 = \partial_r T^r{}_r + \Gamma^\mu_{\mu r} T^r{}_r - \Gamma^\alpha_{r\beta} T^\beta_\alpha = 0. \tag{24.21}$$

With $\Gamma^0_{0r} = \Phi'(r)$, $\Gamma^r_{rr} = \Lambda'(r)$, $\Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r} = 1/r$, this simplifies to:

$$\frac{\mathrm{d}p}{\mathrm{d}r} + \Big(\rho(r)c^2 + p(r)\Big)\,\Phi'(r) = 0 \implies \frac{\mathrm{d}p}{\mathrm{d}r} = -\Big(\rho(r)c^2 + p(r)\Big)\frac{\mathrm{d}\Phi}{\mathrm{d}r}. \tag{24.22}$$

24.3.3 Derivation of the Master TOV Equation

Substituting the potential gradient (24.20) into the hydrostatic relation (24.22):

Theorem: The Tolman–Oppenheimer–Volkoff (TOV) Equation

In general relativity, the hydrostatic equilibrium of a static, spherically symmetric fluid star is governed by:

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G\left(\rho(r) + \frac{p(r)}{c^2}\right)\left(m(r) + \frac{4\pi r^3 p(r)}{c^2}\right)}{r^2\left(1 - \frac{2Gm(r)}{c^2 r}\right)}} \tag{24.23}$$

Key Concept: Physical Interpretation of Relativistic Corrections in the TOV Equation

The TOV equation represents the exact relativistic generalization of the classical Newtonian hydrostatic equation $\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G\rho m(r)}{r^2}$. It contains three distinct general-relativistic enhancement factors:

1. **Inertial Mass Density of Pressure $\left(\rho + \frac{p}{c^2}\right)$:** By mass-energy equivalence, pressure contributes to the effective inertial mass density that must be supported against gravity.
2. **Gravitational Source of Pressure $\left(m(r) + \frac{4\pi r^3 p}{c^2}\right)$:** In general relativity, pressure itself gravitates (acts as a source of spacetime curvature via $T^\mu{}_\mu = -\rho c^2 + 3p$).
3. **Spatial Curvature Metric Factor $\left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}$:** Curved spatial geometry increases the effective proper radial distance, intensifying the required pressure gradient.

Because every relativistic correction has the same sign, general relativity strictly *destabilizes* massive compact stars compared to Newtonian gravity.

24.3.4 Comparison: Non-Relativistic vs. Relativistic Hydrostatic Gradients

To directly illuminate the stark physical divergence between Newtonian stars and general-relativistic stars, we write both hydrostatic equilibrium equations side-by-side:

$$\left(\frac{\mathrm{d}p}{\mathrm{d}r}\right)_{\text{Newton}} = -\frac{Gm(r)\rho(r)}{r^2}, \tag{24.24}$$

$$\left(\frac{\mathrm{d}p}{\mathrm{d}r}\right)_{\text{TOV}} = -\frac{Gm(r)\rho(r)}{r^2}\underbrace{\left[1 + \frac{p(r)}{\rho(r)c^2}\right]}_{\mathcal{C}_1>1}\underbrace{\left[1 + \frac{4\pi r^3 p(r)}{m(r)c^2}\right]}_{\mathcal{C}_2>1}\underbrace{\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}}_{\mathcal{C}_3>1}. \tag{24.25}$$

Physical Property	**Newtonian Star (Euler)**	**Relativistic Star (TOV)**
Hydrostatic Law	$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}$	$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\,\mathcal{C}_1\,\mathcal{C}_2\,\mathcal{C}_3$
Inertial Resistance to Gravity	Rest mass density ρ	Effective enthalpy density $\rho + p/c^2$
Gravitational Source	Rest mass enclosed $m(r)$	Rest mass plus pressure work: $m(r) + \frac{4\pi r^3 p}{c^2}$
Underlying Geometry	Euclidean flat 3-space	Spatially curved: $\mathrm{d}\ell^2_{\text{proper}} = \mathcal{C}_3\,\mathrm{d}r^2$
High-Pressure Limit ($p \to \infty$)	$\mathrm{d}p/\mathrm{d}r \sim \mathcal{O}(1)$ with respect to p (pressure supports arbitrary mass if stiff)	$\mathrm{d}p/\mathrm{d}r \sim -p^2$ (pressure creates vastly more gravity than support)
Maximum Stable Mass	No intrinsic ceiling for cold matter (arbitrarily high mass stable if EoS stiff enough)	Existence of a finite maximum mass (**TOV Limit**); collapse to a black hole is inevitable

Table 24.1: Detailed comparison between the classical hydrostatic balance obtained from the Euler equation and the relativistic hydrostatic balance governed by the TOV equation.

This comparison highlights why the Tolman–Oppenheimer–Volkoff limit is intrinsically relativistic: in Newtonian theory, an increased core pressure purely provides outward support against gravity. In general relativity, because pressure is also a source of gravity (via $\mathcal{C}_2$) and increases the mass-energy to be supported (via $\mathcal{C}_1$), pumping up core pressure triggers diminishing returns. Beyond a critical compactness, increasing p amplifies the inward gravitational pull faster than the outward pressure gradient, precipitating unavoidable gravitational collapse.

24.4 The Exact Interior Schwarzschild Solution

24.4.1 Incompressible (Constant-Density) Stellar Model

We consider an idealized stellar model of uniform mass density:

$$\rho(r) = \begin{cases} \rho_0 = \text{constant}, & r \le R, \\ 0, & r > R. \end{cases} \tag{24.26}$$

The enclosed mass function (24.16) evaluates to:

$$m(r) = 4\pi\rho_0 \int_0^r r'^2\,\mathrm{d}r' = \frac{4}{3}\pi\rho_0 r^3 = M\left(\frac{r}{R}\right)^3, \qquad (r \le R), \tag{24.27}$$

where $M \equiv m(R) = \frac{4}{3}\pi\rho_0 R^3$ is the total stellar mass.

The radial metric component (24.17) becomes:

$$e^{-2\Lambda(r)} = 1 - \frac{2GMr^2}{c^2R^3} = 1 - \frac{r^2}{R_0^2}, \qquad R_0 \equiv \sqrt{\frac{c^2R^3}{2GM}}. \tag{24.28}$$

24.4.2 Analytic Integration of the Pressure Profile

Substituting $\rho(r) = \rho_0$ and $m(r) = \frac{4}{3}\pi\rho_0 r^3$ into the TOV equation (24.23):

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G\left(\rho_0 + \frac{p}{c^2}\right)\left(\frac{4}{3}\pi\rho_0 r^3 + \frac{4\pi r^3 p}{c^2}\right)}{r^2\left(1 - \frac{8\pi G\rho_0 r^2}{3c^2}\right)} = -\frac{4\pi G r}{3}\frac{\left(\rho_0 + \frac{p}{c^2}\right)\left(\rho_0 + \frac{3p}{c^2}\right)}{1 - \frac{r^2}{R_0^2}}. \tag{24.29}$$

Separating variables:

$$\frac{\mathrm{d}p}{(\rho_0 c^2 + p)\,(\rho_0 c^2 + 3p)} = -\frac{4\pi G}{3c^4}\frac{r\,\mathrm{d}r}{1 - \frac{r^2}{R_0^2}} = -\frac{1}{2\rho_0 c^2 R_0^2}\frac{r\,\mathrm{d}r}{1 - \frac{r^2}{R_0^2}}. \tag{24.30}$$

Using partial fractions:

$$\frac{1}{(\rho_0 c^2 + p)\,(\rho_0 c^2 + 3p)} = \frac{1}{2\rho_0 c^2}\left[\frac{3}{\rho_0 c^2 + 3p} - \frac{1}{\rho_0 c^2 + p}\right]. \tag{24.31}$$

Multiplying by $2\rho_0 c^2$:

$$\left[\frac{3}{\rho_0 c^2 + 3p} - \frac{1}{\rho_0 c^2 + p}\right]\mathrm{d}p = -\frac{r\,\mathrm{d}r}{R_0^2\left(1 - \frac{r^2}{R_0^2}\right)}. \tag{24.32}$$

Integrating both sides:

$$\ln\left(\frac{\rho_0 c^2 + 3p(r)}{\rho_0 c^2 + p(r)}\right) = \ln\sqrt{1 - \frac{r^2}{R_0^2}} + \ln C_0 = \ln\left(C_0\sqrt{1 - \frac{r^2}{R_0^2}}\right). \tag{24.33}$$

Exponentiating:

$$\frac{\rho_0 c^2 + 3p(r)}{\rho_0 c^2 + p(r)} = C_0\sqrt{1 - \frac{r^2}{R_0^2}}. \tag{24.34}$$

The boundary condition that pressure vanishes at the stellar surface, $p(R) = 0$, fixes C_0:

$$\frac{\rho_0 c^2 + 0}{\rho_0 c^2 + 0} = 1 = C_0\sqrt{1 - \frac{R^2}{R_0^2}} \implies C_0 = \frac{1}{\sqrt{1 - \frac{2GM}{c^2 R}}}. \tag{24.35}$$

Substituting C_0 back into Eq. (24.34):

$$\frac{\rho_0 c^2 + 3p(r)}{\rho_0 c^2 + p(r)} = \frac{\sqrt{1 - \frac{2GMr^2}{c^2 R^3}}}{\sqrt{1 - \frac{2GM}{c^2 R}}}. \tag{24.36}$$

Solving algebraically for $p(r)$:

Theorem: Exact Pressure Profile for Constant-Density Star

The internal pressure of an incompressible relativistic star of mass M and radius R is:

$$\boxed{p(r) = \rho_0 c^2\left[\frac{\sqrt{1 - \frac{2GM}{c^2 R}} - \sqrt{1 - \frac{2GMr^2}{c^2 R^3}}}{3\sqrt{1 - \frac{2GMr^2}{c^2 R^3}} - \sqrt{1 - \frac{2GM}{c^2 R}}}\right]} \tag{24.37}$$

24.4.3 Integration of the Temporal Metric Potential $e^{\Phi(r)}$

From the hydrostatic conservation relation (24.22):

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{1}{\rho_0 c^2 + p(r)}\frac{\mathrm{d}p}{\mathrm{d}r}. \tag{24.38}$$

Integrating from r to R:

$$\Phi(R) - \Phi(r) = -\ln\left(\frac{\rho_0 c^2 + p(R)}{\rho_0 c^2 + p(r)}\right) = \ln\left(\frac{\rho_0 c^2 + p(r)}{\rho_0 c^2}\right). \tag{24.39}$$

Matching to the exterior Schwarzschild metric at the surface $r = R$:

$$e^{2\Phi(R)} = 1 - \frac{2GM}{c^2 R} \implies e^{\Phi(R)} = \sqrt{1 - \frac{2GM}{c^2 R}}. \tag{24.40}$$

Therefore:

$$e^{\Phi(r)} = e^{\Phi(R)} \left(\frac{\rho_0 c^2}{\rho_0 c^2 + p(r)} \right) = \sqrt{1 - \frac{2GM}{c^2 R}} \left(1 + \frac{p(r)}{\rho_0 c^2} \right)^{-1}. \tag{24.41}$$

Substituting $p(r)$ from Eq. (24.37):

$$\begin{aligned} 1 + \frac{p(r)}{\rho_0 c^2} &= 1 + \frac{\sqrt{1 - \frac{2GM}{c^2 R}} - \sqrt{1 - \frac{2GMr^2}{c^2 R^3}}}{3\sqrt{1 - \frac{2GMr^2}{c^2 R^3}} - \sqrt{1 - \frac{2GM}{c^2 R}}} \\ &= \frac{2\sqrt{1 - \frac{2GMr^2}{c^2 R^3}}}{3\sqrt{1 - \frac{2GMr^2}{c^2 R^3}} - \sqrt{1 - \frac{2GM}{c^2 R}}}. \end{aligned} \tag{24.42}$$

Inverting and multiplying by $\sqrt{1 - \frac{2GM}{c^2 R}}$:

$$e^{\Phi(r)} = \frac{3}{2}\sqrt{1 - \frac{2GM}{c^2 R}} - \frac{1}{2}\sqrt{1 - \frac{2GMr^2}{c^2 R^3}}. \tag{24.43}$$

Theorem: The Complete Interior Schwarzschild Line Element

The interior spacetime metric of an incompressible fluid star of mass M and radius R is:

$$\boxed{\mathrm{d}s^2 = -\left[\frac{3}{2}\sqrt{1 - \frac{2GM}{c^2 R}} - \frac{1}{2}\sqrt{1 - \frac{2GMr^2}{c^2 R^3}} \right]^2 c^2 \mathrm{d}t^2 + \left(1 - \frac{2GMr^2}{c^2 R^3} \right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2} \tag{24.44}$$

24.5 Buchdahl's Theorem and Stellar Boundary Conditions

24.5.1 Central Pressure Divergence

Evaluating the exact pressure profile (24.37) at the stellar center $r = 0$:

$$p_c \equiv p(0) = \rho_0 c^2 \left[\frac{\sqrt{1 - \frac{2GM}{c^2 R}} - 1}{3 - \sqrt{1 - \frac{2GM}{c^2 R}}} \right] = \rho_0 c^2 \left[\frac{1 - \sqrt{1 - \frac{2GM}{c^2 R}}}{3\sqrt{1 - \frac{2GM}{c^2 R}} - 1} \right]. \tag{24.45}$$

For the physical central pressure to remain positive and finite ($0 \le p_c < \infty$), the denominator in Eq. (24.45) must be strictly positive:

$$3\sqrt{1 - \frac{2GM}{c^2 R}} - 1 > 0 \implies \sqrt{1 - \frac{2GM}{c^2 R}} > \frac{1}{3}. \tag{24.46}$$

Squaring both sides:

$$1 - \frac{2GM}{c^2 R} > \frac{1}{9} \implies \frac{2GM}{c^2 R} < \frac{8}{9} \implies R > \frac{9}{8}\left(\frac{2GM}{c^2} \right). \tag{24.47}$$

Theorem: Buchdahl's Theorem (1959)

For any static, spherically symmetric relativistic fluid star with non-increasing mass density ($\frac{\mathrm{d}\rho}{\mathrm{d}r} \leq 0$), the stellar radius R must strictly exceed 9/8 of its Schwarzschild radius:

$$\boxed{R > \frac{9}{8} r_s = \frac{9}{4} \frac{GM}{c^2} = 2.25 \, \frac{GM}{c^2}} \tag{24.48}$$

If $R \leq \frac{9}{8} r_s$, no physical equation of state can sustain hydrostatic equilibrium, and the star must undergo runaway gravitational collapse to a black hole.

24.5.2 The Israel Junction Conditions at the Stellar Surface

The interior solution ($r < R$) and exterior vacuum Schwarzschild solution ($r > R$) meet across the timelike hypersurface Σ at $r = R$. Smooth matching requires the **Darmois–Israel Junction Conditions**:

1. **First Fundamental Form (Induced Metric Continuity):** The induced 3D metric h_{ab} on Σ must be continuous across $r = R$:

$$[h_{ab}]^+_- \equiv h_{ab}(R^+) - h_{ab}(R^-) = 0. \tag{24.49}$$

Checking the components:

$$g_{tt}(R^-) = -\left[\frac{3}{2}\sqrt{1 - \frac{2GM}{c^2 R}} - \frac{1}{2}\sqrt{1 - \frac{2GM}{c^2 R}}\right]^2 = -\left(1 - \frac{2GM}{c^2 R}\right) = g_{tt}(R^+), \tag{24.50}$$

$$g_{rr}(R^-) = \left(1 - \frac{2GM}{c^2 R}\right)^{-1} = g_{rr}(R^+), \qquad g_{\theta\theta}(R^\pm) = R^2, \quad g_{\phi\phi}(R^\pm) = R^2 \sin^2\theta. \tag{24.51}$$

The metric potentials are C^0 continuous across the boundary.

2. **Second Fundamental Form (Extrinsic Curvature Continuity):** In the absence of a singular boundary surface layer (surface stress-energy $S_{ab} = 0$), the extrinsic curvature K_{ab} must be continuous:

$$[K_{ab}]^+_- = 0 \iff p(R^-) = 0. \tag{24.52}$$

This proves that the vanishing of pressure at the surface, $p(R) = 0$, is the direct physical consequence of the junction conditions without boundary surface stresses.

Chapter 25

Classical Tests of General Relativity, Planetary Precession, and Gravitational Redshift

The observational verification of General Relativity rests upon precise solar-system tests where spacetime curvature produces measurable departures from Newtonian mechanics. In this chapter, we derive the relativistic Binet equation governing equatorial trajectories in Schwarzschild geometry. We calculate the anomalous perihelion precession of planetary orbits (Mercury) via secular perturbation theory and resolve the century-old Le Verrier anomaly. We then formulate the generalized gravitational deflection of test particles, demonstrating how General Relativity bridges non-relativistic Newtonian Rutherford scattering and the ultra-relativistic factor-of-two doubling for null photons. Finally, we analyze gravitational redshift, relate proper frequencies to conserved Killing energies, and review its precision experimental confirmation in terrestrial and astrophysical contexts.

25.1 The Relativistic Orbital Binet Equation

25.1.1 First-Order Radial and Angular Differential Relations

In Chapter 23, we established that geodesic motion in the equatorial plane ($\theta = \pi/2$) of a static, spherically symmetric spacetime with metric

$$\mathrm{d}s^2 = -f(r)c^2\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2, \qquad f(r) \equiv 1 - \frac{2GM}{c^2 r}, \tag{25.1}$$

admits two exact constants of motion associated with the Killing vector fields $\xi^\mu = (\partial_t)^\mu$ and $\psi^\mu = (\partial_\phi)^\mu$:

$$\mathcal{E} = f(r)c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \text{constant}, \qquad h = r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \text{constant}. \tag{25.2}$$

For a massive test particle ($u^\mu u_\mu = -1$), the radial equation of motion reads:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \mathcal{E}^2 - c^2 f(r)\left(1 + \frac{h^2}{c^2 r^2}\right). \tag{25.3}$$

Similarly, the spatial trajectory relation $\frac{\mathrm{d}\phi}{\mathrm{d}r}$ is:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\mathrm{d}\phi/\mathrm{d}\tau}{\mathrm{d}r/\mathrm{d}\tau} = \frac{h}{r^2}\left[\mathcal{E}^2 - c^2 f(r)\left(1 + \frac{h^2}{c^2 r^2}\right)\right]^{-1/2}. \tag{25.4}$$

25.1.2 Transformation to the Inverse Radial Coordinate $u \equiv 1/r$

To cast the orbit equation into the standard Binet form, we define the reciprocal radius:

$$u(\phi) \equiv \frac{1}{r(\phi)} \implies \frac{\mathrm{d}u}{\mathrm{d}\phi} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi} = -\frac{1}{r^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^{-1}. \tag{25.5}$$

Substituting Eq. (25.4) into Eq. (25.5) yields:

$$\frac{\mathrm{d}u}{\mathrm{d}\phi} = -\frac{1}{h}\sqrt{\mathcal{E}^2 - c^2\left(1 - \frac{2GMu}{c^2}\right)\left(1 + \frac{h^2u^2}{c^2}\right)}. \tag{25.6}$$

Squaring both sides:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{1}{h^2}\left[\mathcal{E}^2 - c^2\left(1 - \frac{2GMu}{c^2}\right)\left(1 + \frac{h^2u^2}{c^2}\right)\right]. \tag{25.7}$$

25.1.3 The Second-Order Relativistic Binet Equation

Differentiating Eq. (25.7) with respect to the azimuthal angle ϕ:

$$\begin{aligned} 2\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\left(\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}\right) &= \frac{1}{h^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\mathcal{E}^2 - c^2\left(1 - \frac{2GMu}{c^2} + \frac{h^2u^2}{c^2} - \frac{2GMh^2u^3}{c^4}\right)\right] \\ &= \frac{1}{h^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\left[2GM - 2h^2u + \frac{6GMh^2}{c^2}u^2\right]. \end{aligned} \tag{25.8}$$

Dividing by $2\frac{\mathrm{d}u}{\mathrm{d}\phi}$ (for non-circular trajectories where $\frac{\mathrm{d}u}{\mathrm{d}\phi} \neq 0$):

Theorem: The Relativistic Orbital Binet Equation

The spatial trajectory $u(\phi) \equiv 1/r(\phi)$ of a test particle in Schwarzschild spacetime is governed by:

$$\boxed{\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2} \tag{25.9}$$

where $\frac{GM}{h^2}$ is the classical Newtonian gravitational attraction, and $\frac{3GM}{c^2}u^2$ is the general-relativistic non-linear correction.

25.2 Classical Test I: Perihelion Precession of Planetary Orbits

25.2.1 Perturbative Expansion around Newtonian Ellipses

In the solar system, planetary orbits are non-relativistic ($v \ll c$) and the central gravitational field is weak ($r \gg GM/c^2$). The ratio of the relativistic term to the Newtonian term is:

$$\epsilon \equiv \frac{\frac{3GM}{c^2}u^2}{\frac{GM}{h^2}} = \frac{3h^2u^2}{c^2} \approx \frac{3v_\phi^2}{c^2} \ll 1. \tag{25.10}$$

We expand the solution in powers of the small parameter ϵ:

$$u(\phi) = u_0(\phi) + u_1(\phi) + \mathcal{O}(\epsilon^2). \tag{25.11}$$

25.2.2 Zeroth-Order Solution: Newtonian Keplerian Orbit

At order $\mathcal{O}(\epsilon^0)$, Eq. (25.9) reduces to the classical Newtonian harmonic oscillator equation:

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 = \frac{GM}{h^2} \equiv \frac{1}{p}, \tag{25.12}$$

where $p \equiv a(1-e^2)$ is the semi-latus rectum, a is the semi-major axis, and e is the orbital eccentricity. The standard solution is a fixed ellipse:

$$u_0(\phi) = \frac{1}{p}\left(1 + e\cos\phi\right). \tag{25.13}$$

25.2.3 First-Order Perturbation and Secular Resonance

At order $\mathcal{O}(\epsilon^1)$, substituting $u_0(\phi)$ into the right-hand side of Eq. (25.9):

$$\begin{aligned}
\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 &= \frac{3GM}{c^2} u_0^2 = \frac{3GM}{c^2 p^2}\left(1 + e\cos\phi\right)^2 \\
&= \frac{3GM}{c^2 p^2}\left(1 + 2e\cos\phi + e^2\cos^2\phi\right) \\
&= \frac{3GM}{c^2 p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right].
\end{aligned} \tag{25.14}$$

Defining $A \equiv \frac{3GM}{c^2 p^2}$, we solve for each forcing term:

1. **Constant Term:** Particular solution $u_{1,\mathrm{const}} = A\left(1 + \frac{e^2}{2}\right)$.
2. **Second Harmonic ($\cos 2\phi$):** Particular solution $u_{1,2\phi} = -\frac{Ae^2}{6}\cos 2\phi$.
3. **Resonant Harmonic ($\cos\phi$):** Because $\cos\phi$ is an eigenmode of the homogeneous operator $\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} + 1\right)$, it acts as a **resonant forcing term**. Evaluating $\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} + 1\right)(\phi\sin\phi) = 2\cos\phi$, the particular solution for $2Ae\cos\phi$ is:

$$u_{1,\mathrm{res}}(\phi) = Ae\,\phi\sin\phi. \tag{25.15}$$

Combining all components, the complete first-order trajectory is:

$$u(\phi) = \frac{1}{p}(1 + e\cos\phi) + A\left(1 + \frac{e^2}{2}\right) + Ae\,\phi\sin\phi - \frac{Ae^2}{6}\cos 2\phi. \tag{25.16}$$

25.2.4 Resummation and the Precession Angle

The secularly growing term $Ae\,\phi\sin\phi$ governs the cumulative perihelion advance:

$$u(\phi) \approx \frac{1}{p}\left[1 + e\cos\phi + Ap\,e\,\phi\sin\phi\right] = \frac{1}{p}\left[1 + e\left(\cos\phi + \mu\,\phi\sin\phi\right)\right], \tag{25.17}$$

where $\mu \equiv Ap = \frac{3GM}{c^2 p} = \frac{3GM}{c^2 a(1-e^2)} \ll 1$.

Using the Taylor expansion $\cos((1-\mu)\phi) = \cos\phi\cos(\mu\phi) + \sin\phi\sin(\mu\phi) \approx \cos\phi + \mu\phi\sin\phi$:

$$u(\phi) \approx \frac{1}{p}\left[1 + e\cos\left((1-\mu)\phi\right)\right]. \tag{25.18}$$

Key Concept: Physical Mechanism of Perihelion Advance

Successive perihelia occur when the cosine argument advances by 2π:

$$(1-\mu)\phi_c = 2\pi \implies \phi_c = \frac{2\pi}{1-\mu} \approx 2\pi(1+\mu). \tag{25.19}$$

The angular advance $\Delta\phi$ per orbital period is:

$$\Delta\phi = \phi_c - 2\pi = 2\pi\mu = \frac{6\pi GM}{c^2 a(1-e^2)}. \tag{25.20}$$

Physical Insight & Motivation: Quantitative Evaluation for Mercury

For the planet Mercury orbiting the Sun ($M_\odot \approx 1.989 \times 10^{30}\,\mathrm{kg}$, $a \approx 5.791 \times 10^{10}\,\mathrm{m}$, $e \approx 0.20563$):

$$\Delta\phi = \frac{6\pi(6.674\times10^{-11})(1.989\times10^{30})}{(3.0\times10^{8})^2(5.791\times10^{10})(1-0.20563^2)} \approx 5.019\times10^{-7}\,\mathrm{rad/orbit} \approx 0.1035''/\mathrm{orbit}. \tag{25.21}$$

Accumulated over a century (≈ 415.2 revolutions):

$$\Delta\phi_{\text{century}} \approx 43.0''/\text{century}. \tag{25.22}$$

This quantitatively resolved the $43''$/century anomaly discovered by Urbain Le Verrier in 1859.

25.3 Classical Test II: Gravitational Deflection of Light and Particles

25.3.1 Trajectory Perturbation for Massive and Null Scattering

Consider a particle with rest mass m, asymptotic speed v_∞, and impact parameter b. From the relativistic Binet equation with $h = bv_\infty/\sqrt{1-v_\infty^2/c^2}$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2. \tag{25.23}$$

Perturbing about the straight-line trajectory $u_0(\phi) = \frac{\cos\phi}{b}$:

$$u(\phi) \approx \frac{\cos\phi}{b} + \frac{GM}{h^2} + \frac{GM}{c^2b^2}\left(1+\sin^2\phi\right). \tag{25.24}$$

25.3.2 Asymptotic Angles and the Master Deflection Formula

At spatial infinity $u(\phi_\infty) = 0$ with $\phi_\infty = \pm(\pi/2 + q)$:

$$0 = -\frac{q}{b} + \frac{GM}{h^2} + \frac{2GM}{c^2b^2} \implies q = \frac{GMb}{h^2} + \frac{2GM}{c^2b}. \tag{25.25}$$

Substituting $h = bv_\infty/\sqrt{1-v_\infty^2/c^2}$:

$$q = \frac{GM}{bv_\infty^2}\left(1-\frac{v_\infty^2}{c^2}\right) + \frac{2GM}{c^2b} = \frac{GM}{bv_\infty^2}\left(1+\frac{v_\infty^2}{c^2}\right). \tag{25.26}$$

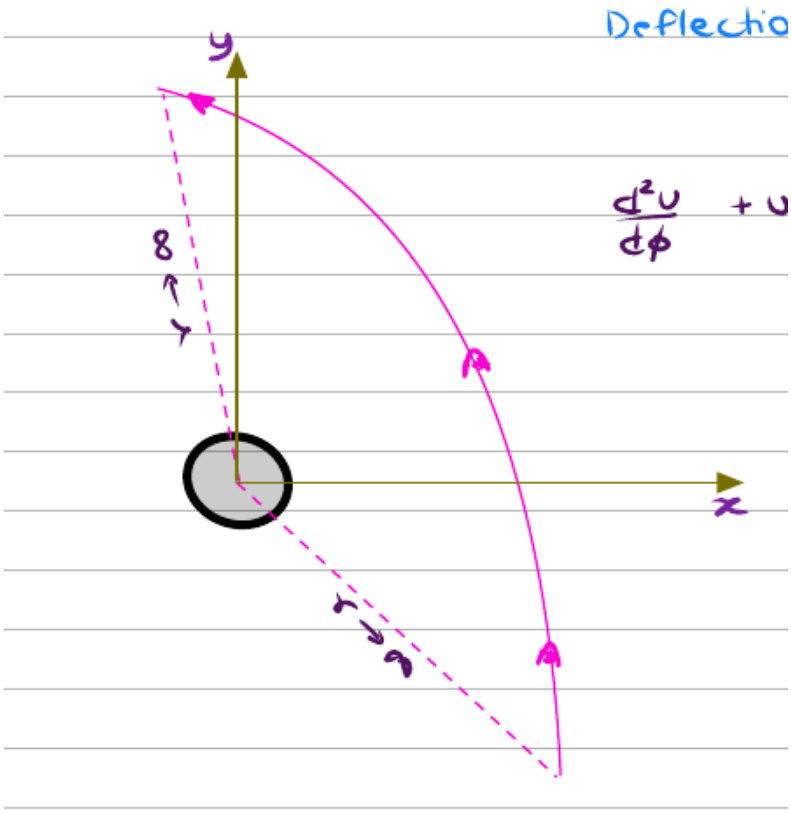


Figure 25.1: Gravitational bending of light rays grazing a mass M with impact parameter b, illustrating the unperturbed asymptote and the total deflection angle $\Delta\theta = 4GM/c^2b$.

Theorem: Generalized Gravitational Deflection Angle

The total deflection angle $\Delta\theta = 2q$ for any test body with asymptotic speed v_∞ is:

$$\boxed{\Delta\theta = \frac{2GM}{bv_\infty^2}\left(1 + \frac{v_\infty^2}{c^2}\right)} \tag{25.27}$$

- **Non-relativistic Newtonian limit ($v_\infty \ll c$):** $\Delta\theta_{\text{Newton}} = \frac{2GM}{bv_\infty^2}$.
- **Ultra-relativistic photon limit ($v_\infty = c$):** $\Delta\theta_{\text{GR}} = \frac{4GM}{c^2 b}$.

25.3.3 Direct Derivation from Geodesics in the Weak-Field Metric

We now derive the general gravitational deflection formula (25.27) directly from the relativistic geodesic equation in the weak-field metric, unifying massive particles with asymptotic speed v_∞ and null photons ($v_\infty = c$) without assuming an ad hoc gravitational force law. Because the deflection is small ($GM/c^2 b \ll 1$), we solve the geodesic equation perturbatively at first order in the metric perturbations and integrate along the unperturbed straight-line path.

1. The Static Weak-Field Metric In a static, weak gravitational field with scalar potentials $|\Phi|/c^2 \ll 1$ and $|\Psi|/c^2 \ll 1$, the isotropic line element is:

$$\mathrm{d}s^2 = -\left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2\Psi}{c^2}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{25.28}$$

For a point mass M in General Relativity, Einstein's field equations require $\Phi = \Psi = -GM/r$.

Consider a test body of rest mass m incident along the $+z$-axis with asymptotic velocity v_∞ and impact parameter b relative to the central mass located at the origin. We choose coordinates such that the unperturbed trajectory lies in the x-z plane at $x = b$, $y = 0$, so that the transverse coordinates are $\mathbf{x}_\perp = (x, y) = (b, 0)$.

Let $x^0 = ct$, and let dots denote differentiation with respect to the particle's proper time τ (or an affine parameter λ for null rays). The geodesic equation is:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0. \tag{25.29}$$

For a transverse index $i \in \{x, y\}$:

$$\ddot{x}^i + \Gamma^i_{00}(\dot{x}^0)^2 + 2\Gamma^i_{0j}\dot{x}^0\dot{x}^j + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0. \tag{25.30}$$

Because the metric (25.28) is static and diagonal, $\Gamma^i_{0j} = 0$.

2. Evaluation of the Christoffel Symbols To first order in the potentials, $g^{ij} \simeq \delta^{ij}$. The temporal Christoffel symbol is:

$$\Gamma^i_{00} = -\frac{1}{2}g^{ij}\partial_j g_{00} = \frac{1}{c^2}\partial_i\Phi. \tag{25.31}$$

For the spatial components:

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\right) = -\frac{1}{c^2}\left(\delta^i_k\partial_j\Psi + \delta^i_j\partial_k\Psi - \delta_{jk}\partial^i\Psi\right). \tag{25.32}$$

For a particle propagating along the $+z$-axis, the only spatial velocity component at zeroth order is $\dot{z}$. Thus, for transverse indices $i \in \{x, y\}$, the spatial Christoffel contribution simplifies to:

$$\Gamma^i_{zz} = \frac{1}{c^2}\partial_i\Psi. \tag{25.33}$$

The transverse geodesic equation (25.30) therefore becomes:

$$\ddot{x}^i + \frac{\partial_i \Phi}{c^2}(\dot{x}^0)^2 + \frac{\partial_i \Psi}{c^2}(\dot{z})^2 = 0. \tag{25.34}$$

3. Velocity Ratio and Transverse Acceleration For a particle moving with asymptotic physical speed $v_\infty = \mathrm{d}z/\mathrm{d}t$ along the z-axis:

$$\dot{x}^0 = c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_\infty c, \qquad \dot{z} = \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_\infty v_\infty, \tag{25.35}$$

where $\gamma_\infty = \left(1 - v_\infty^2/c^2\right)^{-1/2}$. Therefore, the exact kinematic relation between the spatial and temporal velocities is:

$$(\dot{z})^2 = \frac{v_\infty^2}{c^2}(\dot{x}^0)^2 \implies (\dot{x}^0)^2 = \frac{c^2}{v_\infty^2}(\dot{z})^2. \tag{25.36}$$

Note that for a null photon ($v_\infty = c$), $\mathrm{d}s^2 = 0$ gives directly $(\dot{x}^0)^2 = (\dot{z})^2$, completely consistent with taking the $v_\infty \to c$ limit.

Substituting Eq. (25.36) into Eq. (25.34):

$$\ddot{x}^i = -\left[\frac{\partial_i \Phi}{c^2}\left(\frac{c^2}{v_\infty^2}\right) + \frac{\partial_i \Psi}{c^2}\right](\dot{z})^2 = -\frac{(\dot{z})^2}{v_\infty^2}\left[\partial_i \Phi + \frac{v_\infty^2}{c^2}\partial_i \Psi\right]. \tag{25.37}$$

To express the trajectory as a function of the spatial coordinate z along the beam rather than proper time τ (or affine parameter λ), we use the chain rule:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \frac{\mathrm{d}^2 x^i}{\mathrm{d}z^2} + \frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^i}{\mathrm{d}z}. \tag{25.38}$$

Along the unperturbed straight trajectory, $\mathrm{d}x^i/\mathrm{d}z \sim \mathcal{O}(h)$ and $\mathrm{d}^2 z/\mathrm{d}\tau^2 \sim \mathcal{O}(h)$, so the second term is of second order $\mathcal{O}(h^2)$ and may be neglected. Thus:

$$(\dot{z})^2 \frac{\mathrm{d}^2 x^i}{\mathrm{d}z^2} = -\frac{(\dot{z})^2}{v_\infty^2}\left[\partial_i \Phi + \frac{v_\infty^2}{c^2}\partial_i \Psi\right]. \tag{25.39}$$

Canceling $(\dot{z})^2$ gives the general transverse trajectory equation:

$$\boxed{\frac{\mathrm{d}^2 \mathbf{x}_\perp}{\mathrm{d}z^2} = -\frac{1}{v_\infty^2}\left[\nabla_\perp \Phi + \frac{v_\infty^2}{c^2}\nabla_\perp \Psi\right]} \tag{25.40}$$

For General Relativity ($\Phi = \Psi$):

$$\frac{\mathrm{d}^2 \mathbf{x}_\perp}{\mathrm{d}z^2} = -\frac{1}{v_\infty^2}\left(1 + \frac{v_\infty^2}{c^2}\right)\nabla_\perp \Phi. \tag{25.41}$$

This reveals the physical origin of the two terms:

- The leading 1 inside the parenthesis comes directly from Γ^i_{00} (the temporal metric potential g_{00}, representing gravitational time dilation and the Newtonian potential).
- The velocity-dependent factor v_∞^2/c^2 comes directly from Γ^i_{zz} (spatial curvature g_{zz} coupled to the particle's spatial velocity $\dot{z}$).

4. Line-of-Sight Integration and Recovery of the Unified Deflection Formula In the small-angle approximation, the transverse slope $\mathrm{d}\mathbf{x}_\perp/\mathrm{d}z$ represents the trajectory angle. Integrating Eq. (25.41) from asymptotic entrance ($z \to -\infty$) to asymptotic exit ($z \to +\infty$):

$$\left.\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}z}\right|_{+\infty} - \left.\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}z}\right|_{-\infty} = -\frac{1}{v_\infty^2}\left(1+\frac{v_\infty^2}{c^2}\right)\int_{-\infty}^{+\infty} \nabla_\perp \Phi \, \mathrm{d}z. \tag{25.42}$$

For an unperturbed path at impact parameter b along the x-axis, $r(z) = \sqrt{b^2+z^2}$ and $\Phi(r) = -GM/\sqrt{b^2+z^2}$. The transverse gradient is:

$$\nabla_\perp \Phi = \frac{\partial \Phi}{\partial x}\hat{\mathbf{x}} = \frac{GMb}{(b^2+z^2)^{3/2}}\,\hat{\mathbf{x}}. \tag{25.43}$$

Integrating along the z-axis:

$$\int_{-\infty}^{+\infty} \frac{GMb}{(b^2+z^2)^{3/2}}\,\mathrm{d}z = GMb\left[\frac{z}{b^2\sqrt{b^2+z^2}}\right]_{-\infty}^{+\infty} = \frac{2GM}{b}. \tag{25.44}$$

The total deflection angle $\Delta\theta$ is the magnitude of this total angular change:

$$\boxed{\Delta\theta = \frac{2GM}{bv_\infty^2}\left(1+\frac{v_\infty^2}{c^2}\right)} \tag{25.45}$$

Equation (25.45) reproduces the exact master deflection formula (25.27) previously obtained via the Binet orbit equation:

- **Non-relativistic limit ($v_\infty \ll c$):** The spatial curvature term v_∞^2/c^2 is negligible, giving $\Delta\theta_{\text{Newton}} = \frac{2GM}{bv_\infty^2}$.
- **Ultra-relativistic photon limit ($v_\infty \to c$):** Both temporal and spatial metric perturbations contribute equally, doubling the deflection:

$$\Delta\theta_{\text{GR}} = \frac{4GM}{c^2 b}. \tag{25.46}$$

Notice also that because the unperturbed trajectory is straight, the transverse gradient commutes with the line-of-sight integration:

$$\Delta\boldsymbol{\theta} = -\nabla_\perp\left[\frac{1}{v_\infty^2}\left(1+\frac{v_\infty^2}{c^2}\right)\int_{-\infty}^{+\infty}\Phi\,\mathrm{d}z\right]. \tag{25.47}$$

Hence, the total deflection vector is strictly conservative and generated by the transverse gradient of a projected scalar potential.

25.4 Classical Test III: Gravitational Redshift

25.4.1 Locally Measured Frequency vs. Conserved Killing Energy

For a stationary observer with 4-velocity $U^\mu = \frac{\xi^\mu}{\sqrt{-\xi^2}} = \frac{1}{\sqrt{1-\frac{2GM}{c^2 r}}}(c,0,0,0)$ sitting at radius r, the locally measured photon frequency $\omega_{\text{obs}}(r)$ and energy $E_{\text{obs}}(r)$ are given by the frame projection:

$$E_{\text{obs}}(r) = \hbar\omega_{\text{obs}}(r) = -p_\mu U^\mu = \frac{-p_\mu \xi^\mu}{\sqrt{-\xi^2}} = \frac{E_\infty}{\sqrt{1-\frac{2GM}{c^2 r}}}, \tag{25.48}$$

where $E_\infty \equiv -p_0 = \hbar\omega_\infty =$ constant is the conserved Killing energy associated with the static timelike Killing vector field $\xi^\mu = (\partial_t)^\mu$.

25.4.2 Frequency Ratio and Terrestrial Confirmation

The ratio of frequencies measured by a static receiver at radius r_2 and a static emitter at radius r_1 is:

$$\boxed{\frac{\omega_{\text{obs}}(r_2)}{\omega_{\text{obs}}(r_1)} = \frac{\sqrt{-\xi^2(r_1)}}{\sqrt{-\xi^2(r_2)}} = \sqrt{\frac{1-\frac{2GM}{c^2 r_1}}{1-\frac{2GM}{c^2 r_2}}}} \tag{25.49}$$

For an observer at asymptotic infinity ($r_2 \to \infty$):

$$\omega_\infty = \omega_{\text{obs}}(r_1)\sqrt{1-\frac{2GM}{c^2 r_1}} \implies z \equiv \frac{\omega_{\text{obs}}(r_1)-\omega_\infty}{\omega_\infty} = \left(1-\frac{2GM}{c^2 r_1}\right)^{-1/2} - 1 \approx \frac{GM}{c^2 r_1}. \tag{25.50}$$

In the weak-field limit over a terrestrial vertical height $h \equiv r_2 - r_1 \ll R_\oplus$:

$$\frac{\Delta\omega}{\omega} \approx \frac{gh}{c^2}, \tag{25.51}$$

which was experimentally confirmed with precision by the Pound–Rebka experiment (1959) using Mössbauer spectroscopy.

Chapter 26

Relativistic Cosmology: The FLRW Metric, Friedmann Equations, and Cosmic Dynamics

On cosmological scales, the universe exhibits a remarkable degree of spatial homogeneity and isotropy. In this chapter, we develop the relativistic foundations of modern cosmology. We apply the Cosmological Principle to construct the geometry of maximally symmetric spatial 3-hypersurfaces, deriving the **Friedmann–Lemaître–Robertson–Walker (FLRW) metric**. We establish the global topological and volumetric properties of closed ($k = +1$), flat ($k = 0$), and open ($k = -1$) spatial sections. We then apply covariant conservation laws to cosmic perfect fluids, deducing the exact density scaling relations for matter, radiation, and dark energy. Finally, we compute the 4-dimensional Ricci curvature of the expanding spacetime, derive the **Friedmann Equations**, and analyze the dynamical evolution and geometric age of the cosmos.

26.1 The Cosmological Principle and Maximally Symmetric 3-Spaces

26.1.1 The Cosmological Principle

The **Cosmological Principle** states that, when smoothed over sufficiently large scales ($\gtrsim 100\,\mathrm{Mpc}$), the universe is:

1. **Spatially Homogeneous:** Invariant under spatial translations; no preferred observer or position exists.
2. **Spatially Isotropic:** Invariant under spatial rotations about any comoving observer; no preferred direction exists.

These spatial symmetries dictate that spacetime admits a global foliation by a 1-parameter family of spacelike 3-hypersurfaces Σ_t orthogonal to the worldlines of comoving observers:

$$\mathcal{M} = \mathbb{R} \times \Sigma_t, \qquad \mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\sigma^2_{(3)}, \tag{26.1}$$

where t is the cosmic proper time recorded by comoving clocks, $a(t)$ is the dimensionless cosmic scale factor (normalized to $a(t_0) = 1$ today), and $\mathrm{d}\sigma^2_{(3)} = \gamma_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$ is the Riemannian metric of a maximally symmetric 3-space.

26.1.2 Curvature of Maximally Symmetric 3-Spaces

A 3-dimensional Riemannian manifold (Σ, γ) is maximally symmetric if its Riemann curvature tensor ${}^{(3)}R_{\alpha\beta\gamma\delta}$ is entirely determined by its metric and a constant scalar curvature k:

$$ {}^{(3)}R_{\alpha\beta\gamma\delta} = k\left(\gamma_{\alpha\gamma}\gamma_{\beta\delta} - \gamma_{\alpha\delta}\gamma_{\beta\gamma}\right). \tag{26.2} $$

Contracting with $\gamma^{\beta\delta}$:

$$ {}^{(3)}R_{\alpha\gamma} = k\left(3\gamma_{\alpha\gamma} - \gamma_{\alpha\gamma}\right) = 2k\,\gamma_{\alpha\gamma}. \tag{26.3} $$

Taking the trace yields the 3D Ricci scalar curvature:

$$ {}^{(3)}R = \gamma^{\alpha\gamma}{}^{(3)}R_{\alpha\gamma} = 2k(3) = 6k \implies k = \frac{{}^{(3)}R}{6}. \tag{26.4} $$

26.1.3 Exact First-Principles Derivation of the Spatial Metric

To construct the explicit coordinate representation of $\mathrm{d}\sigma^2_{(3)}$, we adopt spherical polar coordinates $(\bar{r}, \theta, \phi)$ exploiting spatial isotropy about the origin:

$$ \mathrm{d}\sigma^2_{(3)} = e^{2\beta(\bar{r})}\mathrm{d}\bar{r}^2 + \bar{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{26.5} $$

The non-vanishing Christoffel symbols for this 3-metric are:

$$ \begin{aligned} &\Gamma^1_{11} = \partial_{\bar{r}}\beta, \qquad \Gamma^1_{22} = -\bar{r}e^{-2\beta}, \qquad \Gamma^1_{33} = -\bar{r}e^{-2\beta}\sin^2\theta, \\ &\Gamma^2_{12} = \frac{1}{\bar{r}}, \qquad \Gamma^3_{13} = \frac{1}{\bar{r}}, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3_{23} = \cot\theta. \end{aligned} \tag{26.6} $$

Computing the 3D Ricci tensor components:

$$ {}^{(3)}R_{\bar{r}\bar{r}} = \frac{2}{\bar{r}}\partial_{\bar{r}}\beta, \qquad {}^{(3)}R_{\theta\theta} = e^{-2\beta}\left(\bar{r}\partial_{\bar{r}}\beta - 1\right) + 1, \qquad {}^{(3)}R_{\phi\phi} = {}^{(3)}R_{\theta\theta}\sin^2\theta. \tag{26.7} $$

Equating ${}^{(3)}R_{\bar{r}\bar{r}} = 2k\gamma_{\bar{r}\bar{r}} = 2ke^{2\beta}$:

$$ \frac{2}{\bar{r}}\partial_{\bar{r}}\beta = 2ke^{2\beta} \implies e^{-2\beta}\partial_{\bar{r}}\beta = k\bar{r} \implies -\frac{1}{2}\partial_{\bar{r}}\left(e^{-2\beta}\right) = k\bar{r}. \tag{26.8} $$

Integrating both sides with respect to $\bar{r}$:

$$ e^{-2\beta(\bar{r})} = -\left(2C + k\bar{r}^2\right). \tag{26.9} $$

Substituting this expression into the angular equation ${}^{(3)}R_{\theta\theta} = 2k\gamma_{\theta\theta} = 2k\bar{r}^2$:

$$ {}^{(3)}R_{\theta\theta} = -(2C + k\bar{r}^2)\left(-\bar{r}\frac{k\bar{r}}{2C + k\bar{r}^2} - 1\right) + 1 = k\bar{r}^2 + 2C + k\bar{r}^2 + 1 = 2k\bar{r}^2 + 2C + 1. \tag{26.10} $$

Demanding consistency ${}^{(3)}R_{\theta\theta} = 2k\bar{r}^2$ requires:

$$ 2C + 1 = 0 \implies C = -\frac{1}{2}. \tag{26.11} $$

Thus, the radial metric potential is determined uniquely:

$$ e^{-2\beta(\bar{r})} = 1 - k\bar{r}^2 \implies e^{2\beta(\bar{r})} = \frac{1}{1 - k\bar{r}^2}. \tag{26.12} $$

Rescaling coordinates such that $k \in \{+1, 0, -1\}$ denotes the discrete sign of curvature, we obtain the standard FLRW metric.

Theorem: The Friedmann–Lemaître–Robertson–Walker (FLRW) Metric

The unique spacetime metric describing a spatially homogeneous and isotropic expanding universe is:

$$\boxed{\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]} \tag{26.13}$$

where $k = +1$ (closed, spherical 3-space), $k = 0$ (flat, Euclidean 3-space), and $k = -1$ (open, hyperbolic 3-space).

26.2 Geometry and Topology of Spatial Hypersurfaces

26.2.1 Hyperspherical Radial Coordinate Transformation

To manifest the underlying geometry of the spatial slices, we define the hyperspherical radial coordinate χ via:

$$\mathrm{d}\chi \equiv \frac{\mathrm{d}r}{\sqrt{1-kr^2}} \implies r(\chi) = S_k(\chi), \tag{26.14}$$

where:

$$S_k(\chi) \equiv \begin{cases} \sin\chi, & k = +1 \quad (\text{Closed } S^3,\ 0 \le \chi \le \pi), \\ \chi, & k = 0 \quad (\text{Flat } \mathbb{R}^3,\ 0 \le \chi < \infty), \\ \sinh\chi, & k = -1 \quad (\text{Open } H^3,\ 0 \le \chi < \infty). \end{cases} \tag{26.15}$$

In (χ, θ, ϕ) coordinates, the FLRW metric assumes the conformal hyperspherical form:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + S_k^2(\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]. \tag{26.16}$$

26.2.2 Volumes and Surface Areas of the 3-Spaces

Key Concept: Geometric Classification and Topology of 3-Slices

1. **Closed Universe ($k = +1$, 3-Sphere S^3):** The spatial sections are compact 3-spheres embedded in $\mathbb{R}^4$. The total spatial volume is finite without boundary:
$$V_{(3)}(t) = a^3(t) \int_0^{\pi} \sin^2\chi \, \mathrm{d}\chi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = a^3(t) \left(\frac{\pi}{2}\right)(2)(2\pi) = 2\pi^2 a^3(t). \tag{26.17}$$
The proper surface area of a concentric 2-sphere at radial coordinate χ is:
$$S_{(2)}(\chi) = 4\pi a^2(t) \sin^2\chi. \tag{26.18}$$
As χ increases from 0 to π, the surface area expands to a maximum of $4\pi a^2(t)$ at the equator $\chi = \pi/2$, and then contracts back to zero at the antipodal pole $\chi = \pi$.
2. **Flat Universe ($k = 0$, Euclidean Space $\mathbb{R}^3$):** The spatial sections are non-compact with infinite total volume $V_{(3)} = \infty$, and the 2-sphere surface area grows monotonically as $S_{(2)}(\chi) = 4\pi a^2(t)\chi^2$.
3. **Open Universe ($k = -1$, Hyperbolic Space H^3):** The spatial sections are non-compact hyperboloids with infinite total volume $V_{(3)} = \infty$, and the 2-sphere surface area diverges exponentially:
$$S_{(2)}(\chi) = 4\pi a^2(t) \sinh^2\chi \xrightarrow{\chi\to\infty} \pi a^2(t) e^{2\chi} \to \infty. \tag{26.19}$$

26.3 Relativistic Cosmic Fluids and Covariant Conservation

26.3.1 Comoving Perfect Fluid Stress-Energy

On cosmological scales, matter and radiation are modeled as a homogeneous, isotropic perfect fluid. In comoving coordinates, the 4-velocity of the cosmic fluid is $u^\mu = (c, 0, 0, 0)$, and the mixed stress-energy tensor is:
$$T^\mu{}_\nu = \mathrm{diag}\left(-\rho(t)c^2,\, p(t),\, p(t),\, p(t)\right), \qquad T \equiv T^\mu{}_\mu = -\rho c^2 + 3p, \tag{26.20}$$
where $\rho(t)$ is the proper mass-energy density and $p(t)$ is the isotropic cosmic pressure.

26.3.2 Covariant Continuity Equation

Applying the covariant energy-momentum conservation law $\nabla_\mu T^\mu{}_0 = 0$:
$$\nabla_\mu T^\mu{}_0 = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, T^\mu{}_0\right) - \frac{1}{2}\left(\partial_0 g_{\alpha\beta}\right) T^{\alpha\beta} = 0. \tag{26.21}$$
With $\sqrt{-g} = c\, a^3(t) \frac{r^2 \sin\theta}{\sqrt{1-kr^2}}$ and $g_{\alpha\beta} = a^2(t)\gamma_{\alpha\beta}$:
$$\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3(-\rho c^2)\right) - \frac{1}{2}(2a\dot{a}\gamma_{\alpha\beta})\left(\frac{p}{a^2}\gamma^{\alpha\beta}\right) = 0 \implies -\dot{\rho}c^2 - 3\frac{\dot{a}}{a}\rho c^2 - 3\frac{\dot{a}}{a}p = 0. \tag{26.22}$$

Theorem: Cosmic Continuity Equation

The cosmic fluid continuity equation governing energy density dilution in an expanding universe is:

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \iff \dot{\rho} + 3H(t)\left(\rho + \frac{p}{c^2}\right) = 0} \tag{26.23}$$

where $H(t) \equiv \frac{\dot{a}}{a}$ is the Hubble parameter.

26.3.3 Barotropic Equations of State and Density Scaling

For a barotropic fluid with linear equation-of-state parameter $w \equiv \frac{p}{\rho c^2}$, the continuity equation integrates directly:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \implies \rho(a) = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}. \tag{26.24}$$

The Dominant Energy Condition (DEC) requires $\rho c^2 \geq |p|$, which enforces $|w| \leq 1$.

Key Concept: Cosmic Density Scaling Regimes

1. **Non-Relativistic Matter / Dust ($w = 0$):** Pressureless dust has $p_{\mathrm{m}} = 0$, giving:

$$\rho_{\mathrm{m}}(a) = \rho_{\mathrm{m},0}\, a^{-3}. \tag{26.25}$$

   The energy density decreases inversely with physical 3-volume ($V \propto a^3$).

2. **Relativistic Radiation and Neutrinos ($w = 1/3$):** From the tracelessness of the electromagnetic stress-energy tensor $T^{\mu}{}_{\mu} = 0 = -\rho_{\mathrm{rad}}c^2 + 3p_{\mathrm{rad}}$, we obtain $p_{\mathrm{rad}} = \frac{1}{3}\rho_{\mathrm{rad}}c^2$, giving:

$$\rho_{\mathrm{rad}}(a) = \rho_{\mathrm{rad},0}\, a^{-4}. \tag{26.26}$$

   The extra factor of a^{-1} originates from the cosmological redshifting of photon energy ($E = h\nu \propto a^{-1}$) combined with volumetric dilution ($n_\gamma \propto a^{-3}$).

3. **Vacuum Energy / Cosmological Constant ($w = -1$):** For dark energy represented by Einstein's cosmological constant Λ:

$$T^{\mu}{}_{\nu} = -\frac{\Lambda c^4}{8\pi G}\delta^{\mu}_{\nu} \implies \rho_\Lambda c^2 = -p_\Lambda = \frac{\Lambda c^4}{8\pi G} = \text{constant}. \tag{26.27}$$

   Thus $w_\Lambda = -1$, and $\rho_\Lambda(a) \propto a^0 = \text{constant}$.

26.4 Derivation of the Friedmann Equations

26.4.1 4-Dimensional Ricci Curvature Components

Evaluating the 4-dimensional connection coefficients for the FLRW metric (26.13):

$$\Gamma^0_{ij} = \frac{a\dot{a}}{c}\gamma_{ij}, \qquad \Gamma^i_{0j} = \frac{\dot{a}}{ca}\delta^i_j, \qquad \Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk}. \tag{26.28}$$

The non-zero components of the 4D Ricci tensor $R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}$ evaluate to:

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{26.29}$$

$$R_{ij} = \left(\frac{a\ddot{a}}{c^2} + 2\frac{\dot{a}^2}{c^2} + 2k \right) \gamma_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2kc^2}{a^2} \right) \frac{g_{ij}}{c^2}. \tag{26.30}$$

Taking the 4D Ricci scalar trace:

$$R = g^{\mu\nu} R_{\mu\nu} = -\frac{1}{c^2} R_{00} + \frac{1}{a^2} \gamma^{ij} R_{ij} = 6 \left[\frac{\ddot{a}}{ac^2} + \left(\frac{\dot{a}}{ac} \right)^2 + \frac{k}{a^2} \right]. \tag{26.31}$$

26.4.2 The Acceleration and Expansion Equations

Einstein's field equations with cosmological constant Λ in trace-reversed form are:

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \Lambda g_{\mu\nu}. \tag{26.32}$$

Evaluating the time-time component ($\mu = 0, \nu = 0$):

$$-3\frac{\ddot{a}}{a} = \frac{8\pi G}{c^4} \left(\rho c^4 - \frac{1}{2} c^2 \left(-\rho c^2 + 3p \right) \right) - \Lambda c^2 = \frac{4\pi G}{c^2} \left(\rho c^2 + 3p \right) - \Lambda c^2. \tag{26.33}$$

Dividing by -3 yields the **Second Friedmann Equation (Acceleration Equation)**:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}} \tag{26.34}$$

Next, evaluating the spatial components ($\mu = i, \nu = j$):

$$\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2kc^2}{a^2} \right) \frac{g_{ij}}{c^2} = \frac{8\pi G}{c^4} \left(p\, g_{ij} - \frac{1}{2} g_{ij} \left(-\rho c^2 + 3p \right) \right) + \Lambda g_{ij} = \frac{4\pi G}{c^4} \left(\rho c^2 - p \right) g_{ij} + \Lambda g_{ij}. \tag{26.35}$$

Multiplying by c^2 and simplifying:

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a} \right)^2 + \frac{2kc^2}{a^2} = 4\pi G \left(\rho - \frac{p}{c^2} \right) + \Lambda c^2. \tag{26.36}$$

Substituting the acceleration equation (26.34) into Eq. (26.36):

$$-\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3} + 2\left(\frac{\dot{a}}{a} \right)^2 + \frac{2kc^2}{a^2} = 4\pi G \left(\rho - \frac{p}{c^2} \right) + \Lambda c^2. \tag{26.37}$$

Rearranging terms:

$$2\left(\frac{\dot{a}}{a} \right)^2 + \frac{2kc^2}{a^2} = 4\pi G \left(\rho - \frac{p}{c^2} + \frac{1}{3}\rho + \frac{p}{c^2} \right) + \frac{2\Lambda c^2}{3} = \frac{16\pi G}{3}\rho + \frac{2\Lambda c^2}{3}. \tag{26.38}$$

Dividing by 2 yields the celebrated **First Friedmann Equation**:

$$\boxed{\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}} \tag{26.39}$$

26.5 Cosmological Parameters and Kinematics

26.5.1 Hubble Parameter, Hubble Time, and Hubble Length

The rate of cosmic expansion is parameterized by the **Hubble parameter**:

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{26.40}$$

At the present epoch $t = t_0$, its value is the **Hubble constant**:

$$H_0 \equiv H(t_0) = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \approx 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \quad (h \approx 0.70). \tag{26.41}$$

Using the conversion $1\,\mathrm{Mpc} = 3.0857 \times 10^{24}\,\mathrm{cm} = 3.0857 \times 10^{22}\,\mathrm{m}$:

- **Hubble Time (t_H):** The characteristic expansion timescale of the cosmos:

$$t_H \equiv H_0^{-1} \approx \frac{1}{70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}} \approx 13.97 \times 10^9\,\mathrm{years} \approx 4.41 \times 10^{17}\,\mathrm{s}. \tag{26.42}$$

- **Hubble Length (d_H):** The characteristic causal scale:

$$d_H \equiv cH_0^{-1} \approx 3.0 \times 10^3\,h^{-1}\,\mathrm{Mpc} \approx 9.25 \times 10^{27}\,h^{-1}\,\mathrm{cm} \approx 4.28\,\mathrm{Gpc}. \tag{26.43}$$

26.5.2 Critical Density and Dimensionless Density Parameters

Setting $k = 0$ and $\Lambda = 0$ in the First Friedmann Equation defines the **critical density** required for spatial flatness:

$$\rho_{\mathrm{crit}}(t) \equiv \frac{3H^2(t)}{8\pi G} \implies \rho_{\mathrm{crit},0} = \frac{3H_0^2}{8\pi G} \approx 1.88 \times 10^{-29}\,h^2\,\mathrm{g\,cm^{-3}} \approx 9.2 \times 10^{-27}\,\mathrm{kg\,m^{-3}}. \tag{26.44}$$

Dividing the Friedmann equation by $H^2(t)$:

$$1 = \frac{8\pi G\rho}{3H^2} - \frac{kc^2}{a^2H^2} + \frac{\Lambda c^2}{3H^2} \equiv \Omega(t) + \Omega_k(t) + \Omega_\Lambda(t), \tag{26.45}$$

where:

$$\Omega_{\mathrm{m}} \equiv \frac{\rho_{\mathrm{m}}}{\rho_{\mathrm{crit}}}, \quad \Omega_{\mathrm{rad}} \equiv \frac{\rho_{\mathrm{rad}}}{\rho_{\mathrm{crit}}}, \quad \Omega_\Lambda \equiv \frac{\Lambda c^2}{3H^2}, \quad \Omega_k \equiv -\frac{kc^2}{a^2H^2}. \tag{26.46}$$

Theorem: The Cosmic Density Sum Rule

At any epoch in cosmic history, the total energy density parameter satisfies:

$$\boxed{\Omega_{\mathrm{total}} - 1 = \frac{kc^2}{a^2H^2}} \tag{26.47}$$

Consequently:

- $k = +1 \iff \Omega_{\mathrm{total}} > 1$ (Closed universe),
- $k = 0 \iff \Omega_{\mathrm{total}} = 1$ (Flat universe),
- $k = -1 \iff \Omega_{\mathrm{total}} < 1$ (Open universe).

Defining an effective **curvature density** $\rho_c \equiv -\frac{3kc^2}{8\pi Ga^2}$, the Friedmann equation assumes the unified form:

$$H^2 = \frac{8\pi G}{3}\left(\sum_i \rho_i + \rho_c\right) \implies 1 = \sum_i \Omega_i + \Omega_c \implies \Omega_c = 1 - \Omega_{\mathrm{total}}. \tag{26.48}$$

26.5.3 Time Evolution of the Expansion Rate $\dot{H}$

Differentiating $H = \dot{a}/a$ with respect to cosmic time:

$$\dot{H} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} - H^2. \tag{26.49}$$

Substituting the acceleration equation and Friedmann equation:

$$\dot{H} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) - \left[\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}\right] = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2}. \tag{26.50}$$

For a universe filled with fluids satisfying the Null Energy Condition ($\rho + p/c^2 \geq 0$) and flat geometry ($k = 0$):

$$\dot{H} = -4\pi G\rho(1+w) \leq 0. \tag{26.51}$$

Thus, the Hubble expansion parameter $H(t)$ decreases monotonically over time. Note that $\dot{H} < 0$ does not imply deceleration ($\ddot{a} < 0$); cosmic acceleration $\ddot{a}/a = \dot{H} + H^2 > 0$ occurs whenever $w < -1/3$.

26.6 Analytic Cosmic Solutions for Flat Universes ($k = 0$)

26.6.1 General Power-Law Expansion

For a flat universe ($k = 0$) dominated by a single fluid component with equation of state $p = w\rho c^2$, the density scales as $\rho(a) \propto a^{-n}$ where $n \equiv 3(1+w)$. The First Friedmann Equation becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-n} \implies \frac{\dot{a}}{a} \propto a^{-n/2} \implies a^{n/2-1}\mathrm{d}a = C\,\mathrm{d}t. \tag{26.52}$$

Integrating from $t = 0$ where $a(0) = 0$:

$$\frac{2}{n}a^{n/2} \propto t \implies a(t) = a_0\left(\frac{t}{t_0}\right)^{2/n} = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}. \tag{26.53}$$

The Hubble parameter and cosmic age are:

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{nt} = \frac{2}{3(1+w)t} \implies t_0 = \frac{2}{3(1+w)H_0}. \tag{26.54}$$

26.6.2 The Standard Cosmological Eras

Key Concept: Special Analytic Solutions

1. **Matter Domination (Einstein–de Sitter, $w = 0, n = 3$):**

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3}, \qquad H(t) = \frac{2}{3t}, \qquad t_0 = \frac{2}{3}H_0^{-1} = \frac{2}{3}t_H \approx 9.3\,\mathrm{Gyr}. \tag{26.55}$$

2. **Radiation Domination ($w = 1/3, n = 4$):**

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{1/2}, \qquad H(t) = \frac{1}{2t}, \qquad t_0 = \frac{1}{2}H_0^{-1} = \frac{1}{2}t_H. \tag{26.56}$$

3. **Vacuum Domination / de Sitter Space ($w = -1, n = 0$):** With $\rho_\Lambda = \text{const}$, the

Friedmann equation gives $H = H_\Lambda \equiv \sqrt{\frac{\Lambda c^2}{3}} = \text{const}$:

$$\frac{\dot{a}}{a} = H_\Lambda \implies a(t) = a_0\, e^{H_\Lambda (t - t_0)}. \tag{26.57}$$

The spacetime is de Sitter space with exponential cosmic expansion and constant horizon distance.

26.7 Geometric Age of the Universe and Tangent Line Analysis

The relationship between the present cosmic age t_0 and the Hubble time H_0^{-1} admits an elegant geometric interpretation from the $a(t)$ trajectory.

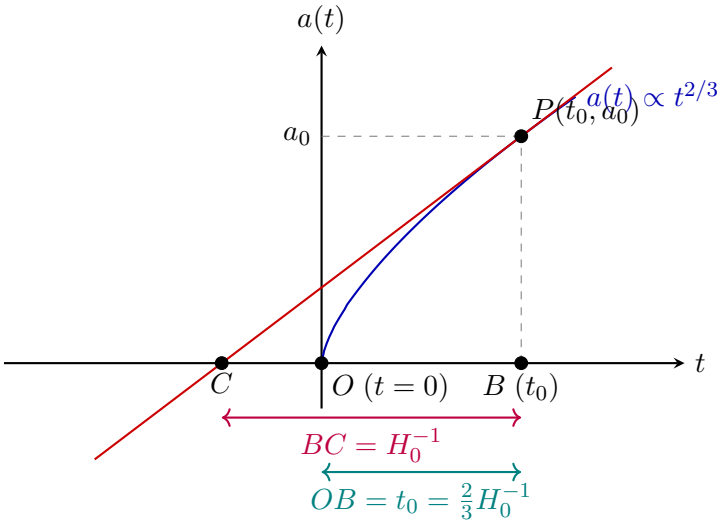


Figure 26.1: Geometric derivation of the cosmic age t_0 for a matter-dominated flat universe. The slope of the tangent line at $P(t_0, a_0)$ is $\dot{a}_0$, intercepting the time axis at point C. The distance $BC = a_0/\dot{a}_0 = H_0^{-1}$ represents the Hubble time, yielding $t_0 = OB = \frac{2}{3}BC = \frac{2}{3}H_0^{-1}$.

Key Concept: Geometric Proof of the Age of the Universe

Consider the tangent line to the scale factor trajectory $a(t)$ at the present epoch $P = (t_0, a_0)$:

1. The height of point P above the time axis is $PB = a_0$.
2. The slope of the tangent line at P is the current expansion rate:
$$\text{Slope} = \dot{a}_0 = \frac{PB}{BC} = \frac{a_0}{BC}. \tag{26.58}$$
3. Therefore, the horizontal base of the tangent triangle BC is:
$$BC = \frac{a_0}{\dot{a}_0} = \frac{1}{H_0} \equiv t_H \quad \text{(the Hubble time)}. \tag{26.59}$$
4. Since $a(t) \propto t^{2/3}$ under matter domination, $\dot{a}_0 = \frac{2}{3}\frac{a_0}{t_0}$, so:
$$H_0 = \frac{\dot{a}_0}{a_0} = \frac{2}{3t_0} \implies OB = t_0 = \frac{2}{3}BC = \frac{2}{3}H_0^{-1}. \tag{26.60}$$

Because gravity decelerates the expansion ($\ddot{a} < 0$), the true age t_0 is strictly less than the naive linear extrapolation age H_0^{-1}.

26.8 Qualitative Dynamics: Evolution of Open, Flat, and Closed Universes

For a matter-dominated universe with zero cosmological constant ($\Lambda = 0$), the First Friedmann Equation can be recast as an energy conservation equation for a unit mass particle in a Newtonian effective potential:

$$\frac{1}{2}\dot{a}^2 - \frac{GM_{\text{eff}}}{a} = -\frac{kc^2}{2}, \qquad M_{\text{eff}} \equiv \frac{4\pi}{3}\rho_0 a_0^3 = \text{constant}. \tag{26.61}$$

The spatial curvature index k plays the role of total energy:

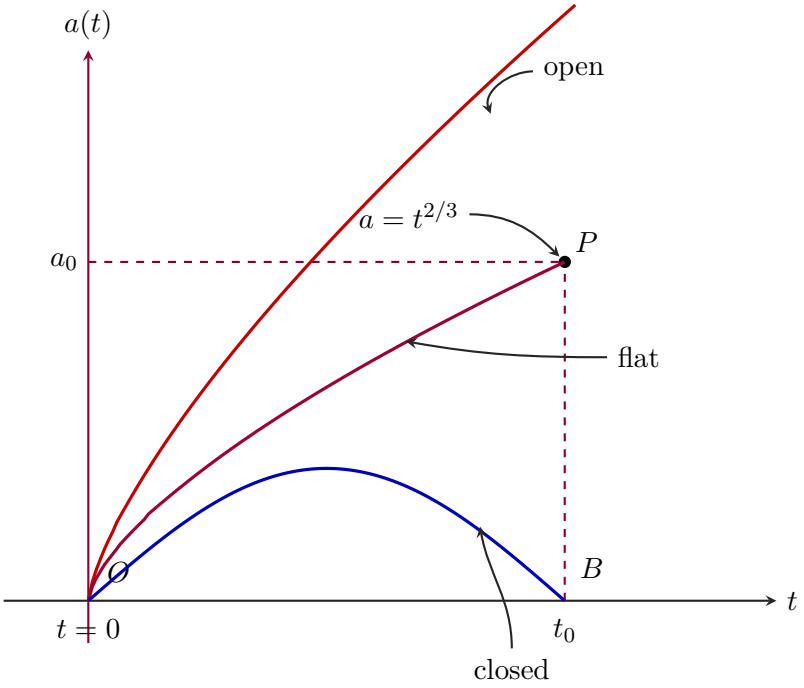


Figure 26.2: Comparative dynamical evolution of the cosmic scale factor $a(t)$ for matter-dominated open ($k = -1$), flat ($k = 0, a \propto t^{2/3}$), and closed ($k = +1$) universes, exactly matching the structure of the lecture notes. The flat curve passes through (t_0, a_0) at point P. The open curve expands fastest and remains above the flat curve, while the closed model reaches a maximum and recollapses to zero size at point $B(t_0)$.

Key Concept: Summary of Geometric Evolutionary Fates

- **Closed Universe ($k = +1, \Omega > 1$, Negative Total Energy):** The gravitational attraction overcomes expansion. The universe expands to a maximum radius $a_{\max} = \frac{8\pi G\rho_0 a_0^3}{3c^2}$ where $\dot{a} = 0$, and then reverses into a symmetric contraction phase, culminating in a **Big Crunch** singularity ($a \to 0$). The parametric solution is a

cycloid:

$$a(\eta) = \frac{a_{\max}}{2}(1 - \cos\eta), \qquad ct(\eta) = \frac{a_{\max}}{2}(\eta - \sin\eta), \qquad 0 \le \eta \le 2\pi. \tag{26.62}$$

- **Flat Universe ($k = 0, \Omega = 1$, Zero Total Energy):** The cosmic expansion is marginally bound, expanding forever with $a(t) \propto t^{2/3}$ and asymptotic coasting velocity $\lim_{t\to\infty} \dot{a}(t) = 0$.
- **Open Universe ($k = -1, \Omega < 1$, Positive Total Energy):** Gravitational attraction is insufficient to halt expansion. The universe expands indefinitely, approaching a free-coasting hyperbolic expansion $\dot{a} \to c$ as $t \to \infty$. The parametric solution is:

$$a(\eta) = A(\cosh\eta - 1), \qquad ct(\eta) = A(\sinh\eta - \eta), \qquad \eta \ge 0. \tag{26.63}$$

In the presence of dark energy ($\Omega_\Lambda \approx 0.7$), the expansion overcomes deceleration at late times, driving the universe into an eternal, accelerating de Sitter expansion phase.

Chapter 27

Tetrad Formalism, Spin Connections, and Fermions in Curved Spacetime

27.1 Foundations of the Tetrad (Vierbein) Formalism

27.1.1 Physical Motivation: Why Orthonormal Frames and Spinors

In standard Riemannian General Relativity, spacetime geometry is described by a coordinate metric tensor $g_{\mu\nu}(x)$ expressed in terms of a holonomic coordinate basis $\{\partial_\mu \equiv \partial/\partial x^\mu\}$ for the tangent space $T_p\mathcal{M}$ and its dual co-basis $\{\mathrm{d}x^\mu\}$ for the cotangent space $T_p^*\mathcal{M}$. While tensor calculus provides a coordinate-invariant description of gravity via General Coordinate Transformations (GCT), coordinate bases present two major limitations:

a. **Geometric and Observational Non-orthonormality:** Coordinate basis vectors are generally neither mutually orthogonal nor of unit length ($g_{\mu\nu} = \partial_\mu \cdot \partial_\nu \neq \eta_{\mu\nu}$). Physical observers carry clocks, gyroscopes, and accelerometers that naturally measure physical quantities (such as velocity, energy-momentum, and electromagnetic fields) with respect to a locally orthonormal, freely falling Minkowski frame where Special Relativity holds locally.

b. **The Spinor Problem (Fermions in Curved Space):** Most fundamentally, **half-integer spin fermions (Dirac and Weyl spinors) cannot couple directly to coordinate metrics $g_{\mu\nu}$ or Christoffel connections $\Gamma^\lambda_{\mu\nu}$.** In quantum field theory and representation theory, spin-1/2 fields transform under the universal double cover $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$ of the Lorentz group $\mathrm{SO}(1,3)$. The infinite-dimensional general linear group $\mathrm{GL}(4,\mathbb{R})$ of general coordinate transformations possesses *no* finite-dimensional unitary or spinorial representations (Bargmann–Wigner theorem). Thus, to introduce Dirac fermions into curved spacetime, one must construct a local orthonormal Minkowski frame at every event $p \in \mathcal{M}$.

27.1.2 Local Orthonormal Axes and the Vierbein

Definition: Tetrad (Vierbein) Basis and Co-Tetrad

A **tetrad** (Greek: *foursome*) or **vierbein** (German: *four legs*) is an ordered set of four linearly independent vector fields $\{e_a\} = \{e_0, e_1, e_2, e_3\}$ that form an orthonormal basis for the tangent space $T_p\mathcal{M}$ at every event $p \in \mathcal{M}$:

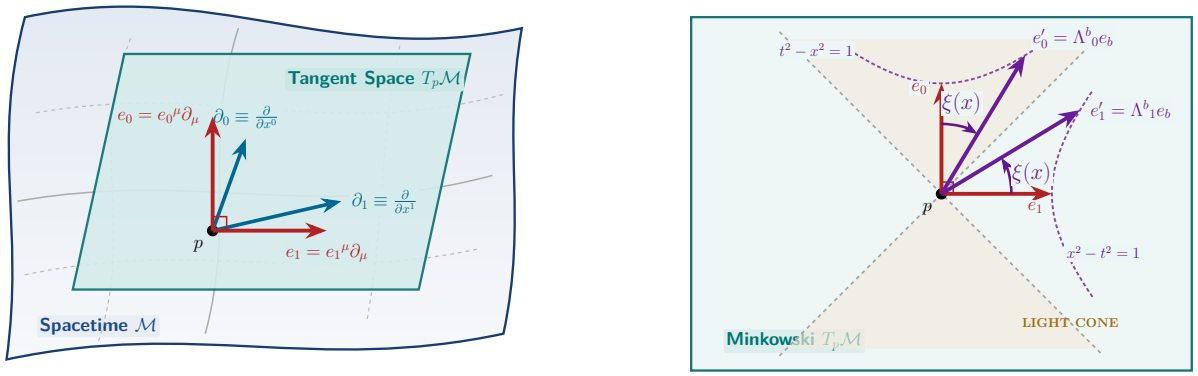
$$e_a \cdot e_b \equiv g(e_a, e_b) = \eta_{ab} = \mathrm{diag}(-1, +1, +1, +1), \tag{27.1}$$

where Latin indices $a, b, c, d \in \{0, 1, 2, 3\}$ denote **local Lorentz (tetrad/frame) indices**, and Greek indices $\mu, \nu, \rho, \sigma \in \{0, 1, 2, 3\}$ denote **spacetime coordinate indices**.
The transformation between the coordinate basis $\{\partial_\mu\}$ and the tetrad basis $\{e_a\}$ is defined by the **vierbein coefficients** ${e^a}_\mu(x)$ and their inverse ${e_a}^\mu(x)$:

$$e_a = {e_a}^\mu(x)\,\partial_\mu, \qquad \partial_\mu = {e^a}_\mu(x)\,e_a, \tag{27.2}$$

$$e^a = {e^a}_\mu(x)\,\mathrm{d}x^\mu, \qquad \mathrm{d}x^\mu = {e_a}^\mu(x)\,e^a. \tag{27.3}$$

The dual basis 1-forms satisfy the duality relation $e^a(e_b) = \delta^a_b$.



(a) Tangent Space & Vierbein
Coordinate Basis $\{\partial_\mu\}$ vs. Orthonormal Frame $\{e_a\}$

(b) Local Lorentz Gauge Freedom
${e'^a}_\mu(x) = \Lambda^a{}_b(x)\,{e^b}_\mu(x) \in \mathrm{SO}(1,3)$

Figure 27.1: The geometric architecture of the tetrad (vierbein) formalism. **(a)** At each event $p \in \mathcal{M}$, the coordinate basis tangent vectors $\{\partial_\mu \equiv \partial/\partial x^\mu\}$ are non-orthonormal ($g_{\mu\nu} = \partial_\mu \cdot \partial_\nu$) and coordinate-dependent. The tetrad basis $\{e_a \equiv {e_a}^\mu\partial_\mu\}$ establishes a rigid, locally orthonormal Minkowski frame ($e_a \cdot e_b = \eta_{ab}$) in the tangent space $T_p\mathcal{M}$. **(b)** The local Lorentz gauge group $\mathrm{SO}(1,3)$ acts internally on the Latin frame indices via point-dependent rotations and boosts $\Lambda^a{}_b(x)$, preserving the Minkowski metric η_{ab} while leaving the underlying spacetime coordinates unaffected.

27.1.3 Dual Transformation Laws: General Coordinate and Local Lorentz Symmetries

The vierbein ${e^a}_\mu(x)$ is a mixed geometric object carrying two distinct types of indices that transform independently under two distinct groups of gauge symmetries:

1. **General Coordinate Transformations (GCT / Diffeomorphisms):** Under a spacetime coordinate transformation $x^\mu \to x'^\mu(x)$, the coordinate basis vectors and dual forms transform via the Jacobian matrix:

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu, \qquad \mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu. \tag{27.4}$$

Because the physical basis vectors e_a and 1-forms e^a are coordinate-independent geometric vectors and 1-forms, their component expansion transforms under GCT as:

$${e'^a}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\,{e^a}_\nu(x), \qquad {e'_a}^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\,{e_a}^\nu(x). \tag{27.5}$$

Thus, the Latin frame index a is completely invariant under coordinate transformations (acting as a GCT scalar), while the Greek index μ transforms as a standard covariant or contravariant tensor index.

2. **Local Lorentz Transformations (LLT / Tangent Space Gauge Group):** At each event $x \in \mathcal{M}$, we can perform an independent, point-dependent Lorentz transformation $\Lambda^a{}_b(x) \in \mathrm{SO}(1,3)$ (comprising 3 spatial rotations and 3 Lorentz boosts) on the local frame vectors that preserves the flat Minkowski metric:

$$\Lambda^a{}_c(x)\,\Lambda^b{}_d(x)\,\eta_{ab} = \eta_{cd} \quad \Longleftrightarrow \quad \Lambda^T(x)\,\eta\,\Lambda(x) = \eta. \tag{27.6}$$

Under this internal local Lorentz gauge transformation, the tetrad basis vectors and co-tetrad 1-forms rotate according to:

$$e'_a(x) = \Lambda^b{}_a(x)\,e_b(x), \qquad e'^a(x) = \Lambda^a{}_b(x)\,e^b(x). \tag{27.7}$$

In component form, the vierbein transforms as:

$$e'^a{}_\mu(x) = \Lambda^a{}_b(x)\,e^b{}_\mu(x), \qquad e'^\mu_a(x) = \Lambda_a{}^b(x)\,e_b{}^\mu(x) \equiv (\Lambda^{-1})^b{}_a(x)\,e_b{}^\mu(x). \tag{27.8}$$

Here the Latin index transforms under the fundamental vector representation of $\mathrm{SO}(1,3)$, while the Greek coordinate index μ is untouched.

3. **Combined (Dual) Transformation Law:** Under a simultaneous coordinate transformation $x \to x'$ and local Lorentz rotation $\Lambda(x)$, the vierbein transforms as a mixed tensor-gauge field:

$$e'^a{}_\mu(x') = \Lambda^a{}_b(x)\,\frac{\partial x^\nu}{\partial x'^\mu}\,e^b{}_\nu(x). \tag{27.9}$$

4. **Invariance of the Metric under Local Lorentz Transformations:** Using the transformation law (27.8) and the defining property (27.6), the spacetime metric reconstructed from the boosted tetrad satisfies:

$$g'_{\mu\nu} = \eta_{ab}e'^a{}_\mu e'^b{}_\nu = \eta_{ab}\,(\Lambda^a{}_c(x)e^c{}_\mu)\left(\Lambda^b{}_d(x)e^d{}_\nu\right) = \left(\eta_{ab}\Lambda^a{}_c\Lambda^b{}_d\right)e^c{}_\mu e^d{}_\nu = \eta_{cd}e^c{}_\mu e^d{}_\nu = g_{\mu\nu}. \tag{27.10}$$

The coordinate metric $g_{\mu\nu}$ is completely invariant under local Lorentz gauge transformations.

Key Concept: Fundamental Concepts and Degree of Freedom Accounting

1. **Why Tetrads?** Physics is most transparent in locally inertial (orthonormal) frames where the laws of Special Relativity apply directly at each event $p \in \mathcal{M}$. Crucially, the tetrad provides the necessary bridge to introduce Clifford Dirac matrices γ^a and spinor representations $\mathrm{SL}(2,\mathbb{C})$ in curved spacetime.

2. **Degree of Freedom Accounting:** A vierbein $e^a{}_\mu(x)$ has $4 \times 4 = 16$ components. The symmetric metric $g_{\mu\nu}(x)$ has $4 \times 5/2 = 10$ independent components. The 6 additional degrees of freedom correspond precisely to the internal **Local Lorentz Transformations** (3 spatial rotations and 3 Lorentz boosts) parameterized by $\Lambda^a{}_b(x) \in \mathrm{SO}(1,3)$ at each event:

$$\underbrace{16}_{\text{Vierbein } e^a{}_\mu} - \underbrace{4}_{\text{GCT } x\to x'} - \underbrace{6}_{\text{LLT } \Lambda(x)} = \mathbf{6} \text{ independent off-shell metric/vierbein DOF.} \tag{27.11}$$

On-shell (after imposing the Einstein field equations and dynamical gauge fixing), this leaves $6 - 4(\text{constraints}) = 2$ propagating physical gravitational wave polarizations $(h_+, h_\times)$.

3. **Directed Derivatives ∂_a:** The directed derivative $\partial_a \equiv e_a{}^\mu\partial_\mu$ is the directional

derivative along the local orthonormal tetrad axis e_a. It represents the rate of change measured by a local observer moving along that axis. Unlike coordinate partial derivatives, directed derivatives **do not commute**: $[\partial_a, \partial_b] = c_{ab}{}^c \partial_c$.

4. **Dual Gauge Invariance:** The coordinate metric $g_{\mu\nu}$ is coordinate-dependent but invariant under local Lorentz gauge transformations. Conversely, tensors expressed purely in Latin tetrad indices (such as the local energy-momentum components $T^{ab} = e^a{}_\mu e^b{}_\nu T^{\mu\nu}$ or local 4-velocity $u^a = e^a{}_\mu u^\mu$) are coordinate scalars (invariant under GCT) but transform covariantly under LLT.

27.1.4 Completeness Relations and Metric Reconstruction

Theorem: Completeness Identities and Metric Factorization

The vierbein matrices satisfy the fundamental **completeness and orthogonality identities**:

$$e^a{}_\mu(x)\, e_b{}^\mu(x) = \delta^a_b \qquad \text{(orthonormality in tangent space)}, \tag{27.12}$$

$$e^a{}_\mu(x)\, e_a{}^\nu(x) = \delta^\nu_\mu \qquad \text{(completeness on spacetime manifold)}. \tag{27.13}$$

The curved spacetime metric $g_{\mu\nu}(x)$ and its inverse $g^{\mu\nu}(x)$ are factorized through the flat Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$:

$$g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x)\, e^b{}_\nu(x), \qquad g^{\mu\nu}(x) = \eta^{ab}\, e_a{}^\mu(x)\, e_b{}^\nu(x). \tag{27.14}$$

Conversely, the Minkowski metric is obtained by projecting with the inverse vierbeins:

$$\eta_{ab} = g_{\mu\nu}(x)\, e_a{}^\mu(x)\, e_b{}^\nu(x). \tag{27.15}$$

The spacetime invariant interval $\mathrm{d}s^2$ factorizes as a sum of squared tetrad 1-forms:

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = \eta_{ab}\, e^a\, e^b = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2. \tag{27.16}$$

Proof. Consider any tangent vector field $V = V^\mu \partial_\mu = V^a e_a$. Its coordinate invariant length squared is:

$$g(V,V) = g_{\mu\nu} V^\mu V^\nu = \eta_{ab} V^a V^b = \eta_{ab}\, (e^a{}_\mu V^\mu) \left(e^b{}_\nu V^\nu\right) = \left(\eta_{ab} e^a{}_\mu e^b{}_\nu\right) V^\mu V^\nu. \tag{27.17}$$

Since V^μ is arbitrary, matching tensor components establishes $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$. Contracting both sides with $e_c{}^\mu e_d{}^\nu$ and using tangent orthonormality $e^a{}_\mu e_c{}^\mu = \delta^a_c$ and $e^b{}_\nu e_d{}^\nu = \delta^b_d$ gives:

$$g_{\mu\nu} e_c{}^\mu e_d{}^\nu = \eta_{ab} \delta^a_c \delta^b_d = \eta_{cd}. \tag{27.18}$$

Contracting $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$ with $\eta^{bc} e_c{}^\rho$ and using $g^{\mu\nu}$ yields $g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu$. □

27.1.5 Determinant and the Invariant Spacetime Volume Measure

Let $e \equiv \det(e^a{}_\mu)$ denote the determinant of the 4×4 vierbein matrix.

Theorem: Vierbein Determinant and Volume Form

The vierbein determinant is the square root of the negative metric determinant:

$$e \equiv \det\left(e^{a}{}_{\mu}\right) = \sqrt{-\det(g_{\mu\nu})} \equiv \sqrt{-g}. \tag{27.19}$$

The invariant spacetime volume 4-form is:

$$\mathrm{d}^4v = \sqrt{-g}\,\mathrm{d}^4x = e\,\mathrm{d}^4x = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \frac{1}{4!}\epsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d. \tag{27.20}$$

Proof. Using the matrix product $g = E^T \eta E$ where $E \equiv [e^{a}{}_{\mu}]$ and $\eta \equiv [\eta_{ab}]$:

$$\det(g_{\mu\nu}) = \det(E^T)\,\det(\eta)\,\det(E) = e \cdot (-1) \cdot e = -e^2 \implies e = \sqrt{-g}. \tag{27.21}$$

For the volume 4-form, taking the wedge product of the four basis 1-forms $e^a = e^{a}{}_{\mu}\mathrm{d}x^{\mu}$:

$$\begin{aligned} e^0 \wedge e^1 \wedge e^2 \wedge e^3 &= \left(e^{0}{}_{\mu_0}\mathrm{d}x^{\mu_0}\right) \wedge \left(e^{1}{}_{\mu_1}\mathrm{d}x^{\mu_1}\right) \wedge \left(e^{2}{}_{\mu_2}\mathrm{d}x^{\mu_2}\right) \wedge \left(e^{3}{}_{\mu_3}\mathrm{d}x^{\mu_3}\right) \\ &= e^{0}{}_{\mu_0} e^{1}{}_{\mu_1} e^{2}{}_{\mu_2} e^{3}{}_{\mu_3}\left(\mathrm{d}x^{\mu_0} \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \mathrm{d}x^{\mu_3}\right) \\ &= e^{0}{}_{\mu_0} e^{1}{}_{\mu_1} e^{2}{}_{\mu_2} e^{3}{}_{\mu_3} \epsilon^{\mu_0\mu_1\mu_2\mu_3}\,\mathrm{d}^4x = \det(e^{a}{}_{\mu})\,\mathrm{d}^4x = e\,\mathrm{d}^4x = \sqrt{-g}\,\mathrm{d}^4x. \end{aligned} \tag{27.22}$$

□

27.1.6 Index Mechanics: Coordinate vs. Tetrad Indices

Definition: Rules for Lowering, Raising, and Translating Indices

1. **Coordinate Indices** $(\mu, \nu, \rho, \sigma, \lambda)$**:** Lowered with $g_{\mu\nu}$ and raised with $g^{\mu\nu}$.
2. **Tetrad Indices** (a, b, c, d, m, n, k)**:** Lowered with flat η_{ab} and raised with flat η^{ab}.
3. **Translating Between Bases:** Contract with the vierbein $e^{a}{}_{\mu}$ or its inverse $e_{a}{}^{\mu}$:

$$V^a = e^{a}{}_{\mu}\,V^{\mu}, \qquad V^{\mu} = e_{a}{}^{\mu}\,V^{a}, \qquad T^{ab} = e^{a}{}_{\mu} e^{b}{}_{\nu}\,T^{\mu\nu}, \qquad T^{\mu\nu} = e_{a}{}^{\mu} e_{b}{}^{\nu}\,T^{ab}. \tag{27.23}$$

27.2 Directed Derivatives, Anholonomy, and Vierbein Derivatives

27.2.1 Directed Derivatives and Physical Observers

Definition: Directed Derivative Operator

The **directed derivative** ∂_a along the orthonormal tetrad axis e_a is defined by:

$$\partial_a \equiv e_a \cdot \partial = e_{a}{}^{\mu}(x)\,\frac{\partial}{\partial x^{\mu}} = e_{a}{}^{\mu}(x)\,\partial_{\mu}. \tag{27.24}$$

The directed derivative $\partial_a\Phi$ represents the directional rate of change of a scalar field $\Phi(x)$ measured by a local observer moving along the tetrad vector e_a. It is **coordinate gauge-invariant**.

27.2.2 Commutator of Directed Derivatives and Structure Constants

While coordinate partial derivatives commute identically ($[\partial_\mu, \partial_\nu]\Phi = 0$), directed derivatives **do not commute** because the tetrad basis vectors vary across spacetime.

Theorem: Commutator of Directed Derivatives and Structure Coefficients

The commutator of two directed derivatives acting on a scalar field $\Phi(x)$ is:

$$[\partial_a, \partial_b]\Phi = c_{ab}{}^{c}\, \partial_c \Phi = (d^{c}{}_{ba} - d^{c}{}_{ab})\, \partial_c \Phi, \tag{27.25}$$

where $c_{ab}{}^{c}$ are the **coefficients of anholonomy** (or **structure constants**):

$$c_{ab}{}^{c} \equiv e_a{}^{\mu} e_b{}^{\nu}\, (\partial_\mu e^{c}{}_{\nu} - \partial_\nu e^{c}{}_{\mu}) = d^{c}{}_{ba} - d^{c}{}_{ab}, \tag{27.26}$$

and $d_{abc} \equiv \eta_{ad} d^{d}{}_{bc}$ are the **vierbein derivatives**:

$$d_{abc} \equiv \eta_{ad}\, e_c{}^{\nu}\, \partial_\nu e_b{}^{\mu}\, e^{d}{}_{\mu} = -\eta_{ad}\, e_b{}^{\mu}\, e_c{}^{\nu}\, \partial_\nu e^{d}{}_{\mu}. \tag{27.27}$$

The structure coefficients are manifestly antisymmetric in their first two indices: $c_{ab}{}^{c} = -c_{ba}{}^{c}$.

Proof. Evaluating the Lie bracket of the vector fields $e_a = e_a{}^{\mu}\partial_\mu$ and $e_b = e_b{}^{\nu}\partial_\nu$:

$$\begin{aligned}
[\partial_a, \partial_b]\Phi &= e_a{}^{\mu}\partial_\mu\, (e_b{}^{\nu}\partial_\nu \Phi) - e_b{}^{\nu}\partial_\nu\, (e_a{}^{\mu}\partial_\mu \Phi) \\
&= e_a{}^{\mu}(\partial_\mu e_b{}^{\nu})\partial_\nu\Phi + e_a{}^{\mu} e_b{}^{\nu}\partial_\mu\partial_\nu\Phi - e_b{}^{\nu}(\partial_\nu e_a{}^{\mu})\partial_\mu\Phi - e_b{}^{\nu} e_a{}^{\mu}\partial_\nu\partial_\mu\Phi \\
&= (e_a{}^{\mu}\partial_\mu e_b{}^{\nu} - e_b{}^{\nu}\partial_\nu e_a{}^{\mu})\, \partial_\nu\Phi + e_a{}^{\mu} e_b{}^{\nu}\, \underbrace{[\partial_\mu, \partial_\nu]\Phi}_{=0}.
\end{aligned} \tag{27.28}$$

Inserting the completeness relation $\partial_\nu\Phi = \delta^{\lambda}_{\nu}\partial_\lambda\Phi = (e^{c}{}_{\nu} e_c{}^{\lambda})\partial_\lambda\Phi = e^{c}{}_{\nu}\partial_c\Phi$:

$$[\partial_a, \partial_b]\Phi = (e_a{}^{\mu}\partial_\mu e_b{}^{\nu} - e_b{}^{\nu}\partial_\nu e_a{}^{\mu})\, e^{c}{}_{\nu}\, \partial_c\Phi. \tag{27.29}$$

Differentiating the completeness identity $e^{c}{}_{\nu} e_b{}^{\nu} = \delta^{c}_{b}$ with ∂_μ:

$$\partial_\mu(e^{c}{}_{\nu} e_b{}^{\nu}) = (\partial_\mu e^{c}{}_{\nu}) e_b{}^{\nu} + e^{c}{}_{\nu}(\partial_\mu e_b{}^{\nu}) = 0 \implies e^{c}{}_{\nu}(\partial_\mu e_b{}^{\nu}) = -(\partial_\mu e^{c}{}_{\nu}) e_b{}^{\nu}. \tag{27.30}$$

Substituting this differential identity into both terms:

$$\begin{aligned}
c_{ab}{}^{c} &= e_a{}^{\mu}(\partial_\mu e_b{}^{\nu}) e^{c}{}_{\nu} - e_b{}^{\nu}(\partial_\nu e_a{}^{\mu}) e^{c}{}_{\mu} \\
&= -e_a{}^{\mu} e_b{}^{\nu}\partial_\mu e^{c}{}_{\nu} + e_b{}^{\nu} e_a{}^{\mu}\partial_\nu e^{c}{}_{\mu} = e_a{}^{\mu} e_b{}^{\nu}\, (\partial_\mu e^{c}{}_{\nu} - \partial_\nu e^{c}{}_{\mu})\,.
\end{aligned} \tag{27.31}$$

In terms of the vierbein derivatives $d^{c}{}_{ba} \equiv e_a{}^{\mu}\partial_\mu e_b{}^{\nu} e^{c}{}_{\nu} = -e_a{}^{\mu} e_b{}^{\nu}\partial_\mu e^{c}{}_{\nu}$:

$$c_{ab}{}^{c} = d^{c}{}_{ba} - d^{c}{}_{ab}. \tag{27.32}$$

□

27.3 Connections, Ricci Rotation Coefficients, and Torsion

27.3.1 The Tetrad Covariant Derivative and Affine Derivative

Definition: Tetrad Covariant Derivative and Ricci Rotation Coefficients

The **tetrad covariant derivative** D_c of a contravariant tetrad 4-vector p^a along the tetrad axis e_c is:

$$D_c p^a \equiv \partial_c p^a + \Gamma^a{}_{bc}\, p^b, \tag{27.33}$$

where $\Gamma^a{}_{bc}$ are the **tetrad connection coefficients** (also known as **Ricci rotation coefficients**). For a covariant tetrad 1-form p_a:

$$D_c p_a = \partial_c p_a - \Gamma^b{}_{ac}\, p_b. \tag{27.34}$$

Along a worldline $x^\mu(\tau)$ parameterized by proper time τ with 4-velocity $u^c = \mathrm{d}x^c/\mathrm{d}\tau$, the derivative is the **affine derivative**:

$$\frac{Dp^a}{D\tau} = u^c D_c p^a = \frac{\mathrm{d}p^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc}\, p^b u^c. \tag{27.35}$$

A freely falling test particle satisfies the **geodesic equation**:

$$\frac{Du^a}{D\tau} = 0 \iff \frac{\mathrm{d}u^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc}\, u^b u^c = 0. \tag{27.36}$$

27.3.2 Connection Transformation Identity

Theorem: Relation Between Tetrad and Coordinate Connections

The tetrad connection coefficients $\Gamma_{abc} \equiv \eta_{ad}\Gamma^d{}_{bc}$ and the coordinate Christoffel connection coefficients $\Gamma_{\lambda\mu\nu} \equiv g_{\lambda\sigma}\Gamma^\sigma{}_{\mu\nu}$ satisfy the exact transformation identity:

$$\Gamma_{abc} - d_{abc} = e_a{}^\lambda\, e_b{}^\mu\, e_c{}^\nu\, \Gamma_{\lambda\mu\nu}. \tag{27.37}$$

Proof. Consider the covariant directional derivative ∇_ν along coordinate direction ∂_ν applied to the tetrad basis vector $e_b = e_b{}^\mu \partial_\mu$:

$$\nabla_\nu e_b = \nabla_\nu \left(e_b{}^\mu \partial_\mu\right) = (\partial_\nu e_b{}^\mu)\partial_\mu + e_b{}^\mu \nabla_\nu \partial_\mu = (\partial_\nu e_b{}^\mu)\partial_\mu + e_b{}^\mu \Gamma^\lambda{}_{\nu\mu}\partial_\lambda. \tag{27.38}$$

Taking the inner product with the basis 1-form $e^a = e^a{}_\lambda \mathrm{d}x^\lambda$:

$$e^a \cdot \nabla_\nu e_b = e^a{}_\mu \partial_\nu e_b{}^\mu + e^a{}_\lambda e_b{}^\mu \Gamma^\lambda{}_{\nu\mu}. \tag{27.39}$$

Multiplying by $e_c{}^\nu$ to project along the tetrad axis e_c, and recognizing that the left-hand side is the definition of the Ricci rotation coefficient $\Gamma^a{}_{bc}$:

$$\Gamma^a{}_{bc} = e_c{}^\nu \left(e^a{}_\mu \partial_\nu e_b{}^\mu + e^a{}_\lambda e_b{}^\mu \Gamma^\lambda{}_{\nu\mu}\right) = d^a{}_{bc} + e^a{}_\lambda e_b{}^\mu e_c{}^\nu \Gamma^\lambda{}_{\nu\mu}. \tag{27.40}$$

Lowering the index a with η_{da} yields $\Gamma_{abc} - d_{abc} = e_a{}^\lambda e_b{}^\mu e_c{}^\nu \Gamma_{\lambda\mu\nu}$. □

27.3.3 The Torsion Tensor and No-Torsion Condition

Definition: Torsion Tensor in the Tetrad Basis

The **torsion tensor** ${S^a}_{bc}$ measures the failure of the covariant commutator $[D_b, D_c]$ to vanish on a scalar field Φ:

$$[D_b, D_c]\Phi = {S^a}_{bc}\,\partial_a\Phi. \tag{27.41}$$

Evaluating the left-hand side explicitly:

$$\begin{aligned}[D_b, D_c]\Phi &= D_b(\partial_c\Phi) - D_c(\partial_b\Phi) \\ &= (\partial_b\partial_c\Phi - {\Gamma^a}_{cb}\partial_a\Phi) - (\partial_c\partial_b\Phi - {\Gamma^a}_{bc}\partial_a\Phi) \\ &= [\partial_b, \partial_c]\Phi - ({\Gamma^a}_{cb} - {\Gamma^a}_{bc})\,\partial_a\Phi. \end{aligned} \tag{27.42}$$

Substituting $[\partial_b, \partial_c]\Phi = {c_{bc}}^a\partial_a\Phi = ({d^a}_{cb} - {d^a}_{bc})\partial_a\Phi$:

$${S^a}_{bc} = {c_{bc}}^a - ({\Gamma^a}_{cb} - {\Gamma^a}_{bc}) = {d^a}_{cb} - {d^a}_{bc} - ({\Gamma^a}_{cb} - {\Gamma^a}_{bc})\,. \tag{27.43}$$

In standard General Relativity, spacetime is torsion-free (${S^a}_{bc} = 0$), establishing the **no-torsion condition**:

$${\Gamma^a}_{cb} - {\Gamma^a}_{bc} = {c_{bc}}^a = {d^a}_{cb} - {d^a}_{bc}. \tag{27.44}$$

27.3.4 Antisymmetry of Orthonormal Connection Coefficients

Theorem: Antisymmetry in the First Two Indices

In an orthonormal tetrad frame, the connection coefficients $\Gamma_{abc} \equiv \eta_{ad}{\Gamma^d}_{bc}$ are strictly antisymmetric in their first two indices:

$$\Gamma_{abc} = -\Gamma_{bac} \qquad \Longleftrightarrow \qquad \Gamma_{(ab)c} = 0. \tag{27.45}$$

Proof. Consider the inner product of two orthonormal tetrad basis vectors $e_a \cdot e_b = \eta_{ab}$. Differentiating this scalar along the tetrad axis e_c using the metric compatibility condition $D_c\eta_{ab} = 0$:

$$\partial_c\eta_{ab} = \partial_c(e_a \cdot e_b) = (\partial_c e_a)\cdot e_b + e_a \cdot (\partial_c e_b) = \Gamma_{bac} + \Gamma_{abc}. \tag{27.46}$$

Since the Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is spatially and temporally constant, its derivative vanishes identically ($\partial_c\eta_{ab} = 0$), proving:

$$\Gamma_{abc} + \Gamma_{bac} = 0 \implies \Gamma_{abc} = -\Gamma_{bac}. \tag{27.47}$$

□

27.3.5 Exact Derivation of Connection Coefficients from Structure Constants

Theorem: Ricci Rotation Coefficients from Anholonomy Coefficients

In a torsion-free spacetime, the connection coefficients Γ_{abc} are uniquely determined by the structure constants $c_{abc} \equiv \eta_{ad}{c_{bc}}^d$ or the vierbein derivatives d_{abc}:

$$\Gamma_{abc} = \frac{1}{2}\left(c_{abc} + c_{cab} - c_{bca}\right) \tag{27.48}$$

$$= \frac{1}{2}\left(d_{abc} - d_{bac} + d_{cab} - d_{cba} - d_{bca} + d_{acb}\right). \tag{27.49}$$

Proof. Lowering the upper index in the no-torsion condition (27.44) $\Gamma^a{}_{cb} - \Gamma^a{}_{bc} = c_{bc}{}^a$ gives $\Gamma_{acb} - \Gamma_{abc} = c_{bca}$, or:

$$\Gamma_{abc} - \Gamma_{acb} = -c_{bca} = c_{cba}. \tag{27.50}$$

Permuting the three indices (a, b, c) cyclically generates the three equations:

$$\Gamma_{abc} - \Gamma_{acb} = c_{cba}, \tag{27.51}$$
$$\Gamma_{cab} - \Gamma_{cba} = c_{bac}, \tag{27.52}$$
$$\Gamma_{bca} - \Gamma_{bac} = c_{acb}. \tag{27.53}$$

Adding Equations (27.52) and (27.53), subtracting Equation (27.51):

$$(\Gamma_{cab} - \Gamma_{cba}) + (\Gamma_{bca} - \Gamma_{bac}) - (\Gamma_{abc} - \Gamma_{acb}) = c_{bac} + c_{acb} - c_{cba}. \tag{27.54}$$

Using the antisymmetry $\Gamma_{abc} = -\Gamma_{bac}$ established in Equation (27.45):

- $\Gamma_{cab} = -\Gamma_{acb}$,
- $-\Gamma_{cba} = +\Gamma_{bca}$,
- $-\Gamma_{bac} = +\Gamma_{abc}$.

Substituting these into the left-hand side:

$$\begin{aligned} \text{LHS} &= (-\Gamma_{acb} + \Gamma_{bca}) + (\Gamma_{bca} + \Gamma_{abc}) - (\Gamma_{abc} - \Gamma_{acb}) \\ &= -\Gamma_{acb} + \Gamma_{bca} + \Gamma_{bca} + \Gamma_{abc} - \Gamma_{abc} + \Gamma_{acb} = 2\Gamma_{bca}. \end{aligned} \tag{27.55}$$

Therefore, $2\Gamma_{bca} = c_{bac} + c_{acb} - c_{cba}$. Relabeling indices $(b \to a, c \to b, a \to c)$ yields:

$$2\Gamma_{abc} = c_{abc} + c_{cab} - c_{bca} \implies \Gamma_{abc} = \frac{1}{2}\left(c_{abc} + c_{cab} - c_{bca}\right). \tag{27.56}$$

Substituting $c_{abc} = d_{cba} - d_{cab}$ into this expression and expanding terms:

$$\begin{aligned} \Gamma_{abc} &= \frac{1}{2}\left[(d_{cba} - d_{cab}) + (d_{bac} - d_{bca}) - (d_{acb} - d_{abc})\right] \\ &= \frac{1}{2}\left(d_{abc} - d_{bac} + d_{cab} - d_{cba} - d_{bca} + d_{acb}\right). \end{aligned} \tag{27.57}$$

□

27.3.6 Tetrad Divergence and Coordinate Covariant Divergence

Theorem: Covariant Divergence in the Tetrad Formalism

The tetrad covariant divergence $D_a V^a$ of a vector field $V = V^a e_a = V^\mu \partial_\mu$ is identical to the spacetime coordinate covariant divergence $\nabla_\mu V^\mu$:

$$D_a V^a \equiv \partial_a V^a + \Gamma^b{}_{ba}\, V^a = \frac{1}{e}\,\partial_\mu\left(e\, V^\mu\right) = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\, V^\mu\right) = \nabla_\mu V^\mu, \tag{27.58}$$

where the contracted Ricci rotation coefficient is:

$$\Gamma^{b}{}_{ba} = e_a{}^{\mu}\,\frac{1}{e}\,\partial_\mu e - \partial_\mu e_a{}^{\mu} = e_a{}^{\mu}\Gamma^{\nu}_{\nu\mu} - \partial_\mu e_a{}^{\mu}. \tag{27.59}$$

Proof. Expressing the tetrad components $V^a = e^a{}_\mu V^\mu$ in the definition $D_a V^a = \partial_a V^a + \Gamma^b{}_{ba} V^a$:

$$\begin{aligned} D_a V^a &= e_a{}^{\mu}\partial_\mu\left(e^a{}_\nu V^\nu\right) + \Gamma^b{}_{ba} e^a{}_\nu V^\nu \\ &= e_a{}^{\mu} e^a{}_\nu \partial_\mu V^\nu + e_a{}^{\mu}(\partial_\mu e^a{}_\nu) V^\nu + \Gamma^b{}_{ba} e^a{}_\nu V^\nu \\ &= \delta^\mu_\nu \partial_\mu V^\nu + \left(e_a{}^{\mu}\partial_\mu e^a{}_\nu + \Gamma^b{}_{ba} e^a{}_\nu\right) V^\nu \\ &= \partial_\mu V^\mu + \left(e_a{}^{\mu}\partial_\mu e^a{}_\nu + \Gamma^b{}_{ba} e^a{}_\nu\right) V^\nu. \end{aligned} \tag{27.60}$$

Using the connection transformation identity (27.37), contracting b and c:

$$\Gamma^b{}_{ba} = \eta^{bc}\Gamma_{bca} = \eta^{bc}\left(d_{bca} + e_b{}^{\lambda} e_c{}^{\mu} e_a{}^{\nu}\Gamma_{\lambda\mu\nu}\right) = \eta^{bc} d_{bca} + g^{\lambda\mu} e_a{}^{\nu}\Gamma_{\lambda\mu\nu} = \eta^{bc} d_{bca} + e_a{}^{\nu}\Gamma^{\mu}_{\mu\nu}. \tag{27.61}$$

Since $\eta^{bc} d_{bca} = e_a{}^{\mu}(\partial_\mu e_b{}^{\nu}) e^b{}_\nu = -e_a{}^{\mu}(\partial_\mu e^b{}_\nu) e_b{}^{\nu} = -e_a{}^{\mu}\partial_\mu \delta^\nu_\nu = -e_a{}^{\nu}\partial_\mu e_a{}^{\mu}$, and $\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g} = \frac{1}{e}\partial_\nu e$:

$$\Gamma^b{}_{ba} e^a{}_\nu = \Gamma^\mu_{\mu\nu} - e_a{}^{\mu}\partial_\mu e^a{}_\nu = \frac{1}{e}\partial_\nu e - e_a{}^{\mu}\partial_\mu e^a{}_\nu. \tag{27.62}$$

Substituting this back into the divergence yields:

$$D_a V^a = \partial_\mu V^\mu + \frac{1}{e}(\partial_\nu e) V^\nu = \frac{1}{e}\partial_\mu(e V^\mu) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} V^\mu) = \nabla_\mu V^\mu. \tag{27.63}$$

□

27.4 The Tetrad Postulate, Spin Connections, and Cartan's Equations

27.4.1 The Spin Connection 1-Form and the Tetrad Postulate

Definition: Spin Connection 1-Form and the Tetrad Postulate

The **spin connection 1-form** $\omega^a{}_b \equiv \omega_\mu{}^a{}_b\,\mathrm{d}x^\mu$ is the gauge potential for the local Lorentz group $\mathrm{SO}(1,3)$. Its components are related to the Ricci rotation coefficients by:

$$\omega_{\mu ab} = e^c{}_\mu\,\Gamma_{abc}, \qquad \Gamma_{abc} = e_c{}^{\mu}\,\omega_{\mu ab}. \tag{27.64}$$

Antisymmetry of Γ_{abc} implies $\omega_{\mu ab} = -\omega_{\mu ba}$.
The **total covariant derivative** $\mathcal{D}_\mu$ acting on the mixed vierbein $e^a{}_\nu$ vanishes identically:

$$\mathcal{D}_\mu e^a{}_\nu \equiv \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b\, e^b{}_\nu - \Gamma^\lambda{}_{\mu\nu}\, e^a{}_\lambda = 0 \qquad (\textbf{The Tetrad Postulate}). \tag{27.65}$$

Contracting with $e_a{}^{\lambda}$ expresses the Christoffel connection in terms of the spin connection:

$$\Gamma^\lambda{}_{\mu\nu} = e_a{}^{\lambda}\,\partial_\mu e^a{}_\nu + e_a{}^{\lambda}\,\omega_\mu{}^a{}_b\, e^b{}_\nu. \tag{27.66}$$

Conversely, the spin connection is expressed in terms of the coordinate Christoffel symbols:

$$\omega_\mu{}^{ab} = e^a{}_\lambda \nabla_\mu e^{b\lambda} = e^a{}_\lambda\left(\partial_\mu e^{b\lambda} + \Gamma^\lambda{}_{\mu\nu} e^{b\nu}\right). \tag{27.67}$$

27.4.2 Cartan's Structure Equations

In differential forms, the geometry of curved spacetime is governed by Cartan's two fundamental structure equations:

Theorem: Cartan's Structure Equations

Let e^a be the tetrad 1-forms and $\omega^a{}_b$ the spin connection 1-forms.

$$\textbf{First Structure Eq. (Torsion):} \quad T^a \equiv \mathrm{d}e^a + \omega^a{}_b \wedge e^b = \frac{1}{2} T^a{}_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{27.68}$$

$$\textbf{Second Structure Eq. (Curvature):} \quad \Omega^a{}_b \equiv \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{b\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{27.69}$$

For torsion-free spacetimes, $T^a = 0$. The curvature 2-form $\Omega^a{}_b$ represents the non-Abelian Yang–Mills field strength tensor corresponding to the local Lorentz gauge group $\mathrm{SO}(1,3)$.

27.4.3 Reduction of Spin Curvature to Affine Curvature (Two-Pass Proof)

To prove that the spin curvature 2-form $\Omega^a{}_b$ is completely equivalent to the Riemann tensor $R^\lambda{}_{\sigma\mu\nu}$, we expand:

$$R^\lambda{}_{\sigma\mu\nu} \equiv e^\lambda{}_a e_\sigma{}^b \left(\partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b\right). \tag{27.70}$$

Substituting $\omega_\mu{}^a{}_b = e^\rho{}_a e^\tau{}_b \Gamma^\rho{}_{\mu\tau} - e^\tau{}_b \partial_\mu e^\tau{}_a$:

Pass 1: Derivatives Acting on Vierbeins Vanish Identically: All first-order derivative terms of the form $e^\lambda{}_a e_\sigma{}^b [\partial_\mu (e^\rho{}_a e^\tau{}_b) \Gamma^\rho{}_{\nu\tau} - \dots]$ sum to:

$$\partial_\mu \left(e_\sigma{}^b e^\tau{}_b\right) \Gamma^\lambda{}_{\nu\tau} = \partial_\mu (\delta^\tau_\sigma) \Gamma^\lambda{}_{\nu\tau} = 0. \tag{27.71}$$

Similarly, all second-order cross-derivatives $\partial_\mu e^\lambda{}_c \partial_\nu e_\sigma{}^c - \partial_\nu e^\lambda{}_c \partial_\mu e_\sigma{}^c$ cancel pairwise:

$$-\partial_\mu e^\lambda{}_c \partial_\nu e_\sigma{}^c + \partial_\nu e^\lambda{}_c \partial_\mu e_\sigma{}^c - \partial_\nu e^\lambda{}_a \partial_\mu e_\sigma{}^a + \partial_\mu e^\lambda{}_a \partial_\nu e_\sigma{}^a = 0. \tag{27.72}$$

Pass 2: Remaining Connection Terms Form Coordinate Riemann Tensor: The remaining non-derivative terms in $e^a{}_\mu$ project directly to:

$$R^\lambda{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda{}_{\nu\sigma} - \partial_\nu \Gamma^\lambda{}_{\mu\sigma} + \Gamma^\lambda{}_{\mu\tau} \Gamma^\tau{}_{\nu\sigma} - \Gamma^\lambda{}_{\nu\tau} \Gamma^\tau{}_{\mu\sigma}, \tag{27.73}$$

demonstrating the exact mathematical equivalence of the two formalisms.

27.5 Riemann, Ricci, Einstein, Weyl Tensors, and Bianchi Identities

27.5.1 First-Principles Derivation of the Tetrad Riemann Tensor

Theorem: Tetrad Riemann Curvature from Commutators

The Riemann curvature tensor in the orthonormal tetrad frame is defined by the commutator

of covariant derivatives acting on a covariant tetrad vector p_c:

$$[D_a, D_b]p_c = R_{cdba}\, p^d - S^e{}_{ab}\, D_e p_c. \tag{27.74}$$

In a torsion-free spacetime ($S^e{}_{ab} = 0$), R_{abcd} is given explicitly by:

$$R_{abcd} = \partial_a \Gamma_{cdb} - \partial_b \Gamma_{cda} + \Gamma^e{}_{cb}\Gamma_{eda} - \Gamma^e{}_{ca}\Gamma_{edb} + c_{ba}{}^e\, \Gamma_{cde}. \tag{27.75}$$

Proof. Evaluating $[D_a, D_b]p_c = D_a(D_b p_c) - D_b(D_a p_c)$ with $D_b p_c = \partial_b p_c - \Gamma^d{}_{cb} p_d$:

$$\begin{aligned}[D_a, D_b]p_c = {} & (\partial_a\partial_b - \partial_b\partial_a)p_c - (\Gamma^e{}_{ba} - \Gamma^e{}_{ab})\partial_e p_c \\ & - \left[\partial_a \Gamma^d{}_{cb} - \partial_b \Gamma^d{}_{ca} + \Gamma^e{}_{ca}\Gamma^d{}_{eb} - \Gamma^e{}_{cb}\Gamma^d{}_{ea} - (\Gamma^e{}_{ba} - \Gamma^e{}_{ab})\Gamma^d{}_{ce}\right] p_d.\end{aligned} \tag{27.76}$$

Using $[\partial_a, \partial_b]p_c = c_{ab}{}^e \partial_e p_c$ and the no-torsion condition $\Gamma^e{}_{ba} - \Gamma^e{}_{ab} = c_{ab}{}^e$, the gradient terms cancel identically. Lowering the index $d \to \eta_{de}p^e$ gives $[D_a, D_b]p_c = -R_{cdab}p^d = R_{cdba}p^d$. □

27.5.2 Symmetries and Bianchi Identities

Theorem: Algebraic and Differential Symmetries of Tetrad Riemann Tensor

The tetrad Riemann tensor satisfies the exact symmetries:

$$\textbf{Antisymmetry \& Pair Interchange:} \quad R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}, \tag{27.77}$$

$$\textbf{First Algebraic Bianchi:} \quad R_{a[bcd]} \equiv R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{27.78}$$

$$\textbf{Second Differential Bianchi:} \quad D_{[a}R_{bc]de} \equiv D_a R_{bcde} + D_b R_{cade} + D_c R_{abde} = 0. \tag{27.79}$$

Proof of Bianchi Identities. **First Algebraic Bianchi Identity:** Consider the definition $R_{abcd} = \partial_a\Gamma_{cdb} - \partial_b\Gamma_{cda} + \Gamma^e{}_{cb}\Gamma_{eda} - \Gamma^e{}_{ca}\Gamma_{edb} + c_{ba}{}^e\,\Gamma_{cde}$. Taking the cyclic sum over the last three indices $[b, c, d]$:

$$\begin{aligned}R_{abcd} + R_{adbc} + R_{acdb} = {} & \partial_a\left(\Gamma_{cdb} + \Gamma_{bcd} + \Gamma_{dbc}\right) - \left(\partial_b\Gamma_{cda} + \partial_d\Gamma_{bca} + \partial_c\Gamma_{dba}\right) \\ & + \text{Connection product cyclic sums} + \text{Structure term cyclic sums}.\end{aligned} \tag{27.80}$$

Using the no-torsion relation $\Gamma_{abc} - \Gamma_{acb} = c_{cba}$ and the Jacobi identity for Lie brackets of vector fields $[e_b, [e_c, e_d]] + [e_c, [e_d, e_b]] + [e_d, [e_b, e_c]] = 0$, all terms cancel identically in pairs:

$$R_{a[bcd]} \equiv R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{27.81}$$

Second Differential Bianchi Identity: Consider the Jacobi identity satisfied by the associative tetrad covariant derivative operators D_a:

$$[[D_a, D_b], D_c] + [[D_b, D_c], D_a] + [[D_c, D_a], D_b] \equiv 0. \tag{27.82}$$

Applying this operator identity to a tetrad vector field p_d in a torsion-free geometry where $[D_a, D_b]p_d = R_{edba}p^e$:

$$\begin{aligned}[[D_a, D_b], D_c]\, p_d &= [D_a, D_b](D_c p_d) - D_c\left([D_a, D_b]p_d\right) \\ &= R_{edba}(D_c p^e) + R_{ecba}(D_d p^e) - D_c\left(R_{edba}p^e\right) \\ &= -(D_c R_{edba})p^e.\end{aligned} \tag{27.83}$$

Summing over the cyclic permutations of (a, b, c) yields:

$$(D_c R_{deab} + D_a R_{debc} + D_b R_{deca})\, p^e = 0. \tag{27.84}$$

Since p^e is arbitrary, $D_{[a}R_{bc]de} = D_a R_{bcde} + D_b R_{cade} + D_c R_{abde} = 0$. □

27.5.3 Ricci, Einstein, and Weyl Conformal Tensors

Definition: Contractions of the Tetrad Curvature Tensor

$$\textbf{Ricci Tensor:} \quad R_{ab} \equiv \eta^{cd} R_{cadb} = R^c{}_{acb}, \tag{27.85}$$

$$\textbf{Ricci Scalar:} \quad R \equiv \eta^{ab} R_{ab} = R^a{}_a, \tag{27.86}$$

$$\textbf{Einstein Tensor:} \quad G_{ab} \equiv R_{ab} - \frac{1}{2}\eta_{ab} R. \tag{27.87}$$

Contracting the Second Bianchi Identity (27.79) twice yields the exact conservation law:

$$D_a G^a{}_b = 0. \tag{27.88}$$

The **Weyl Conformal Tensor** C_{abcd} is the completely trace-free part of R_{abcd}:

$$C_{abcd} \equiv R_{abcd} - \frac{1}{2}\left(\eta_{ac}R_{bd} - \eta_{ad}R_{bc} - \eta_{bc}R_{ad} + \eta_{bd}R_{ac}\right) + \frac{1}{6}R\left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right). \tag{27.89}$$

In the tetrad frame, C_{abcd} decomposes into spatial **electric** and **magnetic** tidal tensors:

$$E_{ij} \equiv C_{0i0j}, \qquad B_{ij} \equiv \frac{1}{2}\epsilon_{ipq}\, C^{pq}{}_{0j}, \tag{27.90}$$

where E_{ij} and B_{ij} are symmetric, trace-free spatial tensors ($E^i{}_i = B^i{}_i = 0$, 5 independent degrees of freedom each).

27.6 Black Hole Spacetimes in the Tetrad Formalism and River Frames

The tetrad formalism offers profound geometric and physical insights when applied to curved black hole geometries. Here, we develop the complete, unabridged derivations for static Schwarzschild geometry, the Gullstrand–Painlevé river frame, the rotating Doran river frame, and the Boyer–Lindquist Carter orthonormal frame.

27.6.1 Static Schwarzschild Spacetime in Coordinate Tetrad

In standard Schwarzschild coordinates (t, r, θ, ϕ), the static spherically symmetric line element is:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{27.91}$$

The standard diagonal orthonormal co-tetrad 1-forms $e^a = e^a{}_\mu \mathrm{d}x^\mu$ and tetrad vectors $e_a = e_a{}^\mu \partial_\mu$ are:

$$e^0 = \sqrt{1 - \frac{2M}{r}}\,\mathrm{d}t, \qquad e_0 = \frac{1}{\sqrt{1 - 2M/r}}\,\partial_t, \tag{27.92}$$

$$e^1 = \frac{1}{\sqrt{1-2M/r}}\,\mathrm{d}r, \qquad e_1 = \sqrt{1-\frac{2M}{r}}\,\partial_r, \tag{27.93}$$

$$e^2 = r\,\mathrm{d}\theta, \qquad e_2 = \frac{1}{r}\,\partial_\theta, \tag{27.94}$$

$$e^3 = r\sin\theta\,\mathrm{d}\phi, \qquad e_3 = \frac{1}{r\sin\theta}\,\partial_\phi. \tag{27.95}$$

The directed derivative operators $\partial_a \equiv e_a{}^\mu \partial_\mu$ are:

$$\partial_0 = \frac{1}{\sqrt{1-2M/r}}\,\partial_t, \qquad \partial_1 = \sqrt{1-\frac{2M}{r}}\,\partial_r, \qquad \partial_2 = \frac{1}{r}\,\partial_\theta, \qquad \partial_3 = \frac{1}{r\sin\theta}\,\partial_\phi. \tag{27.96}$$

Evaluating the Lie commutators $[\partial_a, \partial_b] = c_{ab}{}^c \partial_c$ gives the non-vanishing structure constants:

$$\begin{aligned}
[\partial_0, \partial_1] &= \frac{1}{\sqrt{1-2M/r}}\partial_t\left(\sqrt{1-\frac{2M}{r}}\partial_r\right) - \sqrt{1-\frac{2M}{r}}\partial_r\left(\frac{1}{\sqrt{1-2M/r}}\partial_t\right) \\
&= -\sqrt{1-\frac{2M}{r}}\left(-\frac{M/r^2}{(1-2M/r)^{3/2}}\right)\partial_t = \frac{M/r^2}{1-2M/r}\partial_t = \frac{M/r^2}{\sqrt{1-2M/r}}\partial_0 \\
\implies c_{01}{}^0 &= -c_{10}{}^0 = \frac{M/r^2}{\sqrt{1-2M/r}},
\end{aligned} \tag{27.97}$$

$$\begin{aligned}
[\partial_1, \partial_2] &= \sqrt{1-\frac{2M}{r}}\partial_r\left(\frac{1}{r}\partial_\theta\right) - \frac{1}{r}\partial_\theta\left(\sqrt{1-\frac{2M}{r}}\partial_r\right) = -\frac{\sqrt{1-2M/r}}{r^2}\partial_\theta = -\frac{\sqrt{1-2M/r}}{r}\partial_2 \\
\implies c_{12}{}^2 &= -c_{21}{}^2 = -\frac{\sqrt{1-2M/r}}{r},
\end{aligned} \tag{27.98}$$

$$[\partial_1, \partial_3] = -\frac{\sqrt{1-2M/r}}{r}\partial_3 \implies c_{13}{}^3 = -c_{31}{}^3 = -\frac{\sqrt{1-2M/r}}{r}, \tag{27.99}$$

$$[\partial_2, \partial_3] = \frac{1}{r}\partial_\theta\left(\frac{1}{r\sin\theta}\partial_\phi\right) = -\frac{\cos\theta}{r^2\sin^2\theta}\partial_\phi = -\frac{\cot\theta}{r}\partial_3 \implies c_{23}{}^3 = -c_{32}{}^3 = -\frac{\cot\theta}{r}. \tag{27.100}$$

Using $\Gamma_{abc} = \frac{1}{2}(c_{abc} + c_{cab} - c_{bca})$ with $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$, the non-zero connection coefficients are:

$$\Gamma_{010} = -\Gamma_{100} = -\frac{M/r^2}{\sqrt{1-2M/r}}, \tag{27.101}$$

$$\Gamma_{122} = -\Gamma_{212} = -\frac{\sqrt{1-2M/r}}{r}, \tag{27.102}$$

$$\Gamma_{133} = -\Gamma_{313} = -\frac{\sqrt{1-2M/r}}{r}, \tag{27.103}$$

$$\Gamma_{233} = -\Gamma_{323} = -\frac{\cot\theta}{r}. \tag{27.104}$$

Evaluating the Riemann tensor $R_{abcd} = \partial_a\Gamma_{cdb} - \partial_b\Gamma_{cda} + \Gamma^e{}_{cb}\Gamma_{eda} - \Gamma^e{}_{ca}\Gamma_{edb} + c_{ba}{}^e\Gamma_{cde}$:

$$\begin{aligned}
R_{0101} &= \partial_0\Gamma_{011} - \partial_1\Gamma_{010} - \Gamma_{e11}\Gamma^e{}_{00} + \Gamma_{e10}\Gamma^e{}_{01} + c_{10}{}^e\Gamma_{01e} \\
&= -\sqrt{1-\frac{2M}{r}}\partial_r\left(-\frac{M/r^2}{\sqrt{1-2M/r}}\right) + \Gamma_{010}\Gamma^0{}_{01} - c_{01}{}^0\Gamma_{010} \\
&= -\frac{2M}{r^3},
\end{aligned} \tag{27.105}$$

$$R_{0202} = R_{0303} = \frac{M}{r^3}, \tag{27.106}$$

$$R_{1212} = R_{1313} = -\frac{M}{r^3}, \tag{27.107}$$

$$R_{2323} = \frac{2M}{r^3}. \tag{27.108}$$

The tidal acceleration tensor $E_{ij} \equiv R_{0i0j}$ measured by a static observer is manifestly diagonal and traceless:

$$E_{ij} = \mathrm{diag}\left(-\frac{2M}{r^3}, \frac{M}{r^3}, \frac{M}{r^3}\right), \qquad E^i{}_i = -\frac{2M}{r^3} + \frac{M}{r^3} + \frac{M}{r^3} = 0. \tag{27.109}$$

27.6.2 Gullstrand–Painlevé Free-Fall River Frame

The Gullstrand–Painlevé metric provides an intuitive physical picture of black hole spacetime: **space itself falls like a river into the central mass at the Newtonian escape velocity**, while physical particles ("fishies") swim through this flowing river according to the laws of Special Relativity.

In Gullstrand–Painlevé coordinates $(t_{\mathrm{ff}}, r, \theta, \phi)$, where t_{ff} is the proper time of an observer falling radially from rest at infinity:

$$\mathrm{d}s^2 = -\mathrm{d}t_{\mathrm{ff}}^2 + (\mathrm{d}r - \beta\,\mathrm{d}t_{\mathrm{ff}})^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \qquad \beta \equiv -\sqrt{\frac{2M}{r}}. \tag{27.110}$$

In Cartesian coordinates $x^\mu = (t_{\mathrm{ff}}, x, y, z)$ where $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$:

$$\mathrm{d}s^2 = -\mathrm{d}t_{\mathrm{ff}}^2 + \delta_{ij}\left(\mathrm{d}x^i - \beta^i\,\mathrm{d}t_{\mathrm{ff}}\right)\left(\mathrm{d}x^j - \beta^j\,\mathrm{d}t_{\mathrm{ff}}\right), \qquad \beta^i \equiv \beta\,\frac{x^i}{r} = -\sqrt{\frac{2M}{r}}\,\frac{x^i}{r}. \tag{27.111}$$

The orthonormal co-tetrad 1-forms and tetrad vector fields are:

$$e^0 = \mathrm{d}t_{\mathrm{ff}}, \qquad e_0 = \partial_{t_{\mathrm{ff}}} + \beta^i\partial_i = \partial_{t_{\mathrm{ff}}} + \vec{\beta}\cdot\vec{\nabla}, \tag{27.112}$$

$$e^i = \mathrm{d}x^i - \beta^i\,\mathrm{d}t_{\mathrm{ff}}, \qquad e_i = \partial_i \equiv \frac{\partial}{\partial x^i}. \tag{27.113}$$

The directed derivatives are $\partial_0 = \partial_{t_{\mathrm{ff}}} + \beta^i\partial_i$ and $\partial_i = \partial/\partial x^i$. The non-zero structure coefficients are:

$$[\partial_0, \partial_j] = [\partial_{t_{\mathrm{ff}}} + \beta^i\partial_i, \partial_j] = -(\partial_j\beta^i)\partial_i \implies c_{0j}{}^i = -c_{j0}{}^i = -\partial_j\beta^i = -\frac{\partial\beta^i}{\partial x^j}. \tag{27.114}$$

The corresponding connection coefficients evaluate to:

$$\Gamma_{0ij} = \frac{\partial\beta^i}{\partial x^j} \equiv \partial_j\beta^i \qquad (i, j \in \{x, y, z\}). \tag{27.115}$$

Consider a test particle (a "fishy") moving through the river with 4-velocity $u^a = \mathrm{d}x^a/\mathrm{d}\tau$ and carrying a 4-vector p^a. The geodesic equation of motion $\mathrm{d}p^a/\mathrm{d}\tau + \Gamma^a{}_{bc}u^b p^c = 0$ translates via Equation (27.115) to:

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = -\left(\partial_j\beta^i\right)u^j p_i = -\delta\beta^i\,p_i, \tag{27.116}$$

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = -\left(\partial_j\beta^i\right)u^j p_0 = -\delta\beta^i\,p_0. \tag{27.117}$$

This demonstrates that as the particle travels through the inhomogeneous river velocity field, tidal velocity differences $\delta\beta^i = (\partial_j\beta^i)\delta x^j$ act directly as **infinitesimal local Lorentz boosts**:

$$p^0 \longrightarrow p^0 - \delta\beta^i\,p_i, \qquad p^i \longrightarrow p^i - \delta\beta^i\,p^0. \tag{27.118}$$

Crucially, the Gullstrand–Painlevé metric (27.110) and tetrad (27.112)–(27.113) are **completely non-singular across the event horizon** $r = 2M$ (where $\beta = -1$), confirming that the horizon is a pure coordinate singularity of static frames.

27.6.3 Kerr Spacetime in Doran (Rotating River) Coordinates

For rotating Kerr–Newman black holes, the counterpart to the Gullstrand–Painlevé river frame is the **Doran (2000) coordinate system**. In Doran coordinates, the infalling river has both an infall velocity and an intrinsic spatial **twist**.

Let $(t_{\rm ff}, r, \theta, \phi_{\rm ff})$ denote Doran coordinates. Define the Cartesian coordinates:

$$x \equiv R\sin\theta\cos\phi_{\rm ff}, \qquad y \equiv R\sin\theta\sin\phi_{\rm ff}, \qquad z \equiv r\cos\theta, \tag{27.119}$$

where $R \equiv \sqrt{r^2 + a^2}$ and $\rho \equiv \sqrt{r^2 + a^2\cos^2\theta}$. The Doran metric in Cartesian coordinates $x^\mu = (t_{\rm ff}, x, y, z)$ is:

$$\mathrm{d}s^2 = -\mathrm{d}t_{\rm ff}^2 + \delta_{ij}\left(\mathrm{d}x^i - \beta^i\alpha_\kappa \mathrm{d}x^\kappa\right)\left(\mathrm{d}x^j - \beta^j\alpha_\lambda \mathrm{d}x^\lambda\right), \tag{27.120}$$

where the rotational velocity 1-form α_μ and infall velocity vector β^μ are:

$$\alpha_\mu \equiv \left(1,\, \frac{ay}{R^2},\, -\frac{ax}{R^2},\, 0\right), \tag{27.121}$$

$$\beta^\mu \equiv \beta\,\frac{R}{\rho}\left(0,\, \frac{xr}{R\rho},\, \frac{yr}{R\rho},\, \frac{zR}{r\rho}\right), \qquad \beta \equiv -\frac{\sqrt{2Mr - Q^2}}{R}. \tag{27.122}$$

The vectors satisfy the exact orthogonality relation $\alpha_\mu\beta^\mu = 0$. The vierbeins and inverse vierbeins are:

$$e^a{}_\mu = \delta^a_\mu + \alpha^a\beta_\mu, \qquad e_a{}^\mu = \delta^\mu_a - \alpha^\mu\beta_a. \tag{27.123}$$

The connection coefficients are given by the Cartesian partial derivatives of an antisymmetric **river bivector field** ω_{km}:

$$\Gamma_{kmn} = -\frac{\partial\omega_{km}}{\partial x^n}, \qquad \omega_{km} \equiv \alpha_k\beta_m - \alpha_m\beta_k + \epsilon_{t_{\rm ff}kmi}\,\zeta^i, \tag{27.124}$$

where $\zeta^i = (0, 0, \zeta)$ points along the rotation axis (z-axis) with magnitude:

$$\zeta(r) \equiv a\int_\infty^r \frac{\beta(r')}{R^2(r')}\,\mathrm{d}r'. \tag{27.125}$$

The electric components of ω_{km} constitute the velocity vector $\beta^i = \omega_{t_{\rm ff}i}$, while the spatial magnetic components constitute the **twist vector**:

$$\mu^i \equiv \frac{1}{2}\epsilon_{t_{\rm ff}ikm}\omega^{km} = \vec{\alpha}\times\vec{\beta} + \vec{\zeta}. \tag{27.126}$$

In matrix form, the river field is:

$$\omega_{km} = \begin{pmatrix} 0 & -\beta_x & -\beta_y & -\beta_z \\ \beta_x & 0 & -\mu_z & \mu_y \\ \beta_y & \mu_z & 0 & -\mu_x \\ \beta_z & -\mu_y & \mu_x & 0 \end{pmatrix}. \tag{27.127}$$

A particle swimming in the Doran river undergoes both **Lorentz boosts** (generated by $\delta\vec{\beta}$) and **spatial rotations / frame-dragging precessions** (generated by the twist $\delta\vec{\mu}$).

27.6.4 Kerr–Newman Spacetime in Boyer–Lindquist Coordinates (Carter Orthonormal Frame)

In Boyer–Lindquist coordinates (t, r, θ, ϕ), the Kerr–Newman line element is:

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\,\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)\,\mathrm{d}\phi - a\,\mathrm{d}t\right]^2, \tag{27.128}$$

where $\Delta(r) \equiv r^2 - 2Mr + a^2 + Q^2 = (r^2+a^2)(1-\beta^2)$ and $\rho^2(r,\theta) \equiv r^2 + a^2\cos^2\theta$.

The canonical **Carter orthonormal tetrad** 1-forms $e^a = e^a{}_\mu \mathrm{d}x^\mu$ aligned with the principal null congruences are:

$$e^0 = \frac{\sqrt{\Delta}}{\rho}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right), \tag{27.129}$$

$$e^1 = \frac{\rho}{\sqrt{\Delta}}\,\mathrm{d}r, \tag{27.130}$$

$$e^2 = \rho\,\mathrm{d}\theta, \tag{27.131}$$

$$e^3 = \frac{\sin\theta}{\rho}\left[(r^2+a^2)\,\mathrm{d}\phi - a\,\mathrm{d}t\right]. \tag{27.132}$$

The inverse tetrad vector fields $e_a = e_a{}^\mu \partial_\mu$ are:

$$e_0 = \frac{r^2+a^2}{\rho\sqrt{\Delta}}\,\partial_t + \frac{a}{\rho\sqrt{\Delta}}\,\partial_\phi, \tag{27.133}$$

$$e_1 = \frac{\sqrt{\Delta}}{\rho}\,\partial_r, \tag{27.134}$$

$$e_2 = \frac{1}{\rho}\,\partial_\theta, \tag{27.135}$$

$$e_3 = \frac{a\sin\theta}{\rho}\,\partial_t + \frac{1}{\rho\sin\theta}\,\partial_\phi. \tag{27.136}$$

With respect to this Carter frame, the electromagnetic field tensor F_{ab} has only non-vanishing radial components, expressed compactly in complex form as:

$$E_r + iB_r = \frac{Q}{(r - ia\cos\theta)^2} \implies E_r = \frac{Q(r^2 - a^2\cos^2\theta)}{\rho^4}, \quad B_r = \frac{2Qar\cos\theta}{\rho^4}. \tag{27.137}$$

The Einstein tensor G_{ab} is diagonal:

$$G_{ab} = \frac{Q^2}{\rho^4}\,\mathrm{diag}(1,-1,1,1). \tag{27.138}$$

The Weyl tensor C_{abcd} is completely determined by a single complex Newman–Penrose scalar $\Psi_2 \equiv C$:

$$-\frac{1}{2}C_{0101} = \frac{1}{2}C_{2323} = C_{0202} = C_{0303} = -C_{1212} = -C_{1313} = \mathrm{Re}(C), \tag{27.139}$$

$$\frac{1}{2}C_{0123} = C_{0213} = -C_{0312} = \mathrm{Im}(C), \tag{27.140}$$

where the complex Weyl scalar is:

$$C = -\frac{1}{(r - ia\cos\theta)^3}\left[M - \frac{Q^2}{r + ia\cos\theta}\right]. \tag{27.141}$$

In the uncharged Schwarzschild limit ($a \to 0, Q \to 0$), $C = -M/r^3$, recovering $\mathrm{Re}(C) = -M/r^3, \mathrm{Im}(C) = 0$ and matching the static tidal tensor (27.109).

27.7 Action Principles, Vierbein Variations, and Weak-Field Theory

27.7.1 Variation of the Gravitational Action with Respect to the Vierbein

We now perform the full variation of the Einstein–Hilbert action directly with respect to the vierbein field ${e_a}^\lambda$. The gravitational action is:

$$I_G = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}. \tag{27.142}$$

Theorem: Variation of the Metric and Determinant in Terms of the Vierbein

The variation of the inverse metric and the metric determinant with respect to the vierbein variation $\delta {e_a}^\lambda$ are:

$$\delta g^{\mu\nu} = \delta\left({e_a}^\mu {e_b}^\nu \eta^{ab}\right) = \delta {e_a}^\mu e^{a\nu} + e^{a\mu}\delta {e_a}^\nu = \left(g^{\mu\lambda}{e_a}^\nu + g^{\nu\lambda}{e_a}^\mu\right)\delta e^a{}_\lambda, \tag{27.143}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\left(g^{\mu\lambda}{e_a}^\nu + g^{\nu\lambda}{e_a}^\mu\right)\delta e^a{}_\lambda = \sqrt{-g}\,{e_a}^\lambda\,\delta e^a{}_\lambda. \tag{27.144}$$

In terms of contravariant vierbein variations $\delta e^a{}_\lambda = -e^a{}_\mu e_b{}^\lambda \delta e_b{}^\mu$, the variation of the gravitational action is:

$$\delta I_G = \frac{1}{8\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left(R^\mu{}_\lambda - \frac{1}{2}\delta^\mu_\lambda R\right){e_a}^\lambda\,\delta e^a{}_\mu = \frac{1}{8\pi G}\int \mathrm{d}^4x\,e\,G^\mu{}_\lambda\,{e_a}^\lambda\,\delta e^a{}_\mu. \tag{27.145}$$

Proof. Varying the integrand of I_G:

$$\delta I_G = \frac{1}{16\pi G}\int \mathrm{d}^4x\left[(\delta\sqrt{-g})\,g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}(\delta R_{\mu\nu})\right]. \tag{27.146}$$

The Palatini identity $\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}$ integrates to a boundary term by Gauss's theorem and vanishes under standard boundary conditions. Substituting Equations (27.143) and (27.144):

$$\begin{aligned}
\delta I_G &= \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left[R\left({e_a}^\lambda\delta e^a{}_\lambda\right) + R_{\mu\nu}\left(g^{\mu\lambda}{e_a}^\nu + g^{\nu\lambda}{e_a}^\mu\right)\delta e^a{}_\lambda\right] \\
&= \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left[R\,{e_a}^\lambda + R^\lambda{}_\nu {e_a}^\nu + R_\mu{}^\lambda {e_a}^\mu\right]\delta e^a{}_\lambda \\
&= \frac{1}{8\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left(R^\mu{}_\lambda - \frac{1}{2}\delta^\mu_\lambda R\right){e_a}^\lambda\,\delta e^a{}_\mu.
\end{aligned} \tag{27.147}$$

□

27.7.2 Matter Source Action and the Full Einstein Equations

The variation of the matter source action I_M with respect to the vierbein is defined by:

$$\delta I_M \equiv \int \mathrm{d}^4x\,\sqrt{-g}\,u^\lambda{}_a\,\delta e^a{}_\lambda = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu}. \tag{27.148}$$

Using $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta} = -(g_{\mu\lambda}e^a{}_\nu + g_{\nu\lambda}e^a{}_\mu)\delta {e_a}^\lambda$:

$$u^\lambda{}_a = -T^\lambda{}_\nu\,{e_a}^\nu \iff T_{\mu\nu} = -u_{\mu a}\,e_\nu{}^a. \tag{27.149}$$

Requiring the total action variation $\delta I = \delta I_G + \delta I_M = 0$ for arbitrary $\delta e^a{}_\lambda$:

$$\frac{1}{8\pi G}\left(R^\mu{}_\lambda - \frac{1}{2}\delta^\mu_\lambda R\right) e_a{}^\lambda + u^\mu{}_a = 0 \iff G^\mu{}_\lambda\, e_a{}^\lambda = 8\pi G\, T^\mu{}_\nu\, e_a{}^\nu. \tag{27.150}$$

Contracting with $e^a{}_\nu$ recovers the complete Einstein field equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}. \tag{27.151}$$

27.7.3 Einstein's 1928 Quadratic Lagrangians and Weak-Field Equations

In his June 1928 paper, Einstein proposed formulating gravity using the torsion field strength:

$$\Lambda^\nu{}_{\alpha\beta} \equiv \frac{1}{2} e_a{}^\nu \left(\partial_\beta e^a{}_\alpha - \partial_\alpha e^a{}_\beta\right). \tag{27.152}$$

He investigated two second-order quadratic Lagrangian densities:

$$\mathcal{L}_1 = h\, g^{\mu\nu} \Lambda^\mu{}_{\alpha\beta} \Lambda^\nu{}_{\beta\alpha} = \frac{h}{4} g^{\mu\nu} e_a{}^\alpha e_b{}^\beta \left(\partial_\beta e^a{}_\mu - \partial_\mu e^a{}_\beta\right)\left(\partial_\alpha e^b{}_\nu - \partial_\nu e^b{}_\alpha\right), \tag{27.153}$$

$$\mathcal{L}_2 = h\, g^{\mu\nu} g^{\alpha\sigma} g^{\beta\tau} \Lambda^\mu{}_{\alpha\beta} \Lambda^\nu{}_{\sigma\tau} = \frac{h}{4} F_{\alpha\beta\nu} F^{\alpha\beta\nu}, \qquad F_{\alpha\beta\nu} \equiv \partial_\alpha k_{\beta\nu} - \partial_\beta k_{\alpha\nu}. \tag{27.154}$$

Expanding perturbatively $e^a{}_\mu = \delta^a_\mu - k^a{}_\mu$ with $k_{\mu\nu} \equiv \eta_{a\nu} k^a{}_\mu$:

1. Varying $\mathcal{L}_1$ yields Einstein's 1928 equation of motion:

$$\partial^2 k^\beta{}_\alpha - \partial_\mu \partial^\beta k^\mu{}_\alpha + \partial^\beta \partial_\mu k_\alpha{}^\mu - \partial_\mu \partial_\alpha k^\beta{}_\mu = 0. \tag{27.155}$$

2. Adding Equation (27.155) to itself with indices α and β exchanged yields the symmetric fluctuation equation:

$$\partial^2 g_{\alpha\beta} - \partial_\mu \partial_\alpha k^\mu{}_\beta - \partial_\mu \partial_\beta k^\mu{}_\alpha = 0, \qquad g_{\alpha\beta} \equiv k_{\alpha\beta} + k_{\beta\alpha}. \tag{27.156}$$

3. Contracting the torsion tensor defines the Einstein vector field:

$$\phi_\mu \equiv \Lambda^\alpha{}_{\mu\alpha} = \frac{1}{2}\delta^\alpha_a \left(\partial_\alpha k^a{}_\mu - \partial_\mu k^a{}_\alpha\right) \implies 2\phi_\mu = \partial_\alpha k^\alpha{}_\mu - \partial_\mu k^\alpha{}_\alpha. \tag{27.157}$$

   Differentiating gives the ancillary identities:

$$-\partial_\mu \partial_\beta k^\mu{}_\alpha + \partial_\alpha \partial_\beta k^\mu{}_\mu = -2\partial_\beta \phi_\alpha, \tag{27.158}$$

$$-\partial_\mu \partial_\alpha k^\mu{}_\beta + \partial_\alpha \partial_\beta k^\mu{}_\mu = -2\partial_\alpha \phi_\beta. \tag{27.159}$$

4. Adding these ancillae to the equation of motion yields:

$$\frac{1}{2}\left(-\partial^2 g_{\alpha\beta} + \partial_\mu \partial_\alpha g^\mu{}_\beta + \partial_\mu \partial_\beta g^\mu{}_\alpha - \partial_\alpha \partial_\beta g^\mu{}_\mu\right) = \partial_\beta \phi_\alpha + \partial_\alpha \phi_\beta. \tag{27.160}$$

When the vector field vanishes ($\phi_\alpha = 0$), Equation (27.160) matches the vacuum linearized Einstein equation $R^{(1)}_{\alpha\beta} = 0$.

27.7.4 Einstein's 1928 Teleparallelism

In his distant parallelism (*Fernparallelismus*) framework, Einstein used the Weitzenböck connection $\Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda \partial_\nu e^a{}_\mu$. Spacetime curvature vanishes identically ($R^\rho{}_{\sigma\mu\nu} = 0$), and gravitation is encoded purely in the torsion tensor $T^\lambda{}_{\mu\nu} = e_a{}^\lambda(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu)$.

27.8 Relativistic Electrodynamics in the Tetrad Formalism

27.8.1 The Electromagnetic Field Bivector

In the orthonormal tetrad frame, the electromagnetic field is an antisymmetric tensor $F_{ab} = -F_{ba}$:

$$F_{ab} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{27.161}$$

In terms of the tetrad 4-potential $A_a = e_a{}^\mu A_\mu$:

$$F_{ab} = \partial_a A_b - \partial_b A_a - c_{ab}{}^c A_c = D_a A_b - D_b A_a. \tag{27.162}$$

27.8.2 Lorentz Force Law and Maxwell's Equations

Theorem: Tetrad Formulation of Relativistic Electrodynamics

(i) Relativistic Lorentz Force Law:

$$m \frac{Du^a}{D\tau} = q\, F^a{}_b\, u^b \iff m\left(\frac{\mathrm{d}u^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc}\, u^b u^c\right) = q\, F^a{}_b\, u^b. \tag{27.163}$$

(ii) Inhomogeneous Maxwell Equations:

$$D_a F^{ab} = 4\pi J^b \iff \partial_a F^{ab} + \Gamma^a{}_{ca} F^{cb} + \Gamma^b{}_{ca} F^{ac} = 4\pi J^b. \tag{27.164}$$

(iii) Homogeneous Maxwell Equations (Bianchi Identity):

$$D_{[a} F_{bc]} = 0 \iff \partial_{[a} F_{bc]} - c_{[ab}{}^d F_{c]d} = 0. \tag{27.165}$$

(iv) Electromagnetic Stress-Energy Tensor and Conservation:

$$T^{ab}_{\mathrm{EM}} = \frac{1}{4\pi}\left(-F^a{}_c F^{bc} + \frac{1}{4}\eta^{ab} F_{cd} F^{cd}\right), \qquad D_a T^{ab}_{\mathrm{EM}} = -F^b{}_a J^a. \tag{27.166}$$

27.9 Dirac Spinors, Clifford Algebra, and Fermions in Curved Spacetime

27.9.1 The Dirac Energy-Momentum Factorization and Clifford Algebra

The relativistic energy-momentum dispersion relation for a free particle of mass m is:

$$E^2 - \vec{p}^2 - m^2 = p^\mu p_\mu - m^2 = 0. \tag{27.167}$$

Following Dirac's original 1928 program, we factorize this quadratic relation into two linear factors:

$$\begin{aligned} p^\mu p_\mu - m^2 &= (\beta^\kappa p_\kappa + m)\left(\gamma^\lambda p_\lambda - m\right) \\ &= \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2 + m\gamma^\lambda p_\lambda - m\beta^\kappa p_\kappa \\ &= \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2 + m\left(\gamma^\lambda - \beta^\lambda\right) p_\lambda. \end{aligned} \tag{27.168}$$

Because the physical energy-momentum relation $p^\mu p_\mu - m^2 = 0$ contains no linear terms in p_μ, the linear coefficient on the right-hand side must vanish identically, requiring:

$$\beta^\kappa = \gamma^\kappa. \tag{27.169}$$

The factorization then becomes:

$$p^\mu p_\mu - m^2 = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2. \tag{27.170}$$

Expanding the double summation explicitly over the spacetime components $\mu, \nu \in \{0, 1, 2, 3\}$:

$$\begin{aligned} p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2 = {} & (\gamma^0)^2 p_0^2 + (\gamma^1)^2 p_1^2 + (\gamma^2)^2 p_2^2 + (\gamma^3)^2 p_3^2 \\ & + (\gamma^0\gamma^1 + \gamma^1\gamma^0) p_0 p_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0) p_0 p_2 + (\gamma^0\gamma^3 + \gamma^3\gamma^0) p_0 p_3 \\ & + (\gamma^1\gamma^2 + \gamma^2\gamma^1) p_1 p_2 + (\gamma^1\gamma^3 + \gamma^3\gamma^1) p_1 p_3 + (\gamma^2\gamma^3 + \gamma^3\gamma^2) p_2 p_3 - m^2. \end{aligned} \tag{27.171}$$

Comparing both sides of Equation (27.171) term by term demonstrates that the coefficients must satisfy:

$$(\gamma^0)^2 = \mathbf{1}_4, \qquad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbf{1}_4, \qquad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \quad (\mu \neq \nu). \tag{27.172}$$

This is packaged into the fundamental **Clifford algebra** anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}_4, \tag{27.173}$$

where in flat Minkowski spacetime $[g^{\mu\nu}] = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The energy-momentum relation factorizes as:

$$p^\mu p_\mu - m^2 = (\gamma^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = 0 \implies \left(\gamma^\lambda p_\lambda - m\right)\psi = 0. \tag{27.174}$$

Under the canonical quantization replacement $p_\mu \to i\partial_\mu$, we obtain the free Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0. \tag{27.175}$$

In the standard **Dirac–Pauli representation**, the 4×4 flat gamma matrices γ^a ($a \in \{0, 1, 2, 3\}$) are constructed from the 2×2 Pauli matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (i = 1, 2, 3), \tag{27.176}$$

where the standard Pauli spin matrices are:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{27.177}$$

Multiplying Equation (27.175) by $(\gamma^0)^{-1} = \gamma^0$ gives:

$$\left[i\partial_t + i(\gamma^0)^{-1}\gamma^j\partial_j - m(\gamma^0)^{-1}\right]\psi = 0 \iff \left(i\partial_t + i\vec{\alpha}\cdot\vec{\nabla} - m\beta\right)\psi = 0, \tag{27.178}$$

where the standard Dirac $\vec{\alpha}$ and β matrices are:

$$\alpha^j \equiv (\gamma^0)^{-1}\gamma^j = \gamma^0\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \qquad \beta \equiv \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \Sigma^j \equiv \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}. \tag{27.179}$$

The generators of the Lorentz group in the spinor representation are $S_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b] = -\frac{i}{2}\sigma_{ab}$. Using the Clifford anticommutation relation $\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbf{1}_4$, their commutators satisfy the $\mathfrak{so}(1,3)$ Lie algebra:

$$\begin{aligned}
[S_{ab}, S_{cd}] &= \frac{1}{16}\,[[\gamma_a, \gamma_b], [\gamma_c, \gamma_d]] \\
&= \frac{1}{16}\,(\gamma_a[\gamma_b, [\gamma_c, \gamma_d]] + [\gamma_a, [\gamma_c, \gamma_d]]\gamma_b) \\
&= \frac{1}{16}\,(\gamma_a(4\eta_{bd}\gamma_c - 4\eta_{bc}\gamma_d) + (4\eta_{ad}\gamma_c - 4\eta_{ac}\gamma_d)\gamma_b) \\
&= \eta_{ad}S_{bc} + \eta_{bc}S_{ad} - \eta_{ac}S_{bd} - \eta_{bd}S_{ac}.
\end{aligned} \tag{27.180}$$

In curved spacetime, the coordinate Dirac gamma matrices $\gamma^\mu(x)$ are obtained by projecting with the inverse vierbein $e_a{}^\mu(x)$:

$$\gamma^\mu(x) \equiv e_a{}^\mu(x)\,\gamma^a \implies \{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)\,\mathbf{1}_4. \tag{27.181}$$

27.9.2 Spinor Covariant Derivative and the Dirac Action

Definition: Covariant Dirac Equation

The spinor covariant derivative is:

$$\nabla_\mu\psi(x) = (\partial_\mu + \Gamma_\mu(x))\,\psi(x) = \left(\partial_\mu + \frac{1}{8}\omega_{\mu ab}(x)[\gamma^a, \gamma^b]\right)\psi(x). \tag{27.182}$$

The action for a massive Dirac field $\psi(x)$ in curved spacetime is:

$$S_{\text{Dirac}} = \int \mathrm{d}^4x\, e\left[\frac{i}{2}\bar\psi\gamma^\mu\nabla_\mu\psi - \frac{i}{2}(\nabla_\mu\bar\psi)\gamma^\mu\psi - m\bar\psi\psi\right], \tag{27.183}$$

yielding the **Covariant Dirac Equation**:

$$(i\gamma^\mu\nabla_\mu - m)\,\psi = \left[ie_a{}^\mu\gamma^a\left(\partial_\mu + \frac{1}{8}\omega_{\mu bc}[\gamma^b, \gamma^c]\right) - m\right]\psi = 0. \tag{27.184}$$

The symmetric Dirac energy-momentum tensor obtained by varying S_{Dirac} with respect to $e_a{}^\mu$ is:

$$T_{\mu\nu} = \frac{i}{2}\left[\bar\psi\gamma_{(\mu}\overleftrightarrow{\nabla}_{\nu)}\psi\right] = \frac{i}{4}\left[\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - (\nabla_\mu\bar\psi)\gamma_\nu\psi - (\nabla_\nu\bar\psi)\gamma_\mu\psi\right], \tag{27.185}$$

with conserved Dirac vector current $\nabla_\mu j^\mu = \nabla_\mu(\bar\psi\gamma^\mu\psi) = 0$.

Theorem: Gordon Decomposition in Curved Spacetime

The conserved curved spacetime Dirac current $j^\mu \equiv \bar\psi\gamma^\mu\psi$ decomposes uniquely into a **convective orbital current** j^μ_{conv} and a **spin magnetization current** j^μ_{spin}:

$$j^\mu = \bar\psi\gamma^\mu\psi = \frac{i}{2m}\bar\psi\overleftrightarrow{\nabla}^\mu\psi - \frac{1}{m}\nabla_\nu\left(\bar\psi\sigma^{\mu\nu}\psi\right) \equiv j^\mu_{\text{conv}} + j^\mu_{\text{spin}}, \tag{27.186}$$

where $\bar\psi\overleftrightarrow{\nabla}^\mu\psi \equiv \bar\psi(\nabla^\mu\psi) - (\nabla^\mu\bar\psi)\psi$ and $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Proof. Multiplying the covariant Dirac equation $i\gamma^\nu\nabla_\nu\psi = m\psi$ on the left by $\bar\psi\gamma^\mu$:

$$m\,\bar\psi\gamma^\mu\psi = i\,\bar\psi\gamma^\mu\gamma^\nu\nabla_\nu\psi. \tag{27.187}$$

Using the identity $\gamma^\mu\gamma^\nu = g^{\mu\nu}\mathbf{1}_4 - 2i\sigma^{\mu\nu}$ in Equation (27.187):

$$m\,j^\mu = i\,\bar\psi g^{\mu\nu}\nabla_\nu\psi + 2\,\bar\psi\sigma^{\mu\nu}\nabla_\nu\psi = i\,\bar\psi\nabla^\mu\psi + 2\,\bar\psi\sigma^{\mu\nu}\nabla_\nu\psi. \tag{27.188}$$

Similarly, multiplying the adjoint equation $-i(\nabla_\nu\bar\psi)\gamma^\nu = m\bar\psi$ on the right by $\gamma^\mu\psi$ and using $\gamma^\nu\gamma^\mu = g^{\mu\nu}\mathbf{1}_4 + 2i\sigma^{\mu\nu}$:

$$m\,j^\mu = -i(\nabla^\mu\bar\psi)\psi + 2(\nabla_\nu\bar\psi)\sigma^{\mu\nu}\psi. \tag{27.189}$$

Adding Equations (27.188) and (27.189):

$$\begin{aligned} 2m\,j^\mu &= i\left[\bar\psi\nabla^\mu\psi - (\nabla^\mu\bar\psi)\psi\right] + 2\left[\bar\psi\sigma^{\mu\nu}\nabla_\nu\psi + (\nabla_\nu\bar\psi)\sigma^{\mu\nu}\psi\right] \\ &= i\,\bar\psi\overset{\leftrightarrow}{\nabla}^{\mu}\psi + 2\,\nabla_\nu\left(\bar\psi\sigma^{\mu\nu}\psi\right) - 2\,\bar\psi(\nabla_\nu\sigma^{\mu\nu})\psi. \end{aligned} \tag{27.190}$$

Because $\nabla_\nu\gamma^\mu = 0$ (covariant constancy), $\nabla_\nu\sigma^{\mu\nu} = 0$. Using antisymmetry $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$, dividing by $2m$ yields:

$$j^\mu = \frac{i}{2m}\bar\psi\overset{\leftrightarrow}{\nabla}^{\mu}\psi - \frac{1}{m}\nabla_\nu\left(\bar\psi\sigma^{\mu\nu}\psi\right). \tag{27.191}$$

□

27.9.3 First-Principles Derivations of the Spinor Connection Term

The explicit form of the spinor connection $\Gamma_\mu(x) = \frac{1}{8}\omega_{\mu ab}(x)[\gamma^a,\gamma^b] = -\frac{i}{4}\omega_{\mu ab}(x)\,\sigma^{ab}$ can be established through two complementary, first-principles approaches: (i) demanding that the spinor covariant derivative transforms covariantly under local Lorentz gauge transformations Spin(1, 3), and (ii) imposing the geometric condition of covariant constancy on the curved Dirac gamma matrices ($\nabla_\mu\gamma^\nu = 0$).

Method I: Derivation from Local Lorentz Gauge Invariance

In flat spacetime, the Dirac equation is invariant under global Lorentz transformations $\psi(x) \to \Lambda_{1/2}\psi(\Lambda^{-1}x)$:

$$\begin{aligned} [i\gamma^\mu\partial_\mu - m]\psi(x) &\xrightarrow{\text{LLT}} [i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\Lambda_{1/2}\psi(\Lambda^{-1}x) \\ &= \Lambda_{1/2}\left[i\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right]\psi(\Lambda^{-1}x) \\ &= \Lambda_{1/2}\left[i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right]\psi(\Lambda^{-1}x) \\ &= \Lambda_{1/2}\left[i\delta^\nu_\sigma\gamma^\sigma\partial_\nu - m\right]\psi(\Lambda^{-1}x) = \Lambda_{1/2}[i\gamma^\nu\partial_\nu - m]\psi(\Lambda^{-1}x). \end{aligned} \tag{27.192}$$

In curved spacetime, the transformation $\Lambda_{1/2}(x) = U(\Lambda(x))$ is local:

$$\partial_\mu\psi'(x) = \partial_\mu\left[\Lambda_{1/2}(x)\,\psi(x)\right] = \Lambda_{1/2}(x)\,\partial_\mu\psi(x) + \left(\partial_\mu\Lambda_{1/2}(x)\right)\psi(x). \tag{27.193}$$

Demanding that the covariant derivative transforms covariantly $\nabla'_\mu\psi'(x) = \Lambda_{1/2}(x)\nabla_\mu\psi(x)$ requires:

$$\Gamma'_\mu(x) = \Lambda_{1/2}(x)\,\Gamma_\mu(x)\,\Lambda_{1/2}^{-1}(x) - \left(\partial_\mu\Lambda_{1/2}(x)\right)\Lambda_{1/2}^{-1}(x). \tag{27.194}$$

For an infinitesimal transformation $\Lambda_{1/2} = \mathbf{1} + \frac{1}{2}\lambda_{ab}S^{ab} = \mathbf{1} + \frac{1}{8}\lambda_{ab}[\gamma^a,\gamma^b]$, the shift is:

$$\delta\Gamma_\mu = \frac{1}{8}\lambda_{ab}\left[\left[\gamma^a,\gamma^b\right],\Gamma_\mu\right] - \frac{1}{8}\left(\partial_\mu\lambda_{ab}\right)\left[\gamma^a,\gamma^b\right]. \tag{27.195}$$

Matching the inhomogeneous gradient term $-\frac{1}{8}(\partial_\mu \lambda_{ab})[\gamma^a, \gamma^b]$ with the transformation law of the geometric spin connection $\delta\omega_{\mu ab} = -\partial_\mu \lambda_{ab} + \lambda_a{}^c \omega_{\mu cb} + \lambda_b{}^c \omega_{\mu ac}$, the spinor connection is uniquely identified as:

$$\Gamma_\mu(x) = \frac{1}{8}\omega_{\mu ab}(x)\left[\gamma^a, \gamma^b\right] = -\frac{i}{4}\omega_{\mu ab}(x)\,\sigma^{ab}. \tag{27.196}$$

Method II: Derivation from the Covariant Constancy of Dirac Matrices

We demand that the curved Dirac gamma matrices $\gamma^\nu(x) = e_a{}^\nu(x)\gamma^a$ are covariantly constant:

$$\nabla_\mu \gamma^\nu \equiv \partial_\mu \gamma^\nu + \Gamma^\nu_{\mu\lambda}\,\gamma^\lambda + [\Gamma_\mu, \gamma^\nu] = 0. \tag{27.197}$$

Substituting $\gamma^\nu = e_a{}^\nu \gamma^a$ and applying the Tetrad Postulate $\partial_\mu e_a{}^\nu + \Gamma^\nu_{\mu\lambda} e_a{}^\lambda = \omega_\mu{}^a{}_b e_a{}^\nu$:

$$[\Gamma_\mu, \gamma^a] = -\omega_\mu{}^a{}_b\,\gamma^b. \tag{27.198}$$

Setting $\Gamma_\mu = c\,\omega_{\mu cd}[\gamma^c, \gamma^d]$ and evaluating the double commutator:

$$\begin{aligned}\Big[\big[\gamma^c, \gamma^d\big], \gamma^a\Big] &= \big[\gamma^c\gamma^d - \gamma^d\gamma^c, \gamma^a\big] \\ &= \gamma^c\{\gamma^d, \gamma^a\} - \{\gamma^c, \gamma^a\}\gamma^d - \gamma^d\{\gamma^c, \gamma^a\} + \{\gamma^d, \gamma^a\}\gamma^c \\ &= 2\eta^{da}\gamma^c - 2\eta^{ca}\gamma^d - 2\eta^{ca}\gamma^d + 2\eta^{da}\gamma^c = 4\left(\eta^{da}\gamma^c - \eta^{ca}\gamma^d\right).\end{aligned} \tag{27.199}$$

Therefore:

$$[\Gamma_\mu, \gamma^a] = 4c\,\omega_{\mu cd}\left(\eta^{da}\gamma^c - \eta^{ca}\gamma^d\right) = 4c\left(\omega_\mu{}^{ad}\gamma_d - \omega_{\mu c}{}^a\gamma^c\right) = -8c\,\omega_\mu{}^a{}_b\,\gamma^b. \tag{27.200}$$

Matching coefficients with Equation (27.198) yields $-8c = -1 \implies c = 1/8$, completing the proof:

$$\Gamma_\mu(x) = \frac{1}{8}\omega_{\mu ab}(x)\left[\gamma^a, \gamma^b\right] = -\frac{i}{4}\omega_{\mu ab}(x)\,\sigma^{ab}. \tag{27.201}$$

27.10 The Dirac Equation in Spherical Coordinates: Exact Tetrad Separation

27.10.1 Spherical Tetrad and Christoffel Symbols

In spherical polar coordinates (t, r, θ, ϕ), the flat Minkowski line element is:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{27.202}$$

The metric tensor and its inverse are:

$$[g_{\mu\nu}] = \mathrm{diag}(-1, 1, r^2, r^2\sin^2\theta), \qquad [g^{\mu\nu}] = \mathrm{diag}(-1, 1, r^{-2}, (r\sin\theta)^{-2}). \tag{27.203}$$

The diagonal orthonormal tetrad $e^a{}_\mu$ and inverse tetrad $e_a{}^\mu$ satisfying $g_{\mu\nu}e_a{}^\mu e_b{}^\nu = \eta_{ab}$ are:

$$e^0{}_t = 1, \qquad e^1{}_r = 1, \qquad e^2{}_\theta = r, \qquad e^3{}_\phi = r\sin\theta, \tag{27.204}$$

$$e_0{}^t = 1, \qquad e_1{}^r = 1, \qquad e_2{}^\theta = \frac{1}{r}, \qquad e_3{}^\phi = \frac{1}{r\sin\theta}. \tag{27.205}$$

The coordinate Dirac matrices $\gamma^\mu = e_a{}^\mu\gamma^a$ evaluate explicitly as 4×4 matrices:

$$\gamma^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^r = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{27.206}$$

$$\gamma^\theta = \frac{1}{r}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^\phi = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{27.207}$$

The Christoffel symbols are $\Gamma^\nu_{\mu\lambda} = \frac{1}{2}g^{\nu\sigma}(g_{\sigma\mu,\lambda} + g_{\sigma\lambda,\mu} - g_{\mu\lambda,\sigma})$. By diagonality and independence of coordinate derivatives, a Christoffel symbol vanishes unless exactly two of the indices $\{\mu, \nu, \lambda\}$ are equal:

$$\Gamma^\nu_{\mu\mu} = -\frac{1}{2}g^{\nu\nu}g_{\mu\mu,\nu} \quad (\mu \neq \nu), \tag{27.208}$$

$$\Gamma^\mu_{\mu\lambda} = \Gamma^\mu_{\lambda\mu} = \frac{1}{2}g^{\mu\mu}g_{\mu\mu,\lambda} \quad (\mu \neq \lambda). \tag{27.209}$$

Evaluating Equations (27.208)–(27.209) yields all non-zero Christoffel symbols:

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^r_{\phi\phi} = -r\sin^2\theta, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \tag{27.210}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{27.211}$$

27.10.2 Spin Connections and Explicit 4×4 Matrix Multiplications

The geometric spin connection coefficients are:

$$\omega_\mu{}^a{}_b = e^a{}_\nu e_b{}^\lambda \Gamma^\nu_{\mu\lambda} - e_b{}^\lambda \partial_\mu e^a{}_\lambda. \tag{27.212}$$

Evaluating Equation (27.212) gives the non-zero mixed-index spin connections:

$$\begin{aligned} \omega_\theta{}^1{}_2 &= -1, & \omega_\theta{}^2{}_1 &= 1, \\ \omega_\phi{}^1{}_3 &= -\sin\theta, & \omega_\phi{}^3{}_1 &= \sin\theta, \\ \omega_\phi{}^2{}_3 &= -\cos\theta, & \omega_\phi{}^3{}_2 &= \cos\theta. \end{aligned} \tag{27.213}$$

Lowering the second Lorentz index via $\omega_{\mu ab} = \eta_{bc}\omega_\mu{}^a{}_c$:

$$\begin{aligned} \omega_\theta{}^{12} &= -1, & \omega_\theta{}^{21} &= 1, \\ \omega_\phi{}^{13} &= -\sin\theta, & \omega_\phi{}^{31} &= \sin\theta, \\ \omega_\phi{}^{23} &= -\cos\theta, & \omega_\phi{}^{32} &= \cos\theta. \end{aligned} \tag{27.214}$$

We now expand the spin connection term in the covariant Dirac equation $(i\gamma^\mu\nabla_\mu - m)\psi = (i\gamma^\mu\partial_\mu + \frac{1}{4}\gamma^\mu\omega_{\mu ab}\sigma^{ab} - m)\psi = 0$, where $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b] = i\gamma_a\gamma_b$ for $a \neq b$:

$$\begin{aligned} \frac{1}{4}\gamma^\mu\omega_{\mu ab}\sigma^{ab} &= \frac{1}{4}\left[\gamma^\theta(\omega_\theta{}^{12}\sigma_{12} + \omega_\theta{}^{21}\sigma_{21}) + \gamma^\phi(\omega_\phi{}^{13}\sigma_{13} + \omega_\phi{}^{31}\sigma_{31} + \omega_\phi{}^{23}\sigma_{23} + \omega_\phi{}^{32}\sigma_{32})\right] \\ &= \frac{1}{4}\left[\gamma^\theta(-\sigma_{12} + \sigma_{21}) + \gamma^\phi(-\sin\theta\,\sigma_{13} + \sin\theta\,\sigma_{31} - \cos\theta\,\sigma_{23} + \cos\theta\,\sigma_{32})\right] \\ &= \frac{1}{4}\left[\gamma^\theta(-2\sigma_{12}) + \gamma^\phi(-2\sin\theta\,\sigma_{13} - 2\cos\theta\,\sigma_{23})\right] \\ &= -\frac{i}{2}\left[\gamma^\theta\gamma_1\gamma_2 + \gamma^\phi(\sin\theta\,\gamma_1\gamma_3 + \cos\theta\,\gamma_2\gamma_3)\right]. \end{aligned} \tag{27.215}$$

We now carry out the explicit 4×4 matrix multiplications:

1. **First Matrix Product ($\gamma^\theta\gamma_1\gamma_2$):**

$$\gamma^\theta\gamma_1\gamma_2 = \frac{1}{r}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{r}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}. \tag{27.216}$$

2. Second Matrix Product ($\gamma^\phi\gamma_1\gamma_3$):

$$\gamma^\phi\gamma_1\gamma_3 = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{r\sin\theta}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}. \tag{27.217}$$

3. Third Matrix Product ($\gamma^\phi\gamma_2\gamma_3$):

$$\gamma^\phi\gamma_2\gamma_3 = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{r\sin\theta}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}. \tag{27.218}$$

Substituting Equations (27.216)–(27.218) into Equation (27.215) gives:

$$\left\{ i\gamma^\mu\partial_\mu - \frac{i}{r}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} - \frac{i\cot\theta}{2r}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} - m \right\}\psi = 0. \tag{27.219}$$

Writing the bispinor in terms of two 2-component spinors $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$:

$$i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \partial_t\psi_1 \\ \partial_t\psi_2 \end{pmatrix} + i\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} \partial_r\psi_1 \\ \partial_r\psi_2 \end{pmatrix} + \frac{i}{r}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \partial_\theta\psi_1 \\ \partial_\theta\psi_2 \end{pmatrix}$$
$$+ \frac{i}{r\sin\theta}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}\begin{pmatrix} \partial_\phi\psi_1 \\ \partial_\phi\psi_2 \end{pmatrix} - \frac{i}{r}\begin{pmatrix} \sigma^1\psi_2 \\ -\sigma^1\psi_1 \end{pmatrix} - \frac{i\cot\theta}{2r}\begin{pmatrix} \sigma^2\psi_2 \\ -\sigma^2\psi_1 \end{pmatrix} - m\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{27.220}$$

Separating the upper and lower 2-spinor components yields:

$$\left[i\sigma^1\partial_r + \frac{i}{r}\sigma^2\partial_\theta + \frac{i}{r\sin\theta}\sigma^3\partial_\phi - \frac{i}{r}\sigma^1 - \frac{i\cot\theta}{2r}\sigma^2 \right]\psi_2 = (-i\partial_t + m)\psi_1, \tag{27.221}$$
$$\left[i\sigma^1\partial_r + \frac{i}{r}\sigma^2\partial_\theta + \frac{i}{r\sin\theta}\sigma^3\partial_\phi - \frac{i}{r}\sigma^1 - \frac{i\cot\theta}{2r}\sigma^2 \right]\psi_1 = (-i\partial_t - m)\psi_2. \tag{27.222}$$

Squaring the 2-spinor operator produces the second-order wave equation:

$$\left[i\sigma^1\partial_r + \frac{i}{r}\sigma^2\partial_\theta + \frac{i}{r\sin\theta}\sigma^3\partial_\phi - \frac{i}{r}\sigma^1 - \frac{i\cot\theta}{2r}\sigma^2 \right]^2\psi_1 = (i\partial_t - m)(i\partial_t + m)\psi_1. \tag{27.223}$$

27.10.3 Relativistic Spin-Orbit Operator $\hat{Q}$ and Separation of Variables

The orbital angular momentum operator is $\vec{L} = -i\vec{r} \times \vec{\nabla}$. Evaluating the cross product:

$$\begin{aligned} \vec{r} \times \vec{L} &= -i\vec{r} \times (\vec{r} \times \vec{\nabla}) = i\vec{r} \times (\vec{\nabla} \times \vec{r}) = i\left[(\vec{\nabla}\vec{r}) \cdot \vec{r} - (\vec{r} \cdot \vec{\nabla})\vec{r}\right] \\ &= ir^2\vec{\nabla} - i(\vec{r} \cdot \vec{\nabla})\vec{r} = ir^2(\vec{\nabla} - \hat{r}\partial_r) \\ \implies -i\vec{\nabla} &= -i\hat{r}\partial_r - \frac{1}{r}(\hat{r} \times \vec{L}). \end{aligned} \tag{27.224}$$

Using the operator product identity $(\vec{\alpha} \cdot \vec{A})(\vec{\Sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B}\mathbf{1}_4 + i\vec{\alpha} \cdot (\vec{A} \times \vec{B})$:

$$\begin{aligned} -i\vec{\alpha} \cdot \vec{\nabla} &= -i(\vec{\alpha} \cdot \hat{r})\partial_r - \frac{1}{r}\vec{\alpha} \cdot (\hat{r} \times \vec{L}) \\ &= -i(\vec{\alpha} \cdot \hat{r})\partial_r - \frac{1}{ir}\left[(\vec{\alpha} \cdot \hat{r})(\vec{\Sigma} \cdot \vec{L}) - \hat{r} \cdot \vec{L}\right] \\ &= -i(\vec{\alpha} \cdot \hat{r})\partial_r + \frac{i}{r}(\vec{\alpha} \cdot \hat{r})(\vec{\Sigma} \cdot \vec{L}) \\ &= -i(\vec{\alpha} \cdot \hat{r})\left(\partial_r + \frac{1}{r}\right) - \frac{1}{r}\beta^2(\vec{\Sigma} \cdot \vec{L} + \mathbf{1}) \\ &= -i(\vec{\alpha} \cdot \hat{r})\left(\partial_r + \frac{1}{r}\right) - \frac{1}{r}\beta^2(2\vec{S} \cdot \vec{L} + \mathbf{1}) \\ &= -i(\vec{\alpha} \cdot \hat{r})\left(\partial_r + \frac{1}{r}\right) - \frac{1}{r}\beta^2(\vec{J}^2 - \vec{L}^2 - \vec{S}^2 + 1) \\ &= -i(\vec{\alpha} \cdot \hat{r})\left(\partial_r + \frac{1}{r}\right) - \frac{1}{r}\beta Q, \end{aligned} \tag{27.225}$$

where the relativistic spin-orbit operator is:

$$Q \equiv \beta(2\vec{S} \cdot \vec{L} + 1) = \beta(\vec{J}^2 - \vec{L}^2 - \vec{S}^2 + 1). \tag{27.226}$$

Evaluating Q^2 explicitly:

$$\begin{aligned} Q^2 &= \beta^2(2\vec{S} \cdot \vec{L} + 1)^2 = 4(\vec{S} \cdot \vec{L})^2 + 4\vec{S} \cdot \vec{L} + 1 = (\vec{\sigma} \cdot \vec{L})^2 + 2\vec{\sigma} \cdot \vec{L} + 1 \\ &= \vec{L}^2 + i\vec{\sigma} \cdot (\vec{L} \times \vec{L}) + 2\vec{\sigma} \cdot \vec{L} + 1 = \vec{L}^2 - \vec{\sigma} \cdot \vec{L} + 2\vec{\sigma} \cdot \vec{L} + 1 \\ &= \vec{L}^2 + \vec{\sigma} \cdot \vec{L} + 1 = \vec{L}^2 + 2\vec{S} \cdot \vec{L} + 1 = \vec{J}^2 - \vec{S}^2 + 1 = \vec{J}^2 - \frac{3}{4} + 1 = \vec{J}^2 + \frac{1}{4}. \end{aligned} \tag{27.227}$$

The free Dirac Hamiltonian is:

$$H_0 = -i(\vec{\alpha} \cdot \hat{r})\left(\partial_r + \frac{1}{r}\right) - \frac{1}{r}\beta Q + m\beta. \tag{27.228}$$

We choose $\{H_0, Q, J_3, \vec{J}^2, \beta P\}$ as the complete set of commuting observables (CSCO), so that:

$$\vec{J}^2\psi = j(j+1)\psi, \quad J_3\psi = m_j\psi, \quad Q^2\psi = \left[j(j+1) + \frac{1}{4}\right]\psi = \left(j + \frac{1}{2}\right)^2\psi. \tag{27.229}$$

The parity eigenstates $\psi^{(\pm)}_{jm_jm_s}$ are defined using the 2-spinor spherical harmonics:

$$\psi^{(+)}_{jm_jm_s} = \begin{pmatrix} i\Phi^{m_jm_s}_{j-1/2} \\ 0 \end{pmatrix}, \qquad \psi^{(-)}_{jm_jm_s} = \begin{pmatrix} 0 \\ i\Phi^{m_jm_s}_{j+1/2} \end{pmatrix}, \tag{27.230}$$

where:

$$\Phi^{m_jm_s}_{j-1/2} = \frac{1}{\sqrt{2j}}\begin{pmatrix} \sqrt{j+m_j}\,Y^{m_j-1/2}_{j-1/2} \\ \sqrt{j-m_j}\,Y^{m_j+1/2}_{j-1/2} \end{pmatrix}, \qquad \Phi^{m_jm_s}_{j+1/2} = \frac{1}{\sqrt{2j+2}}\begin{pmatrix} -\sqrt{j+1-m_j}\,Y^{m_j-1/2}_{j+1/2} \\ \sqrt{j+1+m_j}\,Y^{m_j+1/2}_{j+1/2} \end{pmatrix}. \tag{27.231}$$

Using the identity $(\vec{\sigma}\cdot\hat{r})\Phi^{m_j m_s}_{j\pm1/2} = -\Phi^{m_j m_s}_{j\mp1/2}$, the operator actions evaluate to:

$$i(\vec{\alpha}\cdot\hat{r})\psi^{(+)}_{jm_jm_s} = -\psi^{(-)}_{jm_jm_s}, \qquad i(\vec{\alpha}\cdot\hat{r})\psi^{(-)}_{jm_jm_s} = \psi^{(+)}_{jm_jm_s}, \tag{27.232}$$

$$\beta Q\psi^{(+)}_{jm_jm_s} = -\left(j+\frac{1}{2}\right)\psi^{(+)}_{jm_jm_s}, \qquad \beta Q\psi^{(-)}_{jm_jm_s} = \left(j+\frac{1}{2}\right)\psi^{(-)}_{jm_jm_s}, \tag{27.233}$$

$$m\beta\psi^{(+)}_{jm_jm_s} = m\psi^{(+)}_{jm_jm_s}, \qquad m\beta\psi^{(-)}_{jm_jm_s} = -m\psi^{(-)}_{jm_jm_s}. \tag{27.234}$$

27.10.4 Coupled Radial ODEs, Decoupling, and Spherical Bessel Solutions

We adopt the partial-wave expansion ansatz:

$$\psi_{jm_jm_s}(\vec{r},t) = e^{-i\omega t}\left[A(r)\,\psi^{(+)}_{jm_jm_s}(\Omega) + B(r)\,\psi^{(-)}_{jm_jm_s}(\Omega)\right]. \tag{27.235}$$

Let $k \equiv j+1/2$. The time derivative is $i\partial_t\psi = \omega e^{-i\omega t}[A\psi^{(+)} + B\psi^{(-)}]$. Applying H_0:

$$\begin{aligned} H_0\psi &= \left[-i(\vec{\alpha}\cdot\hat{r})\left(\partial_r + \frac{1}{r}\right) - \frac{1}{r}\beta Q + m\beta\right] e^{-i\omega t}\left[A(r)\psi^{(+)}(\Omega) + B(r)\psi^{(-)}(\Omega)\right] \\ &= e^{-i\omega t}\left\{\left[-\partial_r B(r) - \frac{1-k}{r}B(r) + mA(r)\right]\psi^{(+)}(\Omega)\right. \\ &\qquad \left. + \left[\partial_r A(r) + \frac{1+k}{r}A(r) - mB(r)\right]\psi^{(-)}(\Omega)\right\}. \end{aligned} \tag{27.236}$$

Equating components yields the coupled first-order system:

$$\omega A(r) = -\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{1-k}{r}B(r) + mA(r) \implies \frac{\mathrm{d}A}{\mathrm{d}r} + \frac{1+k}{r}A(r) = (m+\omega)B(r), \tag{27.237}$$

$$\omega B(r) = \frac{\mathrm{d}A}{\mathrm{d}r} + \frac{1+k}{r}A(r) - mB(r) \implies \frac{\mathrm{d}B}{\mathrm{d}r} + \frac{1-k}{r}B(r) = (m-\omega)A(r). \tag{27.238}$$

To decouple the system, we differentiate Equation (27.237) with respect to r:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}A}{\mathrm{d}r} + \frac{1+k}{r}A\right] &= (m+\omega)\frac{\mathrm{d}B}{\mathrm{d}r} \\ \frac{\mathrm{d}^2A}{\mathrm{d}r^2} + \frac{1+k}{r}\frac{\mathrm{d}A}{\mathrm{d}r} - \frac{1+k}{r^2}A &= (m+\omega)\left[(m-\omega)A - \frac{1-k}{r}B\right] \\ &= (m^2-\omega^2)A - \frac{1-k}{r}(m+\omega)B \\ &= (m^2-\omega^2)A - \frac{1-k}{r}\left[\frac{\mathrm{d}A}{\mathrm{d}r} + \frac{1+k}{r}A\right] \\ &= (m^2-\omega^2)A - \frac{1-k}{r}\frac{\mathrm{d}A}{\mathrm{d}r} - \frac{1-k^2}{r^2}A. \end{aligned} \tag{27.239}$$

Collecting like terms:

$$\frac{\mathrm{d}^2A}{\mathrm{d}r^2} + \left(\frac{1+k}{r} + \frac{1-k}{r}\right)\frac{\mathrm{d}A}{\mathrm{d}r} + \left(\frac{1-k^2}{r^2} - \frac{1+k}{r^2}\right)A + (\omega^2-m^2)A = 0, \tag{27.240}$$

which simplifies to the standard spherical Bessel differential equation:

$$\frac{\mathrm{d}^2A}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}A}{\mathrm{d}r} - \frac{k(1+k)}{r^2}A + (\omega^2-m^2)A = 0. \tag{27.241}$$

Repeating the identical decoupling procedure for $B(r)$:

$$\frac{\mathrm{d}^2B}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}B}{\mathrm{d}r} + \frac{k(1-k)}{r^2}B + (\omega^2-m^2)B = 0. \tag{27.242}$$

Setting $p \equiv \sqrt{\omega^2 - m^2}$, the solutions regular at the origin $r = 0$ are:

$$\text{For } k > 0: \quad A(r) = j_k(pr), \qquad B(r) = j_{k-1}(pr), \tag{27.243}$$

$$\text{For } k < 0: \quad A(r) = j_{-k-1}(pr), \quad B(r) = j_{-k}(pr). \tag{27.244}$$

The general wave function expansion is:

$$\psi(\vec{r}, t) = \int \mathrm{d}p\, e^{\pm i\omega t} \sqrt{\frac{2}{\pi}} p \sum_{j, m_j, m_s} \left[b\, j_{k_1}(pr)\, \psi^{(+)}_{j m_j m_s}(\Omega) + d\, j_{k_2}(pr)\, \psi^{(-)}_{j m_j m_s}(\Omega) \right], \tag{27.245}$$

where $(k_1, k_2) = (k, k-1)$ for $k > 0$ and $(-k-1, -k)$ for $k < 0$. The orthonormality relations for the angular and radial basis functions are:

$$\int \psi^{(\pm)\dagger}_{j m_j m_s}(\Omega)\, \psi^{(\pm)}_{j' m'_j m'_s}(\Omega)\, \mathrm{d}\Omega = \delta_{jj'} \delta_{m_j m'_j} \delta_{m_s m'_s}, \tag{27.246}$$

$$\int_0^\infty r^2 j_\ell(pr)\, j_{\ell'}(p'r)\, \mathrm{d}r = \frac{\pi}{2p^2} \delta(p - p') \delta_{\ell\ell'}. \tag{27.247}$$

27.11 Canonical Quantization of Fields in Spherical Coordinates

27.11.1 Canonical Quantization of Dirac Fields

We define the orthonormal spherical mode wavefunctions:

$$u_{j m_j m_s}(p, r, \Omega) \equiv \sqrt{\frac{2}{\pi}} p\, j_{k_1}(pr)\, \psi^{(+)}_{j m_j m_s}(\Omega), \qquad v_{j m_j m_s}(p, r, \Omega) \equiv \sqrt{\frac{2}{\pi}} p\, j_{k_2}(pr)\, \psi^{(-)}_{j m_j m_s}(\Omega), \tag{27.248}$$

satisfying the complete orthonormality relations:

$$\int \mathrm{d}^3x \sqrt{-g}\, u^\dagger_A(p) u_{A'}(p') = \delta(p - p') \delta_{AA'}, \qquad \int \mathrm{d}^3x \sqrt{-g}\, v^\dagger_A(p) v_{A'}(p') = \delta(p - p') \delta_{AA'}, \tag{27.249}$$

where $A \equiv \{j, m_j, m_s\}$ and $\mathrm{d}^3x\sqrt{-g} = r^2 \sin\theta\, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$. The quantized field operator $\hat{\psi}(\vec{r}, t)$ is:

$$\hat{\psi}(\vec{r}, t) = \int \mathrm{d}p \sum_{j, m_j, m_s} \left[e^{-i\omega_p t} u_{j m_j m_s}(p)\, \hat{b}_{j m_j m_s}(p) + e^{i\omega_p t} v_{j m_j m_s}(p)\, \hat{d}^\dagger_{j m_j m_s}(p) \right], \tag{27.250}$$

where $\hat{b}, \hat{b}^\dagger$ and $\hat{d}, \hat{d}^\dagger$ are the fermion creation and annihilation operators satisfying the Canonical Anticommutation Relations (CAR):

$$\left\{ \hat{b}_A(p), \hat{b}^\dagger_{A'}(p') \right\} = \delta(p - p')\, \delta_{AA'}, \qquad \left\{ \hat{d}_A(p), \hat{d}^\dagger_{A'}(p') \right\} = \delta(p - p')\, \delta_{AA'}, \tag{27.251}$$

$$\left\{ \hat{b}_A(p), \hat{d}^\dagger_{A'}(p') \right\} = 0, \qquad \left\{ \hat{b}^\dagger_A(p), \hat{d}_{A'}(p') \right\} = 0. \tag{27.252}$$

The conjugate momentum operator is $\hat{\pi}(\vec{r}, t) = i\hat{\psi}^\dagger(\vec{r}, t)$.

Theorem: Equal-Time Anticommutator and Normal-Ordered Hamiltonian

The field operator and its conjugate momentum satisfy the equal-time canonical anticommutation relations:

$$\left\{ \hat{\psi}(\vec{r}, t), \hat{\pi}(\vec{r}', t) \right\} = i\, \delta^{(3)}(\vec{r} - \vec{r}') = i\, \frac{\delta(r - r')}{r^2}\, \delta(\Omega - \Omega'). \tag{27.253}$$

The total normal-ordered Hamiltonian operator is:

$$:\hat{H}: = \int_0^\infty \mathrm{d}p\, \omega_p \sum_{j,m_j,m_s} \left[\hat{b}^\dagger_{jm_jm_s}(p)\, \hat{b}_{jm_jm_s}(p) + \hat{d}^\dagger_{jm_jm_s}(p)\, \hat{d}_{jm_jm_s}(p)\right]. \tag{27.254}$$

Proof. Evaluating the equal-time anticommutator directly:

$$\begin{aligned}\left\{\hat{\psi}(\vec{r},t), \hat{\pi}(\vec{r}',t)\right\} &= i\left\{\hat{\psi}(\vec{r},t), \hat{\psi}^\dagger(\vec{r}',t)\right\} \\ &= i\int \mathrm{d}p\mathrm{d}p' \sum_{A,A'} \Big[e^{-i(\omega-\omega')t}(u\hat{b}\hat{b}'^\dagger u'^\dagger + u'^\dagger \hat{b}'^\dagger \hat{b} u) \\ &\qquad + e^{i(\omega+\omega')t}(u\hat{b}\hat{d}' v'^\dagger + \dots) + e^{-i(\omega+\omega')t}(v\hat{d}^\dagger \hat{b}'^\dagger u'^\dagger + \dots) \\ &\qquad + e^{i(\omega-\omega')t}(v\hat{d}^\dagger \hat{d}' v'^\dagger + v'^\dagger \hat{d}'\hat{d}^\dagger v)\Big].\end{aligned} \tag{27.255}$$

Using the anticommutation relations (27.251)–(27.252) sets $p' = p$ and $A' = A$:

$$\begin{aligned}\left\{\hat{\psi}(\vec{r},t), \hat{\pi}(\vec{r}',t)\right\} &= i\frac{\delta(r-r')}{r^2}\int \mathrm{d}p \sum_A \left(\{\hat{b}_A, \hat{b}^\dagger_A\} + \{\hat{d}_A, \hat{d}^\dagger_A\}\right)\delta_{AA'} \\ &= i\frac{\delta(r-r')}{r^2}\delta(\Omega-\Omega') = i\delta^{(3)}(\vec{r}-\vec{r}').\end{aligned} \tag{27.256}$$

For the Hamiltonian density $\mathcal{H} = \hat{\pi}\dot{\hat{\psi}} - \mathcal{L} = i\hat{\psi}^\dagger \partial_t \hat{\psi}$:

$$\begin{aligned}H &= \int \mathrm{d}^3x\sqrt{-g}\, i\hat{\psi}^\dagger \partial_t \hat{\psi} \\ &= \int \mathrm{d}^3x\mathrm{d}p\mathrm{d}p' \sum_{A,A'} \omega_{p'} \left(e^{i\omega_p t}\hat{b}^\dagger_A u^\dagger_A + e^{-i\omega_p t}\hat{d}_A v^\dagger_A\right)\left(e^{-i\omega_{p'}t}\hat{b}_{A'}u_{A'} - e^{i\omega_{p'}t}\hat{d}^\dagger_{A'}v_{A'}\right) \\ &= \int \mathrm{d}p\, \omega_p \sum_A \left[\hat{b}^\dagger_A(p)\hat{b}_A(p) - \hat{d}_A(p)\hat{d}^\dagger_A(p)\right].\end{aligned} \tag{27.257}$$

Normal ordering places creation operators on the left:

$$:\hat{H}: = \int \mathrm{d}p\, \omega_p \sum_{j,m_j,m_s} \left[\hat{b}^\dagger_{jm_jm_s}(p)\, \hat{b}_{jm_jm_s}(p) + \hat{d}^\dagger_{jm_jm_s}(p)\, \hat{d}_{jm_jm_s}(p)\right]. \tag{27.258}$$

□

27.11.2 Canonical Quantization of Scalar Fields in Spherical Coordinates

For a real massive scalar field $\phi(t,\vec{x})$, the action is:

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}, \qquad \mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2. \tag{27.259}$$

The Euler–Lagrange equation is $\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + M^2\phi = 0$. In spherical coordinates:

$$-\partial_t^2\phi + \frac{1}{r^2}\partial_r(r^2\partial_r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\phi - M^2\phi = 0. \tag{27.260}$$

Separation of variables $\phi = T(t)R(r)Y(\theta,\phi)$ yields $T(t) = e^{\pm i\omega_p t}$, $Y(\theta,\phi) = Y_{\ell m}(\Omega)$, and $R(r) = j_\ell(pr)$ with $p = \sqrt{\omega_p^2 - M^2}$. The field operator is expanded as:

$$\hat{\phi}(\vec{r},t) = \int_0^\infty \mathrm{d}p \sum_{\ell,m} N_p\, j_\ell(pr)\left[Y_{\ell m}(\Omega)\,\hat{a}_{p\ell m}\,e^{-i\omega_p t} + Y^*_{\ell m}(\Omega)\,\hat{a}^\dagger_{p\ell m}\,e^{i\omega_p t}\right], \tag{27.261}$$

with conjugate momentum $\hat{\pi}(\vec{r},t)=\dot{\hat{\phi}}(\vec{r},t)$. Demanding $[\hat{\phi}(\vec{r},t),\hat{\pi}(\vec{r}',t)]=i\delta^{(3)}(\vec{r}-\vec{r}')$ fixes the normalization:

$$N_p=\frac{1}{\sqrt{2\omega_p}}. \tag{27.262}$$

The Hamiltonian density is $\mathcal{H}=\frac{1}{2}(\hat{\pi}^2+(\vec{\nabla}\hat{\phi})^2+M^2\hat{\phi}^2)=\frac{1}{2}(\hat{\pi}^2-\hat{\phi}\ddot{\hat{\phi}})$. Integrating over space yields:

$$H=\int \mathrm{d}^3x\sqrt{-g}\,\mathcal{H}=\int_0^\infty \mathrm{d}p\,\omega_p\sum_{\ell,m}\left[\hat{a}^\dagger_{p\ell m}\hat{a}_{p\ell m}+\frac{1}{2}\delta(0)\right], \tag{27.263}$$

and normal ordering removes the infinite vacuum zero-point energy:

$$:\hat{H}:=\int_0^\infty \mathrm{d}p\,\omega_p\sum_{\ell,m}\hat{a}^\dagger_{p\ell m}\hat{a}_{p\ell m}. \tag{27.264}$$

27.11.3 Canonical Quantization of Gauge Fields in Spherical Coordinates

For the Maxwell electromagnetic gauge field, the action is:

$$S=-\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu},\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{27.265}$$

yielding the equations of motion $\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta})=0$. In the temporal gauge $A_t=0$, we adopt the multipole expansion:

$$A_t=0, \tag{27.266}$$
$$A_r=h(r,t)\,Y_{\ell m}(\Omega), \tag{27.267}$$
$$A_\theta=k(r,t)\,\partial_\theta Y_{\ell m}(\Omega)+\frac{a(r,t)}{\sin\theta}\partial_\phi Y_{\ell m}(\Omega), \tag{27.268}$$
$$A_\phi=k(r,t)\,\partial_\phi Y_{\ell m}(\Omega)-a(r,t)\sin\theta\,\partial_\theta Y_{\ell m}(\Omega). \tag{27.269}$$

Evaluating the field equations for each component $\nu\in\{0,1,2,3\}$:

1. **Temporal Component ($\nu=0$):**

$$\partial_r(r^2\partial_t h)-\ell(\ell+1)\partial_t k=0. \tag{27.270}$$

2. **Radial Component ($\nu=1$):**

$$r^2\partial_t^2 h+(h-\partial_r k)\ell(\ell+1)=0. \tag{27.271}$$

Combining Equations (27.270) and (27.271) decouples the radial TM function:

$$\partial_r^2(r^2\partial_t h)-\partial_t^2(r^2\partial_t h)-\frac{\ell(\ell+1)}{r^2}(r^2\partial_t h)=0. \tag{27.272}$$

3. **Angular Components ($\nu=2,3$):**

$$\partial_t^2 a-\partial_r^2 a+\frac{\ell(\ell+1)}{r^2}a=0,\qquad \partial_t^2 k+\partial_r h-\partial_r k=0. \tag{27.273}$$

The exact physical radial solutions regular at $r=0$ are:

$$a_{\ell mk}(r,t)=D\sqrt{k}\,r\,j_\ell(kr)\,e^{\pm ikt}, \tag{27.274}$$
$$h_{\ell mk}(r,t)=\mp iC\,\frac{j_\ell(kr)}{\sqrt{k}r}\,e^{\pm ikt}, \tag{27.275}$$

$$k_{\ell mk}(r,t) = \frac{\mp iC}{\ell(\ell+1)\sqrt{k}}\,\partial_r\left(r j_\ell(kr)\right)\,e^{\pm ikt}. \tag{27.276}$$

The canonical commutation relations $[A_j(t,\vec{r}),\Pi_k(t,\vec{r}')] = i\delta_{jk}\delta^{(3)}(\vec{r}-\vec{r}')$ determine the mode normalizations $C = D = \sqrt{\frac{2}{\pi}}$. The normal-ordered Hamiltonian decomposes into transverse electric (TE) and transverse magnetic (TM) photon multipoles:

$$:\hat{H}: = \int_0^\infty \mathrm{d}k\,k \sum_{\ell,m}\left[\hat{a}^\dagger_{E,\ell mk}\hat{a}_{E,\ell mk} + \hat{a}^\dagger_{M,\ell mk}\hat{a}_{M,\ell mk}\right]. \tag{27.277}$$

Chapter 28

Canonical General Relativity: Dirac Constraint Analysis, the Hypersurface-Deformation Algebra, and the Contracted Bianchi Identity

In the preceding chapters, General Relativity was formulated primarily in the four-dimensional spacetime covariant language of differential geometry, culminating in the Einstein–Hilbert variational principle and the geometric field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. However, to address fundamental questions such as the initial-value (Cauchy) problem, the deterministic evolution of gravitational fields in numerical relativity, and the canonical quantization of gravity (e.g., the Wheeler–DeWitt equation and loop quantum gravity), spacetime must be unpacked into a dynamical system evolving in time.

The canonical formulation of General Relativity, initiated by Paul Dirac and completed in the 3+1 foliation by Richard Arnowitt, Stanley Deser, and Charles W. Misner (ADM), reveals that Einstein's theory is a constrained Hamiltonian system with local gauge redundancies. In this chapter, we develop the complete canonical analysis of vacuum General Relativity using the Poisson bracket formalism. We execute Dirac's constraint algorithm step by step, demonstrating that:

1. The vanishing of time derivatives of the lapse function and shift vector produces four primary constraints per spatial point.

2. Requiring the primary constraints to be preserved under dynamical evolution yields four secondary constraints: the scalar (Hamiltonian) constraint and the spatial vector (diffeomorphism / momentum) constraints.

3. The Poisson brackets among these secondary constraints close into the famous **hypersurface-deformation algebra** (the Dirac algebra), whose structure functions depend dynamically on the spatial metric.

4. The algorithm terminates cleanly after the secondary stage: no tertiary constraints appear, and the four Lagrange multipliers remain completely arbitrary gauge functions, leaving precisely two physical propagating degrees of freedom per spatial point (the two polarization states of gravitational radiation).

5. Most profoundly, the closure of the constraint algebra and the termination of the Dirac algorithm are not algebraic accidents. They are the exact canonical manifestations of four-dimensional diffeomorphism invariance and the **contracted Bianchi identity** $\nabla_\mu G^{\mu\nu} = 0$, which acts as an off-shell integrability condition on the dynamical evolution equations.

Throughout this chapter, we work in vacuum General Relativity with metric signature $(-,+,+,+)$, units where $16\pi G = 1$ (such that the Einstein–Hilbert action is $S = \int \mathrm{d}^4x\sqrt{-g}\,R^{(4)}$), and assume either compact Cauchy slices Σ without boundary or asymptotically flat boundary conditions with sufficiently rapid falloff so that all spatial boundary terms integrate to zero.

28.1 Constrained Hamiltonian Systems and Dirac's Algorithm

28.1.1 Singular Lagrangians and Primary Constraints

Consider a classical field theory or mechanical system described by generalized coordinates q^A and velocities $\dot{q}^A$, with action $S = \int \mathrm{d}t\, L(q, \dot{q})$. In standard non-singular Lagrangian mechanics, the canonical momenta

$$p_A \equiv \frac{\partial L}{\partial \dot{q}^A} \tag{28.1}$$

can be uniquely inverted for the velocities $\dot{q}^A = v^A(q,p)$ because the Hessian matrix

$$W_{AB} \equiv \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} = \frac{\partial p_A}{\partial \dot{q}^B} \tag{28.2}$$

is non-singular ($\det W_{AB} \neq 0$). When a theory possesses local gauge symmetries, however, the Lagrangian is necessarily invariant under time-dependent gauge transformations. Consequently, the velocities corresponding to pure gauge directions cannot be uniquely determined from the field configurations, forcing the Hessian matrix to be singular:

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right) = 0. \tag{28.3}$$

Because the map $\dot{q}^A \mapsto p_A$ is not invertible, the canonical momenta are not all independent functions of the velocities. Instead, there exist algebraic relations among the coordinates and momenta that hold without using the equations of motion:

$$\phi_m(q,p) \approx 0, \qquad m = 1, \ldots, M. \tag{28.4}$$

Following Dirac, these relations are called **primary constraints**. The symbol $\approx$ denotes a **weak equality**: the constraint $\phi_m \approx 0$ holds on the constrained submanifold of phase space (the constraint surface Γ_c), but its Poisson brackets with other phase-space functionals must be evaluated *before* setting $\phi_m = 0$.

Definition: Weak and Strong Equalities

An equality $F \approx 0$ is **weak** if $F(q,p)$ vanishes on the constraint surface Γ_c defined by $\phi_m = 0$, but has non-vanishing gradients off Γ_c:

$$F|_{\Gamma_c} = 0, \quad \text{while} \quad \mathrm{d}F \neq 0. \tag{28.5}$$

An equality $F = 0$ is **strong** if F and all its phase-space derivatives vanish on Γ_c, which implies $F = \sum_m c^m(q,p)\phi_m$ with c^m vanishing on Γ_c, or if it holds identically throughout the entire unconstrained phase space.

28.1.2 The Total Hamiltonian and Consistency Conditions

The canonical Hamiltonian obtained via the standard Legendre transform,

$$H_c \equiv p_A \dot{q}^A - L(q,\dot{q}), \tag{28.6}$$

is defined only up to linear combinations of the primary constraints, because any shift $H_c \to H_c + u^m(q,p)\phi_m$ produces identical physical dynamics on Γ_c. To account for all possible unconstrained variations, Dirac introduced the **total Hamiltonian**:

$$H_T \equiv H_c + \lambda^m \phi_m, \tag{28.7}$$

where $\lambda^m(t)$ are arbitrary Lagrange multipliers. The canonical equations of motion for any phase-space observable $F(q,p)$ are generated by Poisson brackets with H_T:

$$\dot{F} = \{F, H_T\} = \{F, H_c\} + \lambda^m \{F, \phi_m\}. \tag{28.8}$$

For the physical theory to be mathematically consistent, the primary constraints must be preserved dynamically for all times. If a system starts on the constraint surface Γ_c, it must not leave it:

$$\dot{\phi}_m = \{\phi_m, H_T\} = \{\phi_m, H_c\} + \lambda^n \{\phi_m, \phi_n\} \approx 0. \tag{28.9}$$

Equation (28.9) presents four mutually exclusive possibilities for each m:

(i) The bracket vanishes identically or reduces to a linear combination of primary constraints ($\{\phi_m, H_T\} = c_m{}^k \phi_k \approx 0$). In this case, the condition is trivially satisfied and imposes no new restrictions.

(ii) The condition cannot be satisfied for any choice of variables, e.g., $1 \approx 0$. This signals that the original Lagrangian equations of motion were inconsistent and have no physical solutions.

(iii) The condition determines some of the previously arbitrary Lagrange multipliers λ^m.

(iv) The condition produces a new relation among coordinates and momenta that is independent of the multipliers λ^m and not a combination of existing primary constraints:

$$\chi_k(q,p) \approx 0. \tag{28.10}$$

Such relations are termed **secondary constraints**.

28.1.3 First-Class vs. Second-Class Constraints

If secondary constraints $\chi_k \approx 0$ appear, Dirac's consistency algorithm must be iterated: we demand $\dot{\chi}_k = \{\chi_k, H_T\} \approx 0$. This may in turn fix additional multipliers, or generate **tertiary constraints**, which then must be preserved, and so forth, until the algorithm terminates.

Once all constraints $\{\Phi_a\} = \{\phi_m, \chi_k, \dots\}$ have been collected, they are classified according to their Poisson bracket algebra:

Definition: First-Class and Second-Class Constraints

A constraint Φ_a is **first-class** if its Poisson bracket with every constraint Φ_b in the theory vanishes weakly on the constraint surface:

$$\{\Phi_a, \Phi_b\} \approx 0 \qquad \forall\, b. \tag{28.11}$$

Otherwise, if there exists at least one constraint with which Φ_a has a non-vanishing bracket on the constraint surface, Φ_a is **second-class**.

First-class constraints are the generators of local gauge symmetries. Each first-class constraint removes *two* dimensions from the physical phase space (one dimension restricted by the constraint itself, and one redundant gauge degree of freedom generated by the bracket). Second-class

constraints, on the other hand, represent unphysical or auxiliary variables; they form conjugate pairs with an invertible constraint matrix $\Delta_{ab} = \{\Phi_a, \Phi_b\}$, each pair removing *two* phase-space dimensions. Second-class constraints can be eliminated identically by replacing Poisson brackets with the **Dirac bracket**:

$$\{A, B\}_{\mathrm{D}} \equiv \{A, B\} - \{A, \Phi_a\}\,(\Delta^{-1})^{ab}\,\{\Phi_b, B\}. \tag{28.12}$$

In the Dirac bracket, all second-class constraints become strong operator identities: $\{\Phi_a, B\}_{\mathrm{D}} \equiv 0$ for all B.

28.2 The 3+1 Foliation and the ADM Gravitational Action

28.2.1 Geometry of the Hypersurface Decomposition

To express the four-dimensional Einstein–Hilbert action in canonical form, spacetime $(\mathcal{M}, g_{\mu\nu})$ is foliated by a one-parameter family of spacelike three-dimensional hypersurfaces Σ_t labelled by a global time coordinate $t = x^0$:

$$\mathcal{M} \cong \mathbb{R} \times \Sigma_t. \tag{28.13}$$

Let $t^\mu = (\partial/\partial t)^\mu = \delta^\mu_0$ denote the time-evolution vector field connecting points with identical spatial coordinates x^i on adjacent slices Σ_t and $\Sigma_{t+\mathrm{d}t}$. The gradient 1-form of the foliation is $\omega_\mu = \partial_\mu t = (1, 0, 0, 0)$, which is timelike since the slices are spacelike. The future-directed unit timelike normal vector to Σ_t is defined by

$$n_\mu \equiv -N\,\partial_\mu t = (-N, 0, 0, 0), \qquad g^{\mu\nu} n_\mu n_\nu = -1, \tag{28.14}$$

where $N(t, x^i) > 0$ is the **lapse function**. The lapse measures the lapse of proper time $\mathrm{d}\tau = N\,\mathrm{d}t$ between adjacent slices along the normal direction.

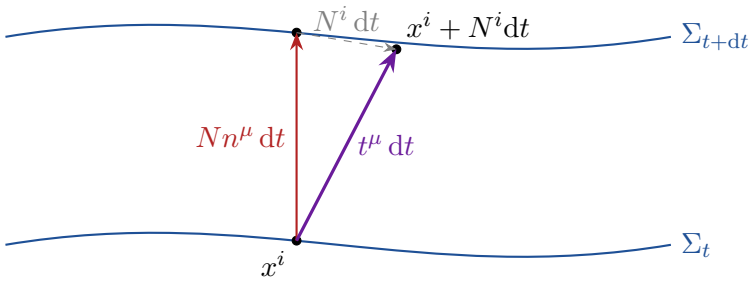


Figure 28.1: The 3+1 geometric decomposition of spacetime. The time vector $t^\mu \partial_\mu = \partial_t$ is decomposed into a normal component $N n^\mu$ parameterized by the lapse function N, and a tangential component N^i parameterized by the shift vector.

The time evolution vector t^μ decomposes into parts orthogonal and tangential to Σ_t:

$$t^\mu = N n^\mu + N^\mu, \qquad n_\mu N^\mu = 0, \tag{28.15}$$

where $N^\mu = (0, N^i)$ is the **shift vector**. The shift vector describes how the spatial coordinate grid shifts tangentially along the slice during the time evolution $\mathrm{d}t$.

28.2.2 Metric and Extrinsic Curvature

The spacetime metric induces a Riemannian 3-metric γ_{ij} on each slice Σ_t. The full four-dimensional metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are expressed in terms of the ADM variables (N, N^i, γ_{ij}) by:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$$

$$= (-N^2 + \gamma_{kl}N^k N^l)\mathrm{d}t^2 + 2\gamma_{ij}N^j\,\mathrm{d}t\,\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{28.16}$$

Comparing components gives:

$$g_{00} = -N^2 + \gamma_{kl}N^k N^l, \qquad g_{0i} = \gamma_{ij}N^j \equiv N_i, \qquad g_{ij} = \gamma_{ij}. \tag{28.17}$$

Spatial indices $i, j, k, \cdots \in \{1, 2, 3\}$ are raised and lowered using the spatial metric γ_{ij} and its inverse γ^{ij}. The contravariant four-metric components are:

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = \frac{N^i}{N^2}, \qquad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2}. \tag{28.18}$$

The unit normal vector in contravariant components reads:

$$n^\mu = g^{\mu\nu}n_\nu = \frac{1}{N}(1, -N^i). \tag{28.19}$$

The determinant of the four-metric factorizes cleanly:

$$\det(g_{\mu\nu}) = -N^2\det(\gamma_{ij}) \implies \sqrt{-g} = N\sqrt{\gamma}, \tag{28.20}$$

where $\gamma \equiv \det(\gamma_{ij})$.

The **extrinsic curvature** K_{ij} of the hypersurface Σ_t measures the rate of change of the spatial metric along the unit normal n^μ, or equivalently, the bending of Σ_t in the embedding four-manifold:

$$K_{ij} \equiv \frac{1}{2}\mathcal{L}_n\gamma_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - D_iN_j - D_jN_i), \tag{28.21}$$

where a dot denotes $\partial_t \equiv \partial/\partial t$, and D_i denotes the torsion-free spatial covariant derivative compatible with γ_{ij} ($D_k\gamma_{ij} = 0$). The trace of the extrinsic curvature is:

$$K \equiv \gamma^{ij}K_{ij} = \frac{1}{2N}\gamma^{ij}(\dot{\gamma}_{ij} - 2D_iN_j) = \nabla_\mu n^\mu. \tag{28.22}$$

28.2.3 The Gauss–Codazzi Curvature Decomposition

The four-dimensional Ricci scalar $R^{(4)}$ is related to the intrinsic three-dimensional Ricci scalar R of Σ_t and its extrinsic curvature through the contracted Gauss–Codazzi equations:

$$R^{(4)} = R + K_{ij}K^{ij} - K^2 - 2\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu). \tag{28.23}$$

Multiplying by the volume measure $\sqrt{-g} = N\sqrt{\gamma}$, the total spacetime divergence integrates to a pure boundary term:

$$\sqrt{-g}\,R^{(4)} = N\sqrt{\gamma}(R + K_{ij}K^{ij} - K^2) - 2\partial_\mu[\sqrt{-g}\,(n^\mu K - a^\mu)], \tag{28.24}$$

where $a^\mu = n^\nu\nabla_\nu n^\mu$ is the four-acceleration of the normal observers. Discarding all boundary terms (as boundary variations vanish under standard Dirichlet conditions or rapid asymptotic falloff), the canonical gravitational action takes the celebrated ADM form:

$$S[N, N^i, \gamma_{ij}] = \int \mathrm{d}t \int \mathrm{d}^3x\,\mathcal{L}_{\mathrm{ADM}}, \tag{28.25}$$

with the Lagrangian density:

$$\mathcal{L}_{\mathrm{ADM}} = N\sqrt{\gamma}\left(R + K_{ij}K^{ij} - K^2\right). \tag{28.26}$$

Physical Insight & Motivation: Absence of Time Derivatives of Lapse and Shift

Inspecting Equation (28.26) together with Equation (28.21) shows that $\dot{\gamma}_{ij}$ appears quadratically inside the extrinsic curvature terms $K_{ij}K^{ij} - K^2$. However, **neither $\dot{N}$ nor $\dot{N}^i$ appears anywhere in $\mathcal{L}_{\mathrm{ADM}}$.**
The lapse N and shift N^i enter the action purely as non-dynamical Lagrange multipliers. This is the exact signature of a singular Lagrangian stemming from local time-reparameterization and spatial diffeomorphism invariance.

28.3 Canonical Momenta and the Legendre Transformation

28.3.1 Primary Constraints

We now introduce canonical momenta conjugate to each of the ten configuration variables (N, N^i, γ_{ij}). Because the Lagrangian density $\mathcal{L}_{\mathrm{ADM}}$ contains no time derivatives of N and N^i, their conjugate momenta vanish identically:

$$\pi_N \equiv \frac{\partial \mathcal{L}_{\mathrm{ADM}}}{\partial \dot{N}} \approx 0, \qquad \pi_i \equiv \frac{\partial \mathcal{L}_{\mathrm{ADM}}}{\partial \dot{N}^i} \approx 0. \tag{28.27}$$

Equations (28.27) constitute **four primary constraints per spatial point**.

28.3.2 Conjugate Momentum of the Spatial Metric

The canonical momentum tensor π^{ij} conjugate to the spatial metric γ_{ij} is obtained by varying with respect to $\dot{\gamma}_{ij}$. From Equation (28.21), we have:

$$\frac{\partial K_{kl}}{\partial \dot{\gamma}_{ij}} = \frac{1}{2N}\delta^{(i}_k \delta^{j)}_l = \frac{1}{4N}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k). \tag{28.28}$$

Varying the kinetic part of the Lagrangian density gives:

$$\begin{aligned} \pi^{ij} &\equiv \frac{\partial \mathcal{L}_{\mathrm{ADM}}}{\partial \dot{\gamma}_{ij}} = \frac{\partial}{\partial \dot{\gamma}_{ij}} \left[N\sqrt{\gamma}(K_{kl}K^{kl} - K^2) \right] \\ &\overset{(1)}{=} N\sqrt{\gamma} \left[2K^{kl}\frac{\partial K_{kl}}{\partial \dot{\gamma}_{ij}} - 2K\gamma^{kl}\frac{\partial K_{kl}}{\partial \dot{\gamma}_{ij}} \right] \\ &\overset{(2)}{=} N\sqrt{\gamma}\,\frac{1}{2N}(2K^{ij} - 2K\gamma^{ij}) = \sqrt{\gamma}(K^{ij} - \gamma^{ij}K). \end{aligned} \tag{28.29}$$

In step (1), we used the chain rule on $K_{kl}K^{kl} = \gamma^{km}\gamma^{ln}K_{kl}K_{mn}$ and $K^2 = (\gamma^{kl}K_{kl})^2$. In step (2), contracting with the symmetrized delta functions yields the result.

Theorem: Inversion of Velocities and Kinetic Quadratic Identities

The canonical momentum π^{ij} can be algebraically inverted to express the velocities $\dot{\gamma}_{ij}$ (via K_{ij}) entirely in terms of canonical variables, yielding:

$$\sqrt{\gamma}\, K_{ij} = \pi_{ij} - \frac{1}{2}\gamma_{ij}\pi, \tag{28.30}$$

where $\pi \equiv \gamma_{ij}\pi^{ij}$ is the trace of the momentum tensor. Furthermore, the kinetic contractions satisfy the exact algebraic identities:

$$\pi^{ij}K_{ij} = \sqrt{\gamma}(K_{ij}K^{ij} - K^2), \tag{28.31}$$

$$\frac{\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2}{\sqrt{\gamma}} = \sqrt{\gamma}(K_{ij}K^{ij} - K^2). \tag{28.32}$$

Proof. Taking the trace of Equation (28.29) by contracting with γ_{ij} (recalling $\gamma_{ij}\gamma^{ij} = 3$ in three spatial dimensions):

$$\pi \equiv \gamma_{ij}\pi^{ij} = \sqrt{\gamma}(K - 3K) = -2\sqrt{\gamma}\,K \implies K = -\frac{\pi}{2\sqrt{\gamma}}. \tag{28.33}$$

Substituting K back into Equation (28.29) and lowering indices:

$$\pi_{ij} = \sqrt{\gamma}K_{ij} - \gamma_{ij}\sqrt{\gamma}\left(-\frac{\pi}{2\sqrt{\gamma}}\right) = \sqrt{\gamma}K_{ij} + \frac{1}{2}\gamma_{ij}\pi, \tag{28.34}$$

which immediately rearranges to Equation (28.30). Because this relationship is algebraically invertible, *no further primary constraints arise from* γ_{ij}.

Next, contracting π^{ij} with K_{ij} gives:

$$\pi^{ij}K_{ij} = \sqrt{\gamma}(K^{ij} - \gamma^{ij}K)K_{ij} = \sqrt{\gamma}(K^{ij}K_{ij} - K^2), \tag{28.35}$$

proving Equation (28.31). Finally, evaluating the momentum square:

$$\pi_{ij}\pi^{ij} = \gamma(K_{ij} - \gamma_{ij}K)(K^{ij} - \gamma^{ij}K) = \gamma(K_{ij}K^{ij} - 2K^2 + 3K^2) = \gamma(K_{ij}K^{ij} + K^2). \tag{28.36}$$

Subtracting $\frac{1}{2}\pi^2 = \frac{1}{2}(-2\sqrt{\gamma}K)^2 = 2\gamma K^2$:

$$\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2 = \gamma(K_{ij}K^{ij} + K^2 - 2K^2) = \gamma(K_{ij}K^{ij} - K^2). \tag{28.37}$$

Dividing by $\sqrt{\gamma}$ confirms Equation (28.32). □

28.3.3 Construction of the Canonical Hamiltonian

We perform the Legendre transform to obtain the canonical Hamiltonian H_c:

$$H_c = \int \mathrm{d}^3x \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_N\dot{N} + \pi_i\dot{N}^i - \mathcal{L}_{\mathrm{ADM}}\right). \tag{28.38}$$

Setting the primary constraints $\pi_N \approx 0$ and $\pi_i \approx 0$, and substituting $\dot{\gamma}_{ij} = 2NK_{ij} + 2D_{(i}N_{j)}$ from Equation (28.21):

$$\begin{aligned} H_c &= \int \mathrm{d}^3x \left[\pi^{ij}(2NK_{ij} + 2D_iN_j) - N\sqrt{\gamma}(R + K_{ij}K^{ij} - K^2)\right] \\ &\overset{(1)}{=} \int \mathrm{d}^3x \left[2N\pi^{ij}K_{ij} + 2\pi^{ij}D_iN_j - N\sqrt{\gamma}R - N\sqrt{\gamma}(K_{ij}K^{ij} - K^2)\right] \\ &\overset{(2)}{=} \int \mathrm{d}^3x \left[2N\sqrt{\gamma}(K_{ij}K^{ij} - K^2) - N\sqrt{\gamma}(K_{ij}K^{ij} - K^2) - N\sqrt{\gamma}R + 2\pi^{ij}D_iN_j\right] \\ &\overset{(3)}{=} \int \mathrm{d}^3x \left[N\left(\frac{\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2}{\sqrt{\gamma}} - \sqrt{\gamma}R\right) + 2\pi^{ij}D_iN_j\right]. \end{aligned} \tag{28.39}$$

In step (1), we used the symmetry $\pi^{ij} = \pi^{ji}$ to write $2\pi^{ij}D_{(i}N_{j)} = 2\pi^{ij}D_iN_j$. In step (2), we substituted the contraction identity (28.31). In step (3), we combined the extrinsic curvature kinetic terms into $N\sqrt{\gamma}(K_{ij}K^{ij} - K^2)$ and applied identity (28.32).

Now we integrate the shift term by parts:

$$2\pi^{ij}D_iN_j = 2D_i(\pi^{ij}N_j) - 2N_jD_i\pi^{ij}. \tag{28.40}$$

Since π^{ij} is a contravariant tensor density of weight $+1$, the quantity $V^i \equiv \pi^{ij} N_j$ is a vector density of weight $+1$. For any vector density of weight $+1$, its covariant divergence equals its ordinary coordinate divergence:

$$D_i V^i = \partial_i V^i. \tag{28.41}$$

Under our boundary conditions, the total divergence $\int \mathrm{d}^3x\, \partial_i(\pi^{ij} N_j) = \oint_{\partial\Sigma} \pi^{ij} N_j \, \mathrm{d}S_i = 0$ vanishes. Therefore:

$$\int \mathrm{d}^3x\, 2\pi^{ij} D_i N_j = -2 \int \mathrm{d}^3x\, N_j D_i \pi^{ij} = \int \mathrm{d}^3x\, N^i \big(-2\gamma_{ij} D_k \pi^{jk} \big). \tag{28.42}$$

Thus, the canonical Hamiltonian simplifies to a linear combination of constraints:

$$H_c = \int \mathrm{d}^3x \left(N\mathcal{H} + N^i \mathcal{H}_i \right), \tag{28.43}$$

where we have defined the **Hamiltonian (scalar) constraint** and the **momentum (vector/diffeomorphism) constraint**:

$$\mathcal{H} \equiv \frac{1}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2 \right) - \sqrt{\gamma} R, \tag{28.44}$$

$$\mathcal{H}_i \equiv -2\gamma_{ij} D_k \pi^{jk} = -2 D_k \pi^k{}_i. \tag{28.45}$$

Key Concept: Linear Dependence on Multipliers and Pure Constraint Hamiltonian

The canonical Hamiltonian H_c is proportional entirely to the lapse N and shift N^i. It contains no independent non-zero term that is free of N or N^i.
When the field equations are satisfied, $H_c \approx 0$ vanishes weakly! This is a universal property of reparameterization-invariant theories: there is no absolute external Newtonian time, and the physical Hamiltonian generates coordinate transformations along the foliation rather than absolute time evolution.

28.4 Dirac's Algorithm: Primary and Secondary Constraints

28.4.1 Phase Space and the Total Hamiltonian

The unconstrained canonical phase space per spatial point is parameterized by twenty variables:

$$\Big(N(x), N^i(x), \gamma_{ij}(x); \quad \pi_N(x), \pi_i(x), \pi^{ij}(x) \Big). \tag{28.46}$$

The non-vanishing fundamental equal-time Poisson brackets are:

$$\{ N(t,x), \pi_N(t,y) \} = \delta^{(3)}(x-y), \tag{28.47}$$

$$\{ N^i(t,x), \pi_j(t,y) \} = \delta^i_j \, \delta^{(3)}(x-y), \tag{28.48}$$

$$\{ \gamma_{ij}(t,x), \pi^{kl}(t,y) \} = \delta^{(k}_i \delta^{l)}_j \, \delta^{(3)}(x-y) = \frac{1}{2}(\delta^k_i \delta^l_j + \delta^l_i \delta^k_j)\, \delta^{(3)}(x-y). \tag{28.49}$$

The **total Hamiltonian** H_T adds the primary constraints with arbitrary Lagrange multipliers $\lambda(t,x)$ and $\lambda^i(t,x)$:

$$H_T = \int \mathrm{d}^3x \left(N\mathcal{H} + N^i \mathcal{H}_i + \lambda \pi_N + \lambda^i \pi_i \right) = \mathcal{H}[N] + \mathcal{H}_i[N^i] + \int \mathrm{d}^3x \left(\lambda \pi_N + \lambda^i \pi_i \right), \tag{28.50}$$

where for any spatial scalar test function $M(x)$ and vector test field $M^i(x)$, we introduce the smeared constraints:

$$\mathcal{H}[M] \equiv \int \mathrm{d}^3x\, M(x) \mathcal{H}(x), \qquad \mathcal{H}_i[M^i] \equiv \int \mathrm{d}^3x\, M^i(x) \mathcal{H}_i(x). \tag{28.51}$$

28.4.2 Generation of Secondary Constraints

We now demand that the four primary constraints $\pi_N \approx 0$ and $\pi_i \approx 0$ be preserved in time. Using Equation (28.8) and the fundamental brackets:

$$\dot{\pi}_N(x) = \{\pi_N(x), H_T\} = \{\pi_N(x), \mathcal{H}[N]\} = -\frac{\delta \mathcal{H}[N]}{\delta N(x)} = -\mathcal{H}(x) \approx 0, \tag{28.52}$$

$$\dot{\pi}_i(x) = \{\pi_i(x), H_T\} = \{\pi_i(x), \mathcal{H}_j[N^j]\} = -\frac{\delta \mathcal{H}_j[N^j]}{\delta N^i(x)} = -\mathcal{H}_i(x) \approx 0. \tag{28.53}$$

Because $\mathcal{H}$ and $\mathcal{H}_i$ depend only on γ_{ij} and π^{ij}, and not on the primary momenta π_N, π_i, the multipliers λ and λ^i do not enter Equations (28.52) and (28.53).

Consequently, preserving the primary constraints does not determine the multipliers. Instead, it forces four new **secondary constraints** to hold at every point on the spatial slice:

$$\mathcal{H}(x) \approx 0, \qquad \mathcal{H}_i(x) \approx 0. \tag{28.54}$$

These are the famous **Hamiltonian constraint** and **momentum constraints** of General Relativity.

28.5 The Hypersurface-Deformation Algebra and Termination of Dirac's Algorithm

28.5.1 Consistency Conditions on Secondary Constraints

The second stage of Dirac's algorithm demands that the secondary constraints themselves remain zero under dynamical evolution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[M] = \{\mathcal{H}[M], H_T\} \approx 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_i[M^i] = \{\mathcal{H}_i[M^i], H_T\} \approx 0, \tag{28.55}$$

for arbitrary fixed test functions $M(x)$ and $M^i(x)$. Because neither $\mathcal{H}[M]$ nor $\mathcal{H}_i[M^i]$ depends on N, N^i or their conjugate momenta π_N, π_i, their brackets with the multiplier terms $\int \mathrm{d}^3x(\lambda\pi_N + \lambda^i\pi_i)$ vanish identically. The consistency conditions reduce entirely to brackets with the canonical Hamiltonian $H_c = \mathcal{H}[N] + \mathcal{H}_i[N^i]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[M] = \{\mathcal{H}[M], \mathcal{H}[N]\} + \{\mathcal{H}[M], \mathcal{H}_i[N^i]\}, \tag{28.56}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_i[M^i] = \{\mathcal{H}_i[M^i], \mathcal{H}[N]\} + \{\mathcal{H}_i[M^i], \mathcal{H}_j[N^j]\}. \tag{28.57}$$

Whether new tertiary constraints appear, or whether the multipliers λ, λ^i are fixed, is decided exclusively by the Poisson brackets of the secondary constraints among themselves.

28.5.2 Spatial Diffeomorphism Algebra: $\{\mathcal{H}_i[M^i], \mathcal{H}_j[N^j]\}$

The geometric meaning of the momentum constraint $\mathcal{H}_i$ is revealed by examining its action on phase-space functionals. We express the smeared momentum constraint by integrating by parts:

$$\begin{aligned}\mathcal{H}_i[M^i] &= -2\int \mathrm{d}^3x\, M_j D_k \pi^{jk} \overset{(1)}{=} 2\int \mathrm{d}^3x\, \pi^{jk} D_k M_j \\ &\overset{(2)}{=} \int \mathrm{d}^3x\, \pi^{jk}(D_j M_k + D_k M_j) = \int \mathrm{d}^3x\, \pi^{jk}\mathcal{L}_M \gamma_{jk}.\end{aligned} \tag{28.58}$$

In (1), we integrated by parts and dropped the surface term. In (2), the symmetry $\pi^{jk} = \pi^{kj}$ allows us to replace $2D_kM_j$ with the Killing-type symmetrized derivative $D_jM_k + D_kM_j = \mathcal{L}_M\gamma_{jk}$,

where $\mathcal{L}_M$ is the three-dimensional Lie derivative along the spatial vector field $M = M^i\partial_i$. Expanding the Lie derivative in coordinate derivatives:

$$\mathcal{H}_i[M^i] = \int \mathrm{d}^3x\, \pi^{jk}(M^l\partial_l\gamma_{jk} + \gamma_{lk}\partial_j M^l + \gamma_{jl}\partial_k M^l) = \int \mathrm{d}^3x\, \pi^{jk}(M^l\partial_l\gamma_{jk} + 2\gamma_{lk}\partial_j M^l). \tag{28.59}$$

The functional derivatives with respect to the canonical variables are:

$$\frac{\delta\mathcal{H}_i[M^i]}{\delta\pi^{jk}} = \mathcal{L}_M\gamma_{jk}, \tag{28.60}$$

$$\frac{\delta\mathcal{H}_i[M^i]}{\delta\gamma_{jk}} = -\partial_l(M^l\pi^{jk}) + \pi^{lk}\partial_l M^j + \pi^{jl}\partial_l M^k = -\mathcal{L}_M\pi^{jk}, \tag{28.61}$$

where $\mathcal{L}_M\pi^{jk}$ is the Lie derivative of a contravariant tensor density of weight $+1$:

$$\mathcal{L}_M\pi^{jk} = M^l\partial_l\pi^{jk} - \pi^{lk}\partial_l M^j - \pi^{jl}\partial_l M^k + \pi^{jk}\partial_l M^l = \partial_l(M^l\pi^{jk}) - \pi^{lk}\partial_l M^j - \pi^{jl}\partial_l M^k. \tag{28.62}$$

From Equations (28.60) and (28.61), the Poisson brackets with the canonical variables are:

$$\{\gamma_{jk}, \mathcal{H}_i[M^i]\} = \mathcal{L}_M\gamma_{jk}, \qquad \{\pi^{jk}, \mathcal{H}_i[M^i]\} = \mathcal{L}_M\pi^{jk}. \tag{28.63}$$

Hence, for any functional $F[\gamma_{jk}, \pi^{lm}]$, $\mathcal{H}_i[M^i]$ is the **infinitesimal generator of spatial diffeomorphisms** along the vector field M^i.

Applying this to the momentum constraint functional $\mathcal{H}_i[N^i]$ itself, whose local density $\mathcal{H}_i$ transforms as a covariant vector density of weight $+1$:

$$\mathcal{L}_M\mathcal{H}_i = M^l\partial_l\mathcal{H}_i + \mathcal{H}_l\partial_i M^l + \mathcal{H}_i\partial_l M^l = \partial_l(M^l\mathcal{H}_i) + \mathcal{H}_l\partial_i M^l. \tag{28.64}$$

Taking the bracket of two smeared momentum constraints:

$$\begin{aligned}\{\mathcal{H}_i[M^i], \mathcal{H}_j[N^j]\} &= \int \mathrm{d}^3x\, M^i\mathcal{L}_N\mathcal{H}_i = \int \mathrm{d}^3x\, M^i[\partial_l(N^l\mathcal{H}_i) + \mathcal{H}_l\partial_i N^l] \\ &\overset{(1)}{=} \int \mathrm{d}^3x \left[-N^l\mathcal{H}_i\partial_l M^i + M^l\mathcal{H}_i\partial_l N^i\right] \\ &= \int \mathrm{d}^3x\, (M^l\partial_l N^i - N^l\partial_l M^i)\mathcal{H}_i = \mathcal{H}_i[[M,N]^i].\end{aligned} \tag{28.65}$$

In (1), we integrated the divergence term by parts and renamed dummy indices. Thus, the momentum constraints form a closed Lie subalgebra isomorphic to the Lie algebra of spatial vector fields:

$$\{\mathcal{H}_i[M^i], \mathcal{H}_j[N^j]\} = \mathcal{H}_i[\mathcal{L}_M N^i]. \tag{28.66}$$

28.5.3 Mixed Bracket: $\{\mathcal{H}[M], \mathcal{H}_i[N^i]\}$

The Hamiltonian constraint density $\mathcal{H}$ is constructed from γ_{ij}, R, and π^{ij} as a scalar density of weight $+1$. Under a spatial diffeomorphism along N^l, its Lie derivative is:

$$\mathcal{L}_N\mathcal{H} = N^l\partial_l\mathcal{H} + \mathcal{H}\partial_l N^l = \partial_l(N^l\mathcal{H}). \tag{28.67}$$

Because the smearing function $M(x)$ in $\mathcal{H}[M]$ is held fixed under the Poisson bracket, the dragging acts entirely on the field density $\mathcal{H}$:

$$\begin{aligned}\{\mathcal{H}[M], \mathcal{H}_i[N^i]\} &= \int \mathrm{d}^3x\, M(x)\,\mathcal{L}_N\mathcal{H} = \int \mathrm{d}^3x\, M\,\partial_l(N^l\mathcal{H}) \\ &\overset{(1)}{=} -\int \mathrm{d}^3x\, (N^l\partial_l M)\mathcal{H} = -\mathcal{H}[N^l\partial_l M].\end{aligned} \tag{28.68}$$

In (1), integration by parts moves the spatial derivative onto the test function M. Anti-symmetry of the Poisson bracket immediately gives:

$$\{\mathcal{H}_i[N^i], \mathcal{H}[M]\} = \mathcal{H}[N^l\partial_l M] = \mathcal{H}[\mathcal{L}_N M]. \tag{28.69}$$

28.5.4 Hamiltonian-Hamiltonian Bracket: $\{\mathcal{H}[M], \mathcal{H}[N]\}$

The Poisson bracket of two Hamiltonian constraints requires the variation of the spatial curvature term. Under an arbitrary variation of the spatial metric $\delta\gamma_{ij}$, the Palatini variation of the three-dimensional Ricci scalar is:

$$\delta R = -R^{ij}\delta\gamma_{ij} + D^i D^j \delta\gamma_{ij} - D_k D^k(\gamma^{ij}\delta\gamma_{ij}). \tag{28.70}$$

Together with $\delta\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\gamma^{ij}\delta\gamma_{ij}$, the variation of the curvature density $\int \mathrm{d}^3x\, M\sqrt{\gamma}R$ yields, after integrating by parts twice:

$$\frac{\delta}{\delta\gamma_{ij}}\int \mathrm{d}^3x\, M\sqrt{\gamma}R = \sqrt{\gamma}M\left(\frac{1}{2}\gamma^{ij}R - R^{ij}\right) + \sqrt{\gamma}(D^iD^jM - \gamma^{ij}D_kD^kM). \tag{28.71}$$

The kinetic part of $\mathcal{H}$ contains no spatial derivatives of γ_{ij}, so its variation with respect to γ_{ij} is simply $M(x)$ multiplied by an ultralocal expression $A^{ij}(\gamma,\pi)$ involving no derivatives of M:

$$\frac{\delta\mathcal{H}[M]}{\delta\gamma_{ij}} = MA^{ij} - \sqrt{\gamma}(D^iD^jM - \gamma^{ij}D_kD^kM). \tag{28.72}$$

Meanwhile, the functional derivative of $\mathcal{H}[N]$ with respect to the momentum is:

$$\frac{\delta\mathcal{H}[N]}{\delta\pi^{ij}} = 2N\frac{\pi_{ij} - \frac{1}{2}\gamma_{ij}\pi}{\sqrt{\gamma}}. \tag{28.73}$$

When evaluating the Poisson bracket

$$\{\mathcal{H}[M],\mathcal{H}[N]\} = \int \mathrm{d}^3x\left(\frac{\delta\mathcal{H}[M]}{\delta\gamma_{ij}}\frac{\delta\mathcal{H}[N]}{\delta\pi^{ij}} - \frac{\delta\mathcal{H}[M]}{\delta\pi^{ij}}\frac{\delta\mathcal{H}[N]}{\delta\gamma_{ij}}\right), \tag{28.74}$$

the terms involving $MNA^{ij}(\pi_{ij}-\frac{1}{2}\gamma_{ij}\pi)$ are completely symmetric under the interchange $M \leftrightarrow N$ and cancel identically in the anti-symmetric bracket. Only the second-derivative terms survive:

$$\begin{aligned}
\{\mathcal{H}[M],\mathcal{H}[N]\} &= -2\int \mathrm{d}^3x\Big[N(D^iD^jM - \gamma^{ij}D_kD^kM) \\
&\qquad - M(D^iD^jN - \gamma^{ij}D_kD^kN)\Big](\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\pi) \\
&\overset{(1)}{=} -2\int \mathrm{d}^3x\,\pi^{ij}(ND_iD_jM - MD_iD_jN) \\
&\overset{(2)}{=} 2\int \mathrm{d}^3x\left[D_i(N\pi^{ij})D_jM - D_i(M\pi^{ij})D_jN\right] \\
&\overset{(3)}{=} 2\int \mathrm{d}^3x\, D_i\pi^{ij}(N\partial_jM - M\partial_jN) \\
&\overset{(4)}{=} \int \mathrm{d}^3x\,\mathcal{H}_k\,\gamma^{kj}(M\partial_jN - N\partial_jM) = \mathcal{H}_k[\gamma^{kj}(M\partial_jN - N\partial_jM)].
\end{aligned} \tag{28.75}$$

Let us justify each transition:

- **In step (1):** For any symmetric tensor $S_{ij} \equiv D_iD_jM$, contracting $(S^{ij} - \gamma^{ij}S^k{}_k)$ with $(\pi_{ij} - \frac{1}{2}\gamma_{ij}\pi)$ yields:
$$(S^{ij} - \gamma^{ij}S)(\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\pi) = S^{ij}\pi_{ij} - \tfrac{1}{2}S\pi - S\pi + \tfrac{3}{2}S\pi = S^{ij}\pi_{ij}, \tag{28.76}$$
where $\gamma^{ij}\gamma_{ij} = 3$ generates the factor of $\frac{3}{2}$, causing the trace terms to cancel completely.
- **In step (2):** We integrate by parts, using that $\pi^{ij}D_jM$ is a vector density.
- **In step (3):** We expand the derivatives: $D_i(N\pi^{ij}) = ND_i\pi^{ij} + \pi^{ij}D_iN$. The cross-terms $\pi^{ij}(D_iND_jM - D_iMD_jN)$ vanish identically due to the symmetry $\pi^{ij} = \pi^{ji}$. Furthermore, on scalar functions, covariant derivatives coincide with partial derivatives ($D_jM = \partial_jM$).
- **In step (4):** We substitute $D_i\pi^{ij} = -\frac{1}{2}\gamma^{kj}\mathcal{H}_k$ from Equation (28.45).

Theorem: The Hypersurface-Deformation Algebra (Dirac Algebra)

The secondary constraints of vacuum General Relativity satisfy the closed Poisson bracket algebra:

$$\{\mathcal{H}_i[M^i], \mathcal{H}_j[N^j]\} = \mathcal{H}_i[\mathcal{L}_M N^i], \tag{28.77}$$

$$\{\mathcal{H}[M], \mathcal{H}_i[N^i]\} = -\mathcal{H}[\mathcal{L}_N M], \tag{28.78}$$

$$\{\mathcal{H}[M], \mathcal{H}[N]\} = \mathcal{H}_k[\gamma^{kj}(M\partial_j N - N\partial_j M)]. \tag{28.79}$$

Physical Insight & Motivation: Field-Dependent Structure Functions and Lie Algebroids

Notice that the right-hand side of Equation (28.79) contains the inverse spatial metric $\gamma^{kj}(x)$, which is a *dynamical field variable* of phase space, not a set of constant structure coefficients.
Consequently, the hypersurface-deformation algebra is **not a genuine infinite-dimensional Lie algebra**, but rather a **Lie algebroid** (or an open gauge algebra). This field-dependence is the fundamental technical reason why canonical quantization of gravity in metric variables is notoriously challenging: the commutator of two quantum Hamiltonian constraints involves operator-valued coefficients that cannot be reordered trivially.

28.5.5 Termination of the Dirac Algorithm

We now return to the consistency conditions (28.56) and (28.57). Inserting the hypersurface-deformation algebra:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[M] = \mathcal{H}_k[\gamma^{kj}(M\partial_j N - N\partial_j M)] - \mathcal{H}[N^l\partial_l M] \approx 0, \tag{28.80}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_i[M^i] = \mathcal{H}[M^l\partial_l N] + \mathcal{H}_i[\mathcal{L}_M N^i] \approx 0. \tag{28.81}$$

Both time derivatives vanish weakly on the secondary constraint surface $\mathcal{H} \approx 0, \mathcal{H}_i \approx 0$!

Key Concept: Termination and Arbitrary Multipliers

The Dirac consistency algorithm terminates completely at this stage:

1. **No Tertiary Constraints:** No additional relations among phase-space variables are generated.
2. **Arbitrary Gauge Multipliers:** The Lagrange multipliers $\lambda(t,x)$ and $\lambda^i(t,x)$ remain completely arbitrary functions of spacetime, reflecting the unrestricted choice of lapse N and shift N^i in foliation-preserving gauge transformations.

28.6 Constraint Classification, Physical Degrees of Freedom, and Dirac Brackets

28.6.1 First-Class Classification and Gauge Invariance

The complete set of constraints consists of eight constraints per spatial point:

$$\Phi_A = \Big(\pi_N, \pi_i; \quad \mathcal{H}, \mathcal{H}_i\Big), \qquad A = 1, \ldots, 8. \tag{28.82}$$

Let us check their mutual Poisson brackets on the constraint surface:

- The primary constraints π_N and π_i commute with each other identically: $\{\pi_N, \pi_M\} = 0$, $\{\pi_N, \pi_i\} = 0$, $\{\pi_i, \pi_j\} = 0$.
- The primary constraints commute with the secondary constraints $\mathcal{H}$ and $\mathcal{H}_i$, because $\mathcal{H}$ and $\mathcal{H}_i$ are built exclusively from γ_{ij} and π^{ij} and do not contain N or N^i:

$$\{\pi_N(x), \mathcal{H}(y)\} = 0, \quad \{\pi_N(x), \mathcal{H}_i(y)\} = 0, \quad \{\pi_j(x), \mathcal{H}(y)\} = 0, \quad \{\pi_j(x), \mathcal{H}_i(y)\} = 0. \tag{28.83}$$

- The secondary constraints close among themselves weakly by the hypersurface-deformation algebra:

$$\{\mathcal{H}(x), \mathcal{H}(y)\} \approx 0, \quad \{\mathcal{H}(x), \mathcal{H}_i(y)\} \approx 0, \quad \{\mathcal{H}_i(x), \mathcal{H}_j(y)\} \approx 0. \tag{28.84}$$

Therefore, every Poisson bracket between any pair of constraints vanishes weakly on Γ_c:

$$\{\Phi_A(x), \Phi_B(y)\} \approx 0 \qquad \forall\, A, B. \tag{28.85}$$

All eight constraints of vacuum General Relativity are first class. They generate the four-dimensional diffeomorphism gauge group of General Relativity.

28.6.2 Counting the Physical Degrees of Freedom

We can now rigorously compute the number of propagating physical degrees of freedom (polarization states) of the gravitational field:

Phase Space Entity	**Variables per Point**	**Count**
Configuration variables	Lapse N (1), Shift N^i (3), Spatial metric γ_{ij} (6)	10
Canonical momenta	π_N (1), π_i (3), π^{ij} (6)	10
Total Unconstrained Phase Space	(q^A, p_A)	**20**
Primary first-class constraints	$\pi_N \approx 0$ (1), $\pi_i \approx 0$ (3)	4
Secondary first-class constraints	$\mathcal{H} \approx 0$ (1), $\mathcal{H}_i \approx 0$ (3)	4
Total First-Class Constraints	N_{FC}	**8**

Each first-class constraint restricts the accessible phase space by one condition and generates a one-parameter gauge transformation, eliminating two phase-space dimensions per spatial point. The dimension of the physical reduced phase space Γ_{phys} per spatial point is:

$$\dim(\Gamma_{\text{phys}}) = 20 - 2 \times N_{\text{FC}} = 20 - 2 \times 8 = 20 - 16 = 4. \tag{28.86}$$

Dividing by 2 to convert phase-space coordinates (coordinates and conjugate momenta) into physical configuration degrees of freedom yields:

$$\text{Physical Degrees of Freedom} = \frac{1}{2}\dim(\Gamma_{\text{phys}}) = \frac{4}{2} = 2. \tag{28.87}$$

These two degrees of freedom correspond precisely to the two transverse-traceless polarization states of the graviton (h_+ and $h_\times$), in exact agreement with the weak-field linearized perturbation analysis of Chapter 18!

28.6.3 Gauge Fixing and the Dirac Bracket

Because there are no second-class constraints in the standard unconstrained ADM formulation, the constraint matrix $\Delta_{AB} = \{\Phi_A, \Phi_B\} \approx 0$ vanishes weakly, and the Dirac bracket coincides with the Poisson bracket:

$$\{A, B\}_{\mathrm{D}} \approx \{A, B\}. \tag{28.88}$$

Non-trivial Dirac brackets emerge only when we make explicit **gauge-fixing choices**. To eliminate gauge redundancies, each first-class constraint must be paired with an appropriate gauge condition.

As a pedagogical example, consider fixing the coordinate foliation to unit lapse and vanishing shift:

$$\chi \equiv N - 1 \approx 0, \qquad \chi^i \equiv N^i \approx 0. \tag{28.89}$$

Now, evaluate the brackets between these gauge conditions and the primary constraints:

$$\{\chi(x), \pi_N(y)\} = \{N(x) - 1, \pi_N(y)\} = \delta^{(3)}(x - y), \tag{28.90}$$

$$\{\chi^i(x), \pi_j(y)\} = \{N^i(x), \pi_j(y)\} = \delta^i_j\, \delta^{(3)}(x - y). \tag{28.91}$$

The pairs (χ, π_N) and (χ^i, π_i) have non-vanishing, invertible Poisson brackets on the constraint surface; they have become **second-class constraint pairs**. Demanding their preservation in time,

$$\dot{\chi} = \{\chi, H_T\} = \lambda = 0, \qquad \dot{\chi}^i = \{\chi^i, H_T\} = \lambda^i = 0, \tag{28.92}$$

uniquely fixes the previously arbitrary multipliers $\lambda = 0$ and $\lambda^i = 0$.

Computing the Dirac bracket for functionals A, B that depend only on the dynamical spatial metric γ_{ij} and its conjugate momentum π^{ij}, every bracket with $\chi, \chi^i, \pi_N, \pi_i$ vanishes identically. Therefore:

$$\{A[\gamma, \pi], B[\gamma, \pi]\}_{\mathrm{D}} = \{A, B\}. \tag{28.93}$$

The sole effect of this gauge fixing is to strongly eliminate N and N^i as canonical variables ($N \equiv 1, N^i \equiv 0, \pi_N \equiv 0, \pi_i \equiv 0$). If one were instead to fix gauges conjugate to the secondary constraints $\mathcal{H}$ and $\mathcal{H}_i$ (such as maximal slicing $K = 0$ or transverse-traceless gauge), the Dirac bracket would deform the fundamental Poisson bracket of the metric and momentum, projecting the system directly onto the two physical radiative degrees of freedom.

28.7 The Contracted Bianchi Identity as an Integrability Condition

28.7.1 The Deep Question: Why Does the Algorithm Terminate?

The explicit Poisson bracket computations in Section 28.5 confirmed that the secondary constraints close among themselves, ensuring that no tertiary constraints emerge. However, an algebraic computation does not explain *why* General Relativity possesses this remarkable property. Why should the messy non-linear variation of the three-dimensional Ricci scalar in $\{\mathcal{H}[M], \mathcal{H}[N]\}$ magically produce exactly the momentum constraint $\mathcal{H}_k$, rather than some higher-derivative scalar quantity that would impose a tertiary constraint and kill the dynamics of the gravitational field?

The deep physical and geometric reason is **four-dimensional diffeomorphism invariance**, whose mathematical embodiment in differential geometry is the **contracted Bianchi identity**. In this section, we prove that the termination of Dirac's algorithm is mathematically equivalent to the contracted Bianchi identity, which acts as an off-shell integrability condition on the Hamiltonian evolution.

28.7.2 Off-Shell Variational Derivatives Along Arbitrary Histories

To uncover this connection, we must compute the time evolution of the constraints along an **arbitrary spacetime metric history** $g_{\mu\nu}(t, x^i) \leftrightarrow (N, N^i, \gamma_{ij})$, *without* assuming that the Einstein field equations hold!

Let $S[N, N^i, \gamma_{ij}] = \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{L}_{\mathrm{ADM}}$ denote the Lagrangian action with density (28.26). In parallel, consider the first-order Hamiltonian action

$$S_H[N, N^i, \gamma_{ij}, \pi^{ij}] \equiv \int \mathrm{d}t \left[\int \mathrm{d}^3x\, \pi^{ij}\dot{\gamma}_{ij} - \mathcal{H}[N] - \mathcal{H}_i[N^i] \right]. \tag{28.94}$$

By the very construction of the canonical Hamiltonian H_c, S_H evaluated at the physical momentum $\pi^{ij} = \sqrt{\gamma}(K^{ij} - \gamma^{ij}K)$ is identical to the Lagrangian action S. Furthermore, this momentum is precisely the solution to the algebraic equation:

$$\frac{\delta S_H}{\delta \pi^{ij}} = \dot{\gamma}_{ij} - \frac{\delta H_c}{\delta \pi^{ij}} = 0. \tag{28.95}$$

When we vary S_H with respect to the configuration variables at this stationary momentum, the chain rule contribution from π^{ij} is multiplied by $\delta S_H/\delta\pi^{ij} = 0$, giving the exact relations:

$$\frac{\delta S}{\delta N} = -\mathcal{H}, \qquad \frac{\delta S}{\delta N^i} = -\mathcal{H}_i, \qquad \frac{\delta S}{\delta \gamma_{ij}} = -\dot{\pi}^{ij} - \frac{\delta H_c}{\delta \gamma_{ij}}. \tag{28.96}$$

Along *any* arbitrary spacetime history:

1. The velocity relation $\dot{\gamma}_{ij} = \frac{\delta H_c}{\delta \pi^{ij}}$ holds identically by definition of the conjugate momentum π^{ij}.

2. The momentum rate of change is given by:

$$\dot{\pi}^{ij} = -\frac{\delta H_c}{\delta \gamma_{ij}} - \frac{\delta S}{\delta \gamma_{ij}}. \tag{28.97}$$

The physical dynamical evolution equations are $\frac{\delta S}{\delta \gamma_{ij}} = 0$. However, in our off-shell analysis, we do *not* set $\frac{\delta S}{\delta \gamma_{ij}}$ to zero!

28.7.3 Off-Shell Time Derivatives of Smeared Constraints

We now evaluate the time derivative of $\mathcal{H}[M]$ along this arbitrary off-shell history:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[M] &= \int \mathrm{d}^3x \left(\frac{\delta \mathcal{H}[M]}{\delta \gamma_{ij}}\dot{\gamma}_{ij} + \frac{\delta \mathcal{H}[M]}{\delta \pi^{ij}}\dot{\pi}^{ij} \right) \\
&\overset{(1)}{=} \int \mathrm{d}^3x \left(\frac{\delta \mathcal{H}[M]}{\delta \gamma_{ij}}\frac{\delta H_c}{\delta \pi^{ij}} - \frac{\delta \mathcal{H}[M]}{\delta \pi^{ij}}\frac{\delta H_c}{\delta \gamma_{ij}} \right) - \int \mathrm{d}^3x\, \frac{\delta \mathcal{H}[M]}{\delta \pi^{ij}}\frac{\delta S}{\delta \gamma_{ij}} \\
&\overset{(2)}{=} \{\mathcal{H}[M], H_c\} - \int \mathrm{d}^3x\, \frac{\delta \mathcal{H}[M]}{\delta \pi^{ij}}\frac{\delta S}{\delta \gamma_{ij}}.
\end{aligned} \tag{28.98}$$

In (1), we substituted $\dot{\gamma}_{ij} = \delta H_c/\delta\pi^{ij}$ and Equation (28.97). In (2), the first integral is recognized as the Poisson bracket with $H_c = \mathcal{H}[N] + \mathcal{H}_i[N^i]$.

From Equation (28.30), the functional derivative with respect to π^{ij} is:

$$\frac{\delta \mathcal{H}[M]}{\delta \pi^{ij}} = 2M\,\frac{\pi_{ij} - \frac{1}{2}\gamma_{ij}\pi}{\sqrt{\gamma}} = 2MK_{ij}. \tag{28.99}$$

Using the brackets from the hypersurface-deformation algebra (28.78) and (28.79):

$$\begin{aligned}\{\mathcal{H}[M], H_c\} &= \{\mathcal{H}[M], \mathcal{H}[N]\} + \{\mathcal{H}[M], \mathcal{H}_i[N^i]\} \\ &= \int \mathrm{d}^3x\, \mathcal{H}_k \gamma^{kj}(M\partial_j N - N\partial_j M) - \int \mathrm{d}^3x\, M\partial_l(N^l\mathcal{H}).\end{aligned} \tag{28.100}$$

Writing $\mathcal{H}^j \equiv \gamma^{jk}\mathcal{H}_k$ and integrating by parts the term $-N\mathcal{H}^j\partial_j M = \partial_j(N\mathcal{H}^j)M - \partial_j(N\mathcal{H}^j M)$:

$$\{\mathcal{H}[M], H_c\} = \int \mathrm{d}^3x\, M\left[\mathcal{H}^j\partial_j N + \partial_j(N\mathcal{H}^j) + \partial_l(N^l\mathcal{H})\right]. \tag{28.101}$$

Substituting this back into Equation (28.98):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[M] = \int \mathrm{d}^3x\, M\left[\mathcal{H}^j\partial_j N + \partial_j(N\mathcal{H}^j) + \partial_l(N^l\mathcal{H}) - 2K_{ij}\frac{\delta S}{\delta\gamma_{ij}}\right]. \tag{28.102}$$

Performing the identical steps for the momentum constraint $\mathcal{H}_i[M^i]$, using $\frac{\delta\mathcal{H}_i[M^i]}{\delta\pi^{jk}} = \mathcal{L}_M\gamma_{jk} = M^l\partial_l\gamma_{jk} + 2\gamma_{lk}\partial_j M^l$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_i[M^i] &= \{\mathcal{H}_i[M^i], H_c\} - \int \mathrm{d}^3x\left(M^l\partial_l\gamma_{jk} + 2\gamma_{lk}\partial_j M^l\right)\frac{\delta S}{\delta\gamma_{jk}} \\ &= \int \mathrm{d}^3x\, M^i\left[\mathcal{H}\partial_i N + \mathcal{H}_l\partial_i N^l + \partial_l(N^l\mathcal{H}_i) + 2\partial_j\left(\gamma_{ik}\frac{\delta S}{\delta\gamma_{jk}}\right) - \partial_i\gamma_{jk}\frac{\delta S}{\delta\gamma_{jk}}\right].\end{aligned} \tag{28.103}$$

Since the test functions $M(x)$ and $M^i(x)$ are completely arbitrary, Equations (28.102) and (28.103) are strictly equivalent to the **local off-shell constraint propagation identities**:

$$\dot{\mathcal{H}} = \partial_l(N^l\mathcal{H}) + \frac{1}{N}\partial_j(N^2\mathcal{H}^j) - 2K_{ij}\frac{\delta S}{\delta\gamma_{ij}}, \tag{28.104}$$

$$\dot{\mathcal{H}}_i = \partial_l(N^l\mathcal{H}_i) + \mathcal{H}_l\partial_i N^l + \mathcal{H}\partial_i N + 2\partial_j\left(\gamma_{ik}\frac{\delta S}{\delta\gamma_{jk}}\right) - \partial_i\gamma_{jk}\frac{\delta S}{\delta\gamma_{jk}}, \tag{28.105}$$

where in Equation (28.104) we simplified $\mathcal{H}^j\partial_j N + \partial_j(N\mathcal{H}^j) = \frac{1}{N}\partial_j(N^2\mathcal{H}^j)$.

Theorem: Off-Shell Constraint Preservation

Equations (28.104) and (28.105) hold identically for **every metric history**, without imposing any equations of motion.
When the dynamical field equations $\frac{\delta S}{\delta\gamma_{ij}} = 0$ are satisfied, the terms involving $\delta S/\delta\gamma_{ij}$ vanish, and the time derivatives $\dot{\mathcal{H}}$ and $\dot{\mathcal{H}}_i$ become **linear and homogeneous in the constraints $\mathcal{H}, \mathcal{H}_i$ and their spatial gradients**:

$$\dot{\mathcal{H}} = \partial_l(N^l\mathcal{H}) + \frac{1}{N}\partial_j(N^2\mathcal{H}^j), \qquad \dot{\mathcal{H}}_i = \partial_l(N^l\mathcal{H}_i) + \mathcal{H}_l\partial_i N^l + \mathcal{H}\partial_i N. \tag{28.106}$$

Consequently, by uniqueness of solutions to first-order linear partial differential systems, if the constraints are satisfied initially on Cauchy slice Σ_0 ($\mathcal{H}|_{t=0} = 0, \mathcal{H}_i|_{t=0} = 0$), they **remain zero for all time** under the dynamical evolution!

28.7.4 Covariant Dictionary and 4D Variational Derivatives

We now map the 3+1 variational derivatives ($\frac{\delta S}{\delta N}, \frac{\delta S}{\delta N^k}, \frac{\delta S}{\delta\gamma_{kl}}$) to the four-dimensional covariant variational derivative $\frac{\delta S}{\delta g_{\alpha\beta}}$. The total variation of the action is:

$$\delta S = \int \mathrm{d}^4x\, \frac{\delta S}{\delta g_{\alpha\beta}}\,\delta g_{\alpha\beta} = \int \mathrm{d}^4x\left[\frac{\delta S}{\delta g_{00}}\delta g_{00} + 2\frac{\delta S}{\delta g_{0k}}\delta g_{0k} + \frac{\delta S}{\delta g_{kl}}\delta g_{kl}\right]. \tag{28.107}$$

From the ADM metric relations (28.17), the variations of the 4D metric components are:

$$\delta g_{00} = -2N\delta N + 2N_k\delta N^k + N^k N^l \delta\gamma_{kl}, \tag{28.108}$$

$$\delta g_{0k} = \gamma_{kl}\delta N^l + N^l\delta\gamma_{kl}, \tag{28.109}$$

$$\delta g_{kl} = \delta\gamma_{kl}. \tag{28.110}$$

Substituting these into Equation (28.107) and equating coefficients with $\delta S = \int \mathrm{d}^4x(\frac{\delta S}{\delta N}\delta N + \frac{\delta S}{\delta N^k}\delta N^k + \frac{\delta S}{\delta\gamma_{kl}}\delta\gamma_{kl})$, we obtain the **dictionary** connecting 4D and 3+1 variations:

$$\frac{\delta S}{\delta N} = -2N\frac{\delta S}{\delta g_{00}} \implies \mathcal{H} = 2N\frac{\delta S}{\delta g_{00}}, \tag{28.111}$$

$$\frac{\delta S}{\delta N^k} = 2N_k\frac{\delta S}{\delta g_{00}} + 2\gamma_{kl}\frac{\delta S}{\delta g_{0l}} \implies \mathcal{H}_k = -2\left(N_k\frac{\delta S}{\delta g_{00}} + \gamma_{kl}\frac{\delta S}{\delta g_{0l}}\right), \tag{28.112}$$

$$\frac{\delta S}{\delta\gamma_{kl}} = \frac{\delta S}{\delta g_{kl}} + N^k\frac{\delta S}{\delta g_{0l}} + N^l\frac{\delta S}{\delta g_{0k}} + N^kN^l\frac{\delta S}{\delta g_{00}}. \tag{28.113}$$

Equations (28.111) and (28.112) reveal that the **secondary constraints are precisely the** 00 **and** $0k$ **components of the four-dimensional field equations**.

Because the chain rule applies to any coordinate increment $\partial_\lambda g_{\alpha\beta}$, we have for each spacetime index $\lambda \in \{0,1,2,3\}$:

$$\frac{\delta S}{\delta g_{\alpha\beta}}\partial_\lambda g_{\alpha\beta} = \frac{\delta S}{\delta N}\partial_\lambda N + \frac{\delta S}{\delta N^k}\partial_\lambda N^k + \frac{\delta S}{\delta\gamma_{kl}}\partial_\lambda\gamma_{kl} = -\mathcal{H}\partial_\lambda N - \mathcal{H}_k\partial_\lambda N^k + \frac{\delta S}{\delta\gamma_{kl}}\partial_\lambda\gamma_{kl}. \tag{28.114}$$

Next, we compute the mixed index combinations $\frac{\delta S}{\delta g_{\alpha\beta}}g_{\beta\lambda}$. Using Equations (28.111)–(28.113):

a. For $\alpha = 0, \lambda = k$**:**

$$\frac{\delta S}{\delta g_{0\beta}}g_{\beta k} = N_k\frac{\delta S}{\delta g_{00}} + \gamma_{kl}\frac{\delta S}{\delta g_{0l}} = -\frac{1}{2}\mathcal{H}_k. \tag{28.115}$$

b. For $\alpha = 0, \lambda = 0$**:**

$$\begin{aligned}\frac{\delta S}{\delta g_{0\beta}}g_{\beta 0} &= (-N^2 + N_kN^k)\frac{\delta S}{\delta g_{00}} + N_l\frac{\delta S}{\delta g_{0l}} = -N^2\frac{\delta S}{\delta g_{00}} + N^k\left(N_k\frac{\delta S}{\delta g_{00}} + \gamma_{kl}\frac{\delta S}{\delta g_{0l}}\right)\\ &= -N^2\left(\frac{\mathcal{H}}{2N}\right) + N^k\left(-\frac{1}{2}\mathcal{H}_k\right) = -\frac{1}{2}N\mathcal{H} - \frac{1}{2}N^k\mathcal{H}_k.\end{aligned} \tag{28.116}$$

c. For $\alpha = j, \lambda = k$**:**

$$\begin{aligned}\frac{\delta S}{\delta g_{j\beta}}g_{\beta k} &= N_k\frac{\delta S}{\delta g_{0j}} + \gamma_{kl}\frac{\delta S}{\delta g_{jl}}\\ &= N_k\frac{\delta S}{\delta g_{0j}} + \gamma_{kl}\left(\frac{\delta S}{\delta\gamma_{jl}} - N^j\frac{\delta S}{\delta g_{0l}} - N^l\frac{\delta S}{\delta g_{0j}} - N^jN^l\frac{\delta S}{\delta g_{00}}\right)\\ &= \gamma_{kl}\frac{\delta S}{\delta\gamma_{jl}} - N^j\left(\gamma_{kl}\frac{\delta S}{\delta g_{0l}} + N_k\frac{\delta S}{\delta g_{00}}\right) = \gamma_{kl}\frac{\delta S}{\delta\gamma_{jl}} + \frac{1}{2}N^j\mathcal{H}_k.\end{aligned} \tag{28.117}$$

d. For $\alpha = j, \lambda = 0$**:** Inverting Equation (28.112) gives $\frac{\delta S}{\delta g_{0j}} = -\frac{1}{2}\mathcal{H}^j - \frac{N^j}{2N}\mathcal{H}$. Then:

$$\begin{aligned}\frac{\delta S}{\delta g_{j\beta}}g_{\beta 0} &= \frac{\delta S}{\delta g_{0j}}g_{00} + \frac{\delta S}{\delta g_{jl}}N_l = N^k\frac{\delta S}{\delta g_{j\beta}}g_{\beta k} - N^2\frac{\delta S}{\delta g_{0j}}\\ &= N^k\left(\gamma_{kl}\frac{\delta S}{\delta\gamma_{jl}} + \frac{1}{2}N^j\mathcal{H}_k\right) - N^2\left(-\frac{1}{2}\mathcal{H}^j - \frac{N^j}{2N}\mathcal{H}\right)\\ &= N^k\gamma_{kl}\frac{\delta S}{\delta\gamma_{jl}} + \frac{1}{2}N^jN^k\mathcal{H}_k + \frac{1}{2}N^2\mathcal{H}^j + \frac{1}{2}NN^j\mathcal{H}.\end{aligned} \tag{28.118}$$

28.7.5 Reconstructing the Four-Vector B_λ

We now construct the covariant vector combination:

$$B_\lambda \equiv \partial_\alpha \left(\frac{\delta S}{\delta g_{\alpha\beta}} g_{\beta\lambda} \right) - \frac{1}{2} \frac{\delta S}{\delta g_{\alpha\beta}} \partial_\lambda g_{\alpha\beta}. \tag{28.119}$$

Let us evaluate B_λ component by component:

Spatial Components: $\lambda = i$

Expanding ∂_α into time and spatial derivatives:

$$\begin{aligned} B_i &= \partial_t \left(\frac{\delta S}{\delta g_{0\beta}} g_{\beta i} \right) + \partial_j \left(\frac{\delta S}{\delta g_{j\beta}} g_{\beta i} \right) - \frac{1}{2} \frac{\delta S}{\delta g_{\alpha\beta}} \partial_i g_{\alpha\beta} \\ &\overset{(1)}{=} -\frac{1}{2} \dot{\mathcal{H}}_i + \partial_j \left(\gamma_{ik} \frac{\delta S}{\delta \gamma_{jk}} + \frac{1}{2} N^j \mathcal{H}_i \right) + \frac{1}{2} \mathcal{H} \partial_i N + \frac{1}{2} \mathcal{H}_k \partial_i N^k - \frac{1}{2} \frac{\delta S}{\delta \gamma_{kl}} \partial_i \gamma_{kl} \\ &\overset{(2)}{=} -\frac{1}{2} \left[\dot{\mathcal{H}}_i - \partial_j (N^j \mathcal{H}_i) - \mathcal{H}_k \partial_i N^k - \mathcal{H} \partial_i N - 2 \partial_j \left(\gamma_{ik} \frac{\delta S}{\delta \gamma_{jk}} \right) + \frac{\delta S}{\delta \gamma_{kl}} \partial_i \gamma_{kl} \right] \\ &\overset{(3)}{=} 0. \end{aligned} \tag{28.120}$$

In (1), we substituted Equations (28.115), (28.117), and (28.114) with $\lambda = i$. In (2), we grouped terms inside the brackets. In (3), comparing with the local off-shell identity (28.105) shows that the term in brackets vanishes identically! Thus:

$$B_i \equiv 0. \tag{28.121}$$

Temporal Component: $\lambda = 0$

To evaluate B_0, it is algebraically cleanest to consider the combination $B_0 - N^i B_i$:

$$\begin{aligned} B_0 &= \partial_t \left(\frac{\delta S}{\delta g_{0\beta}} g_{\beta 0} \right) + \partial_j \left(\frac{\delta S}{\delta g_{j\beta}} g_{\beta 0} \right) - \frac{1}{2} \frac{\delta S}{\delta g_{\alpha\beta}} \partial_t g_{\alpha\beta} \\ &= -\frac{1}{2} \partial_t (N\mathcal{H} + N^k \mathcal{H}_k) + \partial_j \left(N^k \gamma_{kl} \frac{\delta S}{\delta \gamma_{jl}} + \frac{1}{2} N^j N^k \mathcal{H}_k + \frac{1}{2} N^2 \mathcal{H}^j + \frac{1}{2} N N^j \mathcal{H} \right) \\ &\quad + \frac{1}{2} \mathcal{H} \dot{N} + \frac{1}{2} \mathcal{H}_k \dot{N}^k - \frac{1}{2} \frac{\delta S}{\delta \gamma_{kl}} \dot{\gamma}_{kl}. \end{aligned} \tag{28.122}$$

Applying the product rule to $\partial_t(N\mathcal{H} + N^k \mathcal{H}_k) = N\dot{\mathcal{H}} + \dot{N}\mathcal{H} + N^k \dot{\mathcal{H}}_k + \dot{N}^k \mathcal{H}_k$, the terms $\dot{N}\mathcal{H}$ and $\dot{N}^k \mathcal{H}_k$ cancel against the chain rule terms. Expanding the spatial divergence:

$$\begin{aligned} B_0 &= -\frac{1}{2} N \dot{\mathcal{H}} - \frac{1}{2} N^k \dot{\mathcal{H}}_k + N^k \partial_j \left(\gamma_{kl} \frac{\delta S}{\delta \gamma_{jl}} + \frac{1}{2} N^j \mathcal{H}_k \right) + \left(\gamma_{kl} \frac{\delta S}{\delta \gamma_{jl}} + \frac{1}{2} N^j \mathcal{H}_k \right) \partial_j N^k \\ &\quad + \frac{1}{2} \partial_j (N^2 \mathcal{H}^j) + \frac{1}{2} \partial_j (N N^j \mathcal{H}) - \frac{1}{2} \frac{\delta S}{\delta \gamma_{kl}} \dot{\gamma}_{kl}. \end{aligned} \tag{28.123}$$

From Equation (28.120), multiplying B_i by N^i gives:

$$N^i B_i = -\frac{1}{2} N^i \dot{\mathcal{H}}_i + N^i \partial_j \left(\gamma_{ik} \frac{\delta S}{\delta \gamma_{jk}} + \frac{1}{2} N^j \mathcal{H}_i \right) + \frac{1}{2} N^i \mathcal{H} \partial_i N + \frac{1}{2} N^i \mathcal{H}_k \partial_i N^k - \frac{1}{2} N^i \frac{\delta S}{\delta \gamma_{kl}} \partial_i \gamma_{kl}. \tag{28.124}$$

Subtracting $N^i B_i$ from B_0:

$$\begin{aligned} B_0 - N^i B_i &= -\frac{1}{2}N\dot{\mathcal{H}} + \gamma_{kl}\frac{\delta S}{\delta \gamma_{jl}}\partial_j N^k + \frac{1}{2}N^j\mathcal{H}_k\partial_j N^k - \frac{1}{2}N^i\mathcal{H}_k\partial_i N^k \\ &\quad + \frac{1}{2}\partial_j(N^2\mathcal{H}^j) + \frac{1}{2}\partial_j(NN^j\mathcal{H}) - \frac{1}{2}N^i\mathcal{H}\partial_i N - \frac{1}{2}\frac{\delta S}{\delta\gamma_{kl}}(\dot{\gamma}_{kl} - N^i\partial_i\gamma_{kl}) \\ &\overset{(1)}{=} -\frac{1}{2}N\dot{\mathcal{H}} + \frac{1}{2}\partial_j(N^2\mathcal{H}^j) + \frac{1}{2}N\partial_j(N^j\mathcal{H}) - \frac{1}{2}\frac{\delta S}{\delta\gamma_{kl}}(\dot{\gamma}_{kl} - N^i\partial_i\gamma_{kl} - 2\gamma_{il}D_k N^i) \\ &\overset{(2)}{=} -\frac{1}{2}N\left[\dot{\mathcal{H}} - \partial_j(N^j\mathcal{H}) - \frac{1}{N}\partial_j(N^2\mathcal{H}^j) + 2K_{kl}\frac{\delta S}{\delta\gamma_{kl}}\right] \\ &\overset{(3)}{=} 0. \end{aligned} \tag{28.125}$$

In (1), we simplified $\partial_j(NN^j\mathcal{H}) - N^i\mathcal{H}\partial_i N = N\partial_j(N^j\mathcal{H})$. In (2), by Equation (28.21), $\dot{\gamma}_{kl} - \mathcal{L}_N\gamma_{kl} = 2NK_{kl}$. In (3), the bracket is identical to the off-shell identity (28.104), so it vanishes identically!

Since $B_i = 0$ and $N > 0$, we have proven that:

$$B_0 = 0 \quad \text{and} \quad B_i = 0 \iff B_\lambda \equiv 0 \quad \forall\,\lambda \in \{0,1,2,3\}. \tag{28.126}$$

28.7.6 Identification with the Contracted Bianchi Identity

Now comes the ultimate geometric synthesis. What is the four-dimensional covariant meaning of the vector density B_λ?

For *any* symmetric rank-2 contravariant tensor $T^{\alpha\beta} = T^{\beta\alpha}$ in four dimensions, let us compute the four-dimensional covariant divergence $\nabla_\alpha T^\alpha{}_\lambda$:

$$\begin{aligned} \sqrt{-g}\,\nabla_\alpha T^\alpha{}_\lambda &= \sqrt{-g}\left(\partial_\alpha T^\alpha{}_\lambda + \Gamma^\alpha_{\alpha\beta}T^\beta{}_\lambda - \Gamma^\beta_{\alpha\lambda}T^\alpha{}_\beta\right) \\ &\overset{(1)}{=} \partial_\alpha(\sqrt{-g}\,T^\alpha{}_\lambda) - \sqrt{-g}\,\Gamma^\beta_{\alpha\lambda}T^{\alpha\beta} \\ &\overset{(2)}{=} \partial_\alpha(\sqrt{-g}\,T^\alpha{}_\lambda) - \sqrt{-g}\,\frac{1}{2}g^{\beta\sigma}(\partial_\alpha g_{\sigma\lambda} + \partial_\lambda g_{\alpha\sigma} - \partial_\sigma g_{\alpha\lambda})T^{\alpha\beta} \\ &\overset{(3)}{=} \partial_\alpha(\sqrt{-g}\,T^\alpha{}_\lambda) - \frac{1}{2}\sqrt{-g}\,T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}. \end{aligned} \tag{28.127}$$

Let us verify the steps:

- In (1), we used the standard metric identity $\Gamma^\alpha_{\alpha\beta} = \partial_\beta \ln\sqrt{-g} = \frac{1}{\sqrt{-g}}\partial_\beta\sqrt{-g}$ to absorb the connection term into the total derivative: $\sqrt{-g}\,\partial_\alpha T^\alpha{}_\lambda + (\partial_\alpha\sqrt{-g})T^\alpha{}_\lambda = \partial_\alpha(\sqrt{-g}\,T^\alpha{}_\lambda)$.
- In (2), we wrote out the Christoffel symbol $\Gamma^\beta_{\alpha\lambda}$ in terms of metric derivatives.
- In (3), because $T^{\alpha\beta}$ is symmetric ($T^{\alpha\beta} = T^{\beta\alpha}$), the terms $\partial_\alpha g_{\sigma\lambda}T^{\alpha\beta}$ and $-\partial_\sigma g_{\alpha\lambda}T^{\alpha\beta}$ cancel identically upon contraction! Only $-\frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}$ survives.

Because the Einstein–Hilbert action S is a spacetime scalar, its variational derivative $\frac{\delta S}{\delta g_{\alpha\beta}}$ is a tensor density of weight $+1$. Therefore:

$$T^{\alpha\beta} \equiv \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}} \tag{28.128}$$

is a bona fide symmetric rank-2 tensor. Substituting $T^{\alpha\beta}$ into Equation (28.127) and comparing directly with the definition of B_λ in Equation (28.119):

$$B_\lambda = \sqrt{-g}\,\nabla_\alpha\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}g_{\beta\lambda}\right) = \sqrt{-g}\,\nabla_\alpha T^\alpha{}_\lambda. \tag{28.129}$$

Therefore, the statement $B_\lambda \equiv 0$ is strictly equivalent to:

$$\nabla_\alpha \left(\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}} \right) = 0. \tag{28.130}$$

Finally, for the Einstein–Hilbert action $S = \int \mathrm{d}^4x \sqrt{-g}\, R^{(4)}$, the functional derivative yields the vacuum Einstein equations:

$$\frac{\delta S}{\delta g_{\alpha\beta}} = -\sqrt{-g}\, G^{\alpha\beta}, \tag{28.131}$$

where $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R^{(4)}$ is the four-dimensional Einstein tensor. Equation (28.130) becomes:

$$\nabla_\alpha G^{\alpha\beta} = 0. \tag{28.132}$$

Theorem: The Contracted Bianchi Identity as a Canonical Integrability Condition

The absence of tertiary constraints in the Dirac constraint algorithm of General Relativity is mathematically and physically equivalent to the **contracted Bianchi identity** $\nabla_\alpha G^{\alpha\beta} = 0$. The contracted Bianchi identity acts as an **integrability condition** ensuring that the Hamiltonian constraint $\mathcal{H} \approx 0$ and momentum constraints $\mathcal{H}_i \approx 0$ propagate consistently along the dynamical time evolution of the spatial metric.

28.8 Noether's Second Theorem and Diffeomorphism Invariance

28.8.1 Active Diffeomorphisms and Gauge Transformations

To complete the physical picture, we establish how this result connects to Noether's theorems. Under an infinitesimal coordinate transformation (diffeomorphism) generated by an arbitrary vector field $\xi^\mu(x)$:

$$x^\mu \to x^\mu + \xi^\mu(x), \tag{28.133}$$

the metric transforms by its Lie derivative:

$$\delta_\xi g_{\alpha\beta} = \mathcal{L}_\xi g_{\alpha\beta} = \xi^\lambda \partial_\lambda g_{\alpha\beta} + g_{\lambda\beta}\partial_\alpha \xi^\lambda + g_{\alpha\lambda}\partial_\beta \xi^\lambda = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha. \tag{28.134}$$

Because the action $S = \int \mathrm{d}^4x \sqrt{-g}\, R^{(4)}$ is invariant under any diffeomorphism with ξ^μ of compact support, the variation of the action must vanish identically for *every* field configuration:

$$\begin{aligned} 0 = \delta S &= \int \mathrm{d}^4x \, \frac{\delta S}{\delta g_{\alpha\beta}} \delta_\xi g_{\alpha\beta} = \int \mathrm{d}^4x \, \frac{\delta S}{\delta g_{\alpha\beta}} \left(\xi^\lambda \partial_\lambda g_{\alpha\beta} + 2 g_{\beta\lambda} \partial_\alpha \xi^\lambda \right) \\ &\overset{(1)}{=} \int \mathrm{d}^4x \, \xi^\lambda \left[\frac{\delta S}{\delta g_{\alpha\beta}} \partial_\lambda g_{\alpha\beta} - 2\partial_\alpha \left(\frac{\delta S}{\delta g_{\alpha\beta}} g_{\beta\lambda} \right) \right] \\ &\overset{(2)}{=} -2 \int \mathrm{d}^4x \, \xi^\lambda B_\lambda. \end{aligned} \tag{28.135}$$

In (1), we integrated the derivative $\partial_\alpha \xi^\lambda$ by parts, dropping the boundary term since ξ^λ has compact support, and used the symmetry of $\delta S / \delta g_{\alpha\beta}$. In (2), we recognized the expression inside the brackets as $-2B_\lambda$ from Equation (28.119).

Because Equation (28.135) holds for **arbitrary vector fields** $\xi^\lambda(x)$, the integrand must vanish identically at every spacetime point:

$$B_\lambda(x) \equiv 0 \qquad \forall\, x \in \mathcal{M}. \tag{28.136}$$

Key Concept: Noether's Second Theorem and Canonical Integrability

This is **Noether's second theorem** in action:

- When an action possesses a continuous *infinite-dimensional* gauge group parameterized by arbitrary spacetime functions $\xi^\mu(x)$, the Euler–Lagrange equations are not all independent. Instead, they satisfy an off-shell differential identity: the **Noether identity** $\nabla_\alpha G^{\alpha\beta} \equiv 0$.
- In the canonical Hamiltonian formalism, the four arbitrary gauge functions $\xi^\lambda(x)$ manifest as the four arbitrary Lagrange multipliers $\lambda(t,x)$ and $\lambda^i(t,x)$ (the freedom to reparametrize time and shift spatial coordinates).
- The four Noether identities $B_\lambda \equiv 0$ are precisely the four integrability conditions that prevent the secondary constraints from generating tertiary constraints, ensuring that the constraint algorithm terminates cleanly with exactly two physical propagating degrees of freedom.