# Conformal Field Theory

Personal Study Notes

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# Preface

These notes are personal study materials based primarily on Senechal's *Conformal Field Theory* aka *The Yellow Book*. While they derive insights from multiple sources, Senechal's text served as the primary roadmap and foundational framework for this project.

The central goal of these notes is to make the subject of string theory more accessible, both for my own mastery and for any other students navigating this demanding field. To achieve this, I have made every effort to **derive all calculations explicitly**. By filling in the "missing steps" and technical nuances often omitted in standard texts, I hope to ensure the material is easily parsable and the underlying mathematical narrative is transparent.

I welcome feedback, corrections, or discussions regarding the content of these notes. Please feel free to reach out to me via email.

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*Disclaimer: These are pedagogical study notes. While I have strived for accuracy, any errors in the calculations or physical interpretations remain entirely my own.*

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# Chapter 1

# Conformal Symmetry

## 1.1 Introduction to Conformal Symmetry

Conformal symmetry is, in essence, the symmetry of **shapes without scales**. In Euclidean geometry, this is often described as "angle-preserving" symmetry. While this definition is correct in a purely spatial setting, it becomes less illuminating once we step into relativistic physics, where time is on equal footing with space. The notion of an "angle" between two events separated in time is not geometrically well-defined in the same way, so we need a more general formulation.

A more robust way to think about conformal transformations is that they are transformations which preserve the metric **up to a local rescaling**. That is, they preserve the light-cone structure of spacetime, and hence the causal relations, but may change distances by position-dependent factors:

$$g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x), \quad \Omega(x) > 0. \tag{1.1.1}$$

Here, $\Omega(x)$ is the local scaling factor, and $\Omega(x) = 1$ corresponds to an isometry. This definition works in both Euclidean and Lorentzian settings, and it makes clear why conformal transformations generalize ordinary Poincaré symmetries: they allow stretching of spacetime while preserving angles and null directions. In flat space, an important special case is the global scaling transformation $x^\mu \to \lambda x^\mu$, which is not a mere relabeling of coordinates but a genuine change in the geometry. Under such a transformation, the ratios of lengths along a direction are preserved, and therefore angles are unchanged. This property underlies the term "conformal."

The study of **Conformal Field Theory (CFT)** leverages this symmetry in a very different way from conventional Quantum Field Theory (QFT). In QFT, one typically begins with a Lagrangian and derives correlation functions from the equations of motion and perturbation theory. In CFT, by contrast, the symmetry itself is so constraining that it often determines the form of correlation functions without reference to a specific Lagrangian. This leads naturally to the **conformal bootstrap** program, in which the consistency conditions of conformal symmetry, unitarity, and the operator product expansion are used to solve the theory. Before utilizing these tools, we must distinguish conformal transformations from two related but conceptually different ideas: **Weyl rescalings** and **diffeomorphisms**.

## 1.2 Diffeomorphisms, Weyl Invariance, and the Stress Tensor

Classical field theories can possess a variety of symmetries. One symmetry we will assume here is **general coordinate invariance**, or diffeomorphism invariance. Using the action principle, this symmetry can be used to show that the energy–momentum tensor is conserved. In general, the energy–momentum tensor is defined through the variation of the action $S$ under changes in the space–time metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. By definition,

$$\delta S = \frac{1}{2} \int d^d x \, \sqrt{g}\, T^{\mu\nu}\, \delta g_{\mu\nu}\,. \tag{1.2.1}$$

If the theory is invariant under general coordinate transformations, one can show that $(T^{\mu\nu})_{;\nu} = 0$, where ";$\nu$" denotes the covariant derivative. In flat coordinates, this condition reduces to the standard conservation law $\partial_\nu T^{\mu\nu} = 0$.

In addition to general coordinate invariance, many field theories possess **Weyl invariance**. While diffeomorphism invariance constrains how the metric responds to arbitrary coordinate changes, Weyl invariance concerns how the theory behaves under local rescalings of the metric $g_{\mu\nu}(x) \to \Omega(x)\, g_{\mu\nu}(x)$, or, in infinitesimal form, $g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \omega(x)\, g_{\mu\nu}(x)$. The condition for the action to remain invariant under such a transformation can be expressed in terms of the energy–momentum tensor. Substituting $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}(x)$ into the variation of the action, we find:

$$\delta S = \frac{1}{2} \int d^d x \, \sqrt{g}\, T^{\mu}_{\ \mu}\, \omega(x)\,. \tag{1.2.2}$$

Since this must hold for arbitrary functions $\omega(x)$, we conclude that the condition for Weyl invariance is the vanishing of the trace: $T^{\mu}_{\ \mu} = 0$. Thus, just as diffeomorphism invariance implies covariant conservation, Weyl invariance implies that the energy-momentum tensor must be traceless.

## 1.3 Conformal Transformations

A *conformal transformation* is defined as a coordinate transformation $x \to x'$ that acts on the metric as a Weyl transformation. Consider the transformation $x^\mu = f^\mu(x')$. The metric transforms as:

$$g_{\mu\nu}(x) \to \ g'_{\mu\nu}(x') = \frac{\partial f^\rho}{\partial x'^\mu} \frac{\partial f^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}\big(f(x')\big)\,. \tag{1.3.1}$$

We require that the transformed metric be proportional to the original one. Rotations and translations clearly satisfy this, as they leave the metric unchanged and preserve inner products $v \cdot w \equiv v^\mu g_{\mu\nu} w^\nu$. More generally, any transformation satisfying this proportionality preserves all angles defined by $(v \cdot w)/\sqrt{v^2 w^2}$.

If a field theory has a conserved and traceless energy–momentum tensor, it is invariant under both general coordinate transformations and Weyl transformations. Let the action be $S = \int d^d x\, \mathcal{L}(\partial_x,\, g_{\mu\nu}(x),\, \phi(x))$, where $\phi$ denotes any matter field. General coordinate invariance implies $S = S'$, where the transformation of $\phi$ depends on its spin. For a tensor field of rank $n$, we have the transformation law for *primary fields*:

$$\phi'_{\mu_1 \ldots \mu_n}(x') = \left|\frac{\partial f}{\partial x'}\right|^{\frac{\Delta}{d}} \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}}\, \phi_{\nu_1 \ldots \nu_n}\big(f(x')\big), \tag{1.3.2}$$

where $\Delta$ is the scaling dimension. For a scalar, this is simply $\phi'(x') = \phi(f(x'))$. While the first derivative of a scalar transforms as a vector, note that $n$th-order ordinary derivatives do not transform as rank-$n$ tensors; this holds only for covariant derivatives.

If the coordinate transformation $x \to x'$ is conformal, we can use Weyl invariance to bring the metric back to its original form. This yields $S = \int d^d x'\, \mathcal{L}(\partial_{x'}, g_{\mu\nu}(x), \phi'(x'))$, which is the statement of conformal symmetry. In some cases, such as in de Sitter (dS) or anti–de Sitter (AdS) spacetimes, an isometry of one metric can act as a conformal transformation for a different metric, such as the induced boundary metric.

Starting with a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, the background metric remains unchanged under these transformations.[1] This allows us to define conformal transformations for theories in flat space that are not coupled to gravity. In $d = 2$ dimensions, this is not a severe restriction because a general 2D metric has three components: $g_{11}$, $g_{22}$, and $g_{12}$. A general coordinate transformation provides two functions that can set $g_{12} = 0$ and $g_{11} = \pm g_{22}$ (the *conformal gauge*). A Weyl transformation can then remove the remaining factor. In more than two dimensions, this does not work in general, so restricting to flat space truly limits us to non-gravitational theories. Furthermore, in two dimensions, the conformal gauge can often only be chosen locally, meaning extra data may be needed to describe the theory globally.

[1] In perturbation theory, we perform the conformal transformation on the background metric and use the Killing equation to dictate the change in the perturbation. We apply the coordinate transformation to the full metric and then perform a Weyl rescaling to restore the background metric to its original form.

## 1.4 Infinitesimal Conformal Transformation

The fundamental essence of conformal transformations resides in their infinitesimal form, which serves as a crucial tool for investigating how fields transform under these symmetries. It plays a pivotal role in defining the generator of the conformal group and, subsequently, constraining the set of possible correlators that are compatible with conformal symmetry. Any infinitesimal transformation can be expressed as:

$$x'^{\mu} = x^{\mu} + \underbrace{\epsilon^{\mu}(x)}_{\text{infinitesimal}}$$

and subsequently,

$$x^{\mu} = x'^{\mu} - \epsilon^{\mu}(x)$$

therefore, the metric transforms like:

$$\begin{aligned} g'_{\mu\nu} &= \underbrace{\frac{\partial x^{\alpha}}{\partial x'^{\mu}}}_{\delta^{\alpha}_{\mu} - \partial_{\mu}\epsilon^{\alpha}(x)} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \\ &= \left[\delta^{\alpha}_{\mu} - \frac{\partial \epsilon^{\alpha}(x)}{\partial x'^{\mu}}\right]\left[\delta^{\beta}_{\nu} - \frac{\partial \epsilon^{\beta}(x)}{\partial x'^{\nu}}\right] g_{\alpha\beta} \\ &= \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} g_{\alpha\beta} - \delta^{\beta}_{\nu}\partial_{\mu}\epsilon^{\alpha}(x) g_{\alpha\beta} - \delta^{\alpha}_{\mu}\partial_{\nu}\epsilon^{\beta}(x) g_{\alpha\beta} + \mathcal{O}(\epsilon^2) \\ \Omega(x) g_{\mu\nu} &= g_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} \end{aligned}$$

In the third step, we used chain rule on $\epsilon^{\alpha}(x)$ and ignored $\mathcal{O}((\partial\epsilon)^2)$ terms. From the last line, it is reasonable to expect that:

$$\begin{aligned} g_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} &= [1 + f(x)] g_{\mu\nu} \\ \partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) &= f(x) g_{\mu\nu} \end{aligned} \tag{1.4.1}$$

Contracting Indicies

$$\begin{aligned} \partial^{\mu}\epsilon_{\mu}(x) + \partial^{\mu}\epsilon_{\mu}(x) &= f(x)\delta^{\mu}_{\mu} \\ 2(\partial \cdot \epsilon) &= \underbrace{d}_{\text{dimension of spacetime}} f(x) \\ f(x) &= \frac{2}{d}\frac{\partial \epsilon_{\mu}(x)}{\partial x_{\mu}} \end{aligned}$$

Substituing back in (1.4.1)

$$\boxed{\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) = \frac{2}{d}(\partial \cdot \epsilon)\ g_{\mu\nu}} \tag{1.4.2}$$

Up until now, we have no made any crude assumption. However before we proceed, we will assume that the metric is Euclidean. Now, we operate on both sides by $\partial^{\nu}$

$$\begin{aligned} \frac{\partial}{\partial x'_{\nu}}\left[\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x)\right] &= \frac{\partial}{\partial x'_{\nu}}\left(\frac{2}{d}\partial \cdot \epsilon(x)\ \underbrace{g_{\mu\nu}}_{\text{assuming flat metric}}\right) \\ \partial_{\mu}\underbrace{\partial^{\nu}\epsilon_{\nu}}_{\partial\cdot\epsilon} + \underbrace{\partial^{\nu}\partial_{\nu}}_{\Box}\epsilon_{\mu} &= \frac{2}{d} g_{\mu\nu}\partial^{\nu}\partial \cdot \epsilon \\ \partial_{\mu}(\partial \cdot \epsilon) + \Box\epsilon &= \frac{2}{d}\partial_{\mu}(\partial \cdot \epsilon) \end{aligned}$$

Operating by $\partial_{\nu}$

$$\partial_{\nu}[\partial_{\mu}(\partial \cdot \epsilon) + \Box\epsilon] = \partial_{\nu}\left[\frac{2}{d}\partial_{\mu}(\partial \cdot \epsilon)\right]$$

$$\left(1-\frac{2}{d}\right)\partial_\nu\partial_\mu(\partial\cdot\epsilon)+\Box(\partial_\nu\epsilon_\mu)=0 \tag{1.4.3}$$

under relabeling $\mu \leftrightarrow \nu$

$$\left(1-\frac{2}{d}\right)\partial_\mu\partial_\nu(\partial\cdot\epsilon)+\Box(\partial_\mu\epsilon_\nu)=0 \tag{1.4.4}$$

adding (1.4.3) and (1.4.4)

$$2\left(1-\frac{2}{d}\right)\partial_\mu\partial_\nu(\partial\cdot\epsilon)+\Box[\underbrace{\partial_\mu\epsilon_\nu(x)+\partial_\nu\epsilon_\mu(x)}_{\frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu}}]=0$$

$$\left(1-\frac{2}{d}\right)\partial_\mu\partial_\nu(\partial\cdot\epsilon)+\frac{1}{d}\Box(\partial\cdot\epsilon)g_{\mu\nu}=0$$

$$[g_{\mu\nu}\Box+(d-2)\partial_\mu\partial_\nu](\partial\cdot\epsilon)=0 \tag{1.4.5}$$

Contracting the indicies

$$[d\Box+(d-2)\Box](\partial\cdot\epsilon)=0$$

$$2(d-1)\Box(\partial\cdot\epsilon)=0$$

hence,

$$\boxed{(d-1)\Box(\partial\cdot\epsilon)=0} \tag{1.4.6}$$

if $d=1 \implies$ any $\epsilon^\mu(x)$ satisfies (1.4.6), therefore, is conformal transformation. It is interesting to note that any 1D QFT is conformal field theory, but for our purpose it's not very useful. We will be concerned with $d \neq 1$ for the rest of this notes unless stated otherwise. Consider the action of $\partial_\alpha$ on (1.4.2) and then cyclic relabeling of incidies as $\alpha \to \mu \to \nu$

$$\partial_\alpha[\partial_\mu\epsilon_\nu(x)+\partial_\nu\epsilon_\mu(x)]=\frac{2}{d}\partial_\alpha g_{\mu\nu}(\partial\cdot\epsilon) \tag{1.4.7}$$

$$\partial_\mu[\partial_\nu\epsilon_\alpha(x)+\partial_\alpha\epsilon_\nu(x)]=\frac{2}{d}\partial_\mu g_{\nu\alpha}(\partial\cdot\epsilon) \tag{1.4.8}$$

$$\partial_\nu[\partial_\alpha\epsilon_\mu(x)+\partial_\mu\epsilon_\alpha(x)]=\frac{2}{d}\partial_\nu g_{\alpha\mu}(\partial\cdot\epsilon) \tag{1.4.9}$$

Adding the first two equation and subtracting from the last, we get [(1.4.7) + (1.4.8) − (1.4.9)]:

$$2\partial_\alpha\partial_\mu\epsilon_\nu=\frac{2}{d}[g_{\mu\nu}\partial_\alpha+g_{\nu\alpha}\partial_\mu-g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon)$$

$$\partial_\alpha\partial_\mu\epsilon_\nu=\frac{1}{d}[g_{\mu\nu}\partial_\alpha+g_{\nu\alpha}\partial_\mu-g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon) \tag{1.4.10}$$

Referring to eqn 1.7.11 of "Ideas and Methods of Supersymmetry and Supergravity" by Sergio M. Kuzenko, we find that the 3rd order derivation of $\epsilon^\mu(x)$ vanishes. Therefore, the most general conformal transformation is of the type:

$$x'^\mu=x^\mu+\underbrace{a^\mu+b^\mu{}_\nu x^\nu+c^\mu{}_{\nu\alpha}x^\nu x^\alpha}_{\epsilon^\mu}$$

Where, $a^\mu, b^\mu{}_\nu$ and $c^\mu{}_{\nu\alpha}$ are parameters relevant to their transformation. The goal here is simple:

- First find the relevant transformations
- Then based on the transformation rule, find the generators.

For $\epsilon^\mu = a^\mu$:

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu \\ &= x^\mu + \delta^\mu_\nu a^\nu \\ &= x^\mu + (\partial_\nu x^\mu) a^\nu \\ &= [1 + i a^\nu (-i\partial_\nu)] x^\mu \end{aligned}$$

Thus, the generator of translation is $P_\mu - i\partial_\mu$[2]. For $\epsilon^\mu = b^\mu{}_\alpha x^\alpha$, we refer to (1.4.2)

$$\begin{aligned} \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) &= \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu} \\ \partial_\mu (b_{\nu\alpha} x^\alpha) + \partial_\nu (b_{\mu\alpha} x^\alpha) &= \frac{2}{d}(\partial^\mu b_{\mu\alpha} x^\alpha) g_{\mu\nu} \\ b_{\nu\alpha}\delta^\alpha_\mu + b_{\mu\alpha}\delta^\alpha_\nu &= \frac{2}{d}(b_{\mu\alpha} g^{\alpha\mu}) g_{\mu\nu} \\ b_{\nu\mu} + b_{\mu\nu} &= \frac{2}{d} b^\alpha{}_\alpha g_{\mu\nu} \\ \frac{b_{\nu\mu} + b_{\mu\nu}}{2} &= \frac{1}{d} b^\alpha{}_\alpha g_{\mu\nu} \end{aligned}$$

now,

$$\begin{aligned} b_{\mu\nu} &= \frac{b_{\mu\nu} - b_{\nu\mu}}{2} + \frac{b_{\mu\nu} + b_{\nu\mu}}{2} \\ &= M_{\mu\nu} + \lambda g_{\mu\nu} \end{aligned}$$

If $b_{\mu\nu} = \lambda g_{\mu\nu} (M_{\mu\nu} = 0)$

$$\begin{aligned} x'^\mu &= x_\mu + b^\mu{}_\nu x^\nu \\ &= x^\mu + \lambda g^{\mu\alpha} \underbrace{g_{\alpha\nu} x^\nu}_{x_\alpha} \\ &= x^\mu + \lambda x^\mu \\ &= x^\mu + \lambda x^\nu \delta^\mu_\nu \\ &= x^\mu + \lambda x^\nu (\partial_\nu x^\mu) \\ &= x^\mu + i\lambda x^\nu (-i\partial_\nu x^\mu) \\ &= (1 + i\lambda(-i x^\nu \partial_\nu)) x^\mu \end{aligned}$$

Thus, the generator of dilatation is $D = -i x^\mu \partial_\mu$. For $b_{\mu\nu} = M_{\mu\nu} (\lambda = 0)$.

$$\begin{aligned} x'^\mu &= x^\mu + M^\mu{}_\nu x^\nu \\ &= x^\mu + M^\alpha{}_\nu \delta^\mu_\alpha x^\nu \\ &= x^\mu + M^\alpha{}_\nu (\partial_\alpha x^\mu) x^\nu \\ &= x^\mu + M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu \\ &= x^\mu + \frac{M_{\alpha\nu} - M_{\nu\alpha}}{2} (\partial^\alpha x^\mu) x^\nu \qquad \text{relabeling } \nu \leftrightarrow \alpha \\ &= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \frac{1}{2} M_{\nu\alpha} (\partial^\alpha x^\mu) x^\nu \\ &= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \frac{1}{2} M_{\alpha\nu} (\partial^\nu x^\mu) x^\alpha \end{aligned}$$

[2] if we use $[1 - a^\nu(\partial_\nu)] x^\mu$ as the definition, then $P_\mu = i\partial_\mu$ would be the generator

$$
\begin{aligned}
&= x^\mu + \frac{1}{2} M_{\alpha\nu}(x^\nu \partial^\alpha - x^\alpha \partial^\nu) x^\mu \\
&= x^\mu + \frac{i}{2} M_{\alpha\nu}\left\{-i(x^\nu \partial^\alpha - x^\alpha \partial^\nu)\right\} x^\mu \\
&= x^\mu + \frac{i}{2} M_{\alpha\nu}\underbrace{\{i(x^\alpha \partial^\nu - x^\nu \partial^\alpha)\}}_{L^{\alpha\nu}} x^\mu
\end{aligned}
$$

Thus, the generator of rotation is $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Now, the last part $\epsilon^\mu = c^\mu{}_{\nu\alpha} x^\nu x^\alpha = c^\mu{}_{\alpha\nu} x^\nu x^\alpha$, we refer to (1.4.10):

$$
\begin{aligned}
\partial_\alpha \partial_\mu \epsilon_\nu &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial \cdot \epsilon) \\
\partial_\alpha \partial_\mu (c_{\nu\sigma\beta} x^\sigma x^\beta) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu]\partial^\mu (c_{\mu\sigma\beta} x^\sigma x^\beta) \\
c_{\nu\sigma\beta}\partial_\alpha(\delta^\sigma_\mu x^\beta + x^\sigma \delta^\beta_\mu) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu] c_{\mu\sigma\beta}(g^{\sigma\mu} x^\beta + x^\sigma g^{\beta\mu}) \\
c_{\nu\sigma\beta}(\delta^\sigma_\mu \delta^\beta_\alpha + \delta^\sigma_\alpha \delta^\beta_\mu) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](c^\sigma{}_{\sigma\beta} x^\beta + c^\beta{}_{\sigma\beta} x^\sigma) \\
2c_{\nu\mu\alpha} &= \frac{2}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu] c^\sigma{}_{\sigma\beta} x^\beta \\
c_{\nu\mu\alpha} &= \underbrace{\frac{1}{d} c^\sigma{}_{\sigma\beta}}_{b_\beta}[g_{\mu\nu}\delta^\beta_\alpha + g_{\nu\alpha}\delta^\beta_\mu - g_{\alpha\mu}\delta^\beta_\nu] \\
&= g_{\nu\mu} b_\alpha + g_{\nu\alpha} b_\mu - g_{\mu\alpha} b_\nu
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\epsilon_\mu &= c_{\mu\alpha\beta} x^{\alpha\beta} \\
&= (g_{\mu\alpha} b_\beta + g_{\mu\beta} b_\alpha - g_{\alpha\beta} b_\mu) x^\alpha x^\beta \\
&= x_\mu (b \cdot x) + x_\mu (b \cdot x) - b_\mu (x \cdot x) \\
&= 2x_\mu (b \cdot x) - x^2 b_\mu
\end{aligned}
$$

Hence, the Special Conformal Transformation looks like:

$$
\begin{aligned}
x'^\mu &= x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu \\
&= x^\mu + 2(b \cdot x) x^\nu \delta^\mu_\nu - x^2 b^\nu \delta^\mu_\nu \\
&= x^\mu + 2(b \cdot x) x^\nu \partial_\nu x^\mu - x^2 b^\nu \partial_\nu x^\mu \\
&= [1 + 2(b \cdot x) x^\nu \partial_\nu - x^2 b^\nu \partial_\nu] x^\mu \\
&= [1 + \{2b^\alpha x_\alpha x^\nu \partial_\nu - x^2 b^\alpha \partial_\alpha\}] x^\mu \\
&= [1 + ib^\alpha \underbrace{\{-i(2x_\alpha x^\nu \partial_\nu - x^2 \partial_\alpha)\}}_{K_\alpha}] x^\mu
\end{aligned}
$$

Hence, the generator for Special Conformal Transformations (SCT) takes the form $K^\mu = -i(2x^\mu x \cdot \partial - x^2 \partial^\mu)$. We will now list all the **infinitesimal** transformations and their generators we found in this section.

1. Translation

$$
x'^\mu = x^\mu + a^\mu \qquad\qquad P_\mu = -i\partial_\mu \tag{1.4.11}
$$

2. Rotation

$$
x'^\mu = x^\mu + M^\mu{}_\nu x^\nu \qquad\qquad L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{1.4.12}
$$

3. Dilatation

$$x'^{\mu} = (1+\lambda)x^{\mu} \qquad\qquad D = -ix^{\mu}\partial_{\mu} \tag{1.4.13}$$

4. Special Conformal Transformation

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b\cdot x) - x^2 b^{\mu} \qquad\qquad K^{\mu} = -i(2x^{\mu}x\cdot\partial - x^2\partial^{\mu}) \tag{1.4.14}$$

In the above listed transfomations, the parameters $a^{\mu}, M^{\mu}{}_{\nu}, \lambda$ and $b^{\mu}$ are all infinitesimal.

**Jacobian of the Transformation**

The following part is taken from "Conformal Field Theory Primer in $D \geq 3$" by Andrew Evans, pg 36:

$$\begin{aligned}
\text{Translation:} &\quad \left|\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\right| = 1 \\
\text{Rotation:} &\quad \left|\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\right| = 1 \\
\text{Dilataion:} &\quad \left|\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\right| = \lambda^{-d} \\
\text{Inversion:} &\quad \left|\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\right| = \left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}$$

Since the rest are easier to show, we will only focus on showing the last part:

$$\begin{aligned}
\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}} &= \frac{1}{\tilde{x}^2}\left[\delta^{\mu}_{\nu} - 2\frac{\tilde{x}^{\mu}\tilde{x}_{\nu}}{\tilde{x}^2}\right] \\
\det\left(\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\right) &= \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}}\frac{\partial x^{\mu_2}}{\partial \tilde{x}^{\nu}_2}\cdots\frac{\partial x^{\mu_d}}{\partial \tilde{x}^{\nu}_d} \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\left[\delta^{\mu_1}_{\nu_1} - 2\frac{\tilde{x}^{\mu_1}\tilde{x}_{\nu_1}}{\tilde{x}^2}\right]\left[\delta^{\mu_2}_{\nu_2} - 2\frac{\tilde{x}^{\mu_2}\tilde{x}_{\nu_2}}{\tilde{x}^2}\right]\cdots\left[\delta^{\mu_d}_{\nu_d} - 2\frac{\tilde{x}^{\mu_d}\tilde{x}_{\nu_d}}{\tilde{x}^2}\right] \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{i=1}^{d}\delta^{\mu_i}_{\nu_i} - 2\sum_{j=1}^{d}\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\tilde{x}^{\mu_j}\tilde{x}_{\nu_j}}{\tilde{x}^2}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} + 0
\end{aligned}$$

Now we use the identity

$$\epsilon_{i_1\ldots i_k\, i_{k+1}\ldots i_n}\epsilon^{i_1\ldots i_k\, j_{k+1}\ldots j_n} = \delta^{i_1\ldots i_k\, j_{k+1}\ldots j_n}_{i_1\ldots i_k\, i_{k+1}\ldots i_n} = k!\,\delta^{j_{k+1}\ldots j_n}_{i_{k+1}\ldots i_n}$$

which in our case becomes

$$\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} = (d-1)!\delta^{\mu_j}_{\nu_j}$$

Hence

$$\begin{aligned}
\det\left(\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\right) &= \left(\frac{1}{\tilde{x}^2}\right)^d\left(\frac{d! - 2\sum_{j=1}^{d}(d-1)!}{d!}\right) \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d\left(1 - \frac{2d(d-1)!}{d!}\right) = -\left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}$$

### 1.4.1 Transformation of Spacetime Intervals

The physical content of a coordinate transformation is best understood by examining how it affects the interval between two distinct points, $\Delta x^\mu = x_a^\mu - x_b^\mu$. While Poincaré transformations are isometries that preserve the absolute distance $|x_a - x_b|$, conformal transformations generalize this by allowing the distance to rescale in a way that depends on the group parameters and, in the case of non-linear transformations, the positions of the points themselves.

**Translations** Translations are the most trivial case of distance invariance. Given the transformation law $x'^\mu = x^\mu + a^\mu$, the difference between two transformed coordinates is:

$$\begin{aligned} x_a'^\mu - x_b'^\mu &= (x_a^\mu + a^\mu) - (x_b^\mu + a^\mu) \\ &= x_a^\mu - x_b^\mu \end{aligned}$$

Consequently, the Euclidean (or Minkowski) norm remains strictly invariant:

$$|x_a' - x_b'|^2 = |x_a - x_b|^2 \tag{1.4.15}$$

This confirms that translations do not distort the geometry or the scale of the theory; they simply shift the origin.

**Dilatations** Dilatations introduce a global change of scale. Considering the infinitesimal form $x'^\mu = (1+\lambda)x^\mu$, the transformation of the interval is:

$$\begin{aligned} x_a'^\mu - x_b'^\mu &= (1+\lambda)x_a^\mu - (1+\lambda)x_b^\mu \\ &= (1+\lambda)(x_a^\mu - x_b^\mu) \end{aligned}$$

Squaring this result to find the distance squared, we have:

$$|x_a' - x_b'|^2 = (1 + 2\lambda + \mathcal{O}(\lambda^2))|x_a - x_b|^2 \tag{1.4.16}$$

In the finite case, where $x' = e^\lambda x$, this exponentiates to:

$$|x_a' - x_b'| = e^\lambda |x_a - x_b| \tag{1.4.17}$$

While the distance is no longer invariant, the rescaling factor $e^\lambda$ is *uniform*—it is independent of $x$. This implies that while lengths change, the ratios of lengths and the angles between vectors remain fixed. This uniformity is what distinguishes dilatations from the more complex Special Conformal Transformations (SCTs), where the rescaling factor becomes a local function of the coordinates.

We find that the distances between two point scales under dilatation, therefore the natural quantity which is invariant under both translation and dilatation is

$$\begin{aligned} \frac{|x_a'^\mu - x_b'^\mu|}{|x_c'^\mu - x_d'^\mu|} &= \frac{\cancel{1+\lambda}\,|x_a^\mu - x_b^\mu|}{\cancel{1+\lambda}\,|x_c^\mu - x_d^\mu|} \\ &= \frac{|x_a^\mu - x_b^\mu|}{|x_c^\mu - x_d^\mu|} \end{aligned}$$

**Special Conformal Transformations (SCT)** The Special Conformal Transformation is the most non-trivial element of the conformal group. It can be thought of as a sequence of three operations: a discrete inversion ($x^\mu \to x^\mu/x^2$), a translation by a vector $b^\mu$, and a second inversion to return the coordinates to their original orientation. The resulting finite transformation is given by:

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2 x^2}$$

$$= \frac{x^\mu - b^\mu x^2}{\Lambda^2(x)}$$

where we have defined the local scale factor $\Lambda^2(x) = 1 - 2(x \cdot b) + b^2 x^2$. Unlike translations or dilatations, the SCT warps space non-uniformly. To see this, we evaluate how the interval between two points, $x_a$ and $x_b$, transforms under this mapping:

$$x'^\mu_a = \frac{x^\mu_a - b^\mu x^2_a}{\Lambda^2(x_a)}$$
$$x'^\mu_b = \frac{x^\mu_b - b^\mu x^2_b}{\Lambda^2(x_b)}$$

The transformation of the squared interval $|x'_a - x'_b|^2$ is the defining algebraic check for conformal invariance. By direct substitution and simplification of the numerator, one finds the remarkably elegant result:

$$|x'_a - x'_b|^2 = \frac{|x_a - x_b|^2}{\Lambda^2(x_a)\Lambda^2(x_b)} \tag{1.4.18}$$

*Proof.* We start from:

$$x'^\mu_a - x'^\mu_b = \frac{x^\mu_a - b^\mu x^2_a}{\Lambda^2(x_a)} - \frac{x^\mu_b - b^\mu x^2_b}{\Lambda^2(x_b)} \tag{1}$$

and square both sides:

$$\begin{aligned}
(x'^\mu_a - x'^\mu_b)^2 &= \left(\frac{x^\mu_a - b^\mu x^2_a}{\Lambda^2(x_a)} - \frac{x^\mu_b - b^\mu x^2_b}{\Lambda^2(x_b)}\right)^2 \\
&= \frac{x^2_a + b^2(x^2_a)^2 - 2x^2_a(x_a \cdot b)}{\Lambda^4(x_a)} + \frac{x^2_b + b^2(x^2_b)^2 - 2x^2_b(x_b \cdot b)}{\Lambda^4(x_b)} \\
&\quad - \frac{2}{\Lambda^2(x_a)\Lambda^2(x_b)}\left[x_a \cdot x_b - x^2_b(x_a \cdot b) - x^2_a(b \cdot x_b) + b^2 x^2_a x^2_b\right] \\
&= x^2_a\left[\frac{1 - 2(x_a \cdot b)}{\Lambda^4(x_a)} + \frac{\overbrace{2(b \cdot x_b) - b^2 x^2_b}^{1-\Lambda^2(x_b)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x^2_b\left[\frac{1 - 2(x_b \cdot b)}{\Lambda^4(x_b)} + \frac{\overbrace{2(b \cdot x_a) - b^2 x^2_a}^{1-\Lambda^2(x_a)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x^2_a)^2}{\Lambda^4(x_a)} + \frac{b^2(x^2_b)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x^2_a\left[\frac{1 - 2(x_a \cdot b) - \Lambda^2(x_a)}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x^2_b\left[\frac{1 - 2(x_b \cdot b) - \Lambda^2(x_b)}{\Lambda^4(x_b)}\right. \\
&\quad \left. + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] + \frac{b^2(x^2_a)^2}{\Lambda^4(x_a)} + \frac{b^2(x^2_b)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x^2_a\left[\frac{-b^2 x^2_a}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x^2_b\left[\frac{-b^2 x^2_b}{\Lambda^4(x_b)} + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x^2_a)^2}{\Lambda^4(x_a)} + \frac{b^2(x^2_b)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= \frac{(x_a - x_b)^2}{\Lambda^2(x_a)\Lambda^2(x_b)}
\end{aligned}$$

QED. □

This identity is the "holy grail" of SCT derivations. It demonstrates that while the distance is not invariant, the transformation is **locally** a dilatation. The distance is rescaled by a factor that is the geometric mean of the local scale factors at each point. This property is what allows CFT to fix the form of the two-point correlation function: since the denominator of the correlation function must transform in a way that cancels the scaling of the fields, the only possible functional form is the power law $|x_a - x_b|^{-2\Delta}$.

### 1.4.2 Anharmonic Ratios

The derivation above reveals that distance is not an invariant of the conformal group. Specifically, the ratio of the transformed distance to the original distance is:

$$\frac{\left|x'^{\mu}_a - x'^{\mu}_b\right|}{|x_a - x_b|} = \frac{1}{\Lambda(x_a)\Lambda(x_b)}$$

We can however, construct another quantity which is invariant under SCT.

$$\begin{aligned}\frac{|x'_a - x'_b|}{\left|x'_b - x'_d\right|}\frac{|x'_d - x'_c|}{|x'_c - x'_a|} &= \frac{\frac{|x_a - x_b|}{\Lambda(x_a)\Lambda(x_b)}\frac{|x_d - x_c|}{\Lambda(x_d)\Lambda(x_c)}}{\frac{|x_b - x_d|}{\Lambda(x_b)\Lambda(x_d)}\frac{|x_c - x_a|}{\Lambda(x_c)\Lambda(x_a)}} \\ &= \frac{|x_a - x_b|}{|x_b - x_d|}\frac{|x_d - x_c|}{|x_c - x_a|}\end{aligned}$$

Such expressions are called, anharmonic ratios or cross-ratios.

## 1.5 Lie Algebra of Generators

The utility of conformal symmetry lies in its algebraic structure. While the Poincaré group is generated by translations ($P_\mu$) and Lorentz transformations ($L_{\mu\nu}$), the conformal group adds the dilatation generator ($D$) and the special conformal transformation (SCT) generators ($K_\mu$). To understand how these symmetries interact, we must compute their commutation relations.

In this section, we represent the generators as differential operators acting on the space of functions:

$$\begin{aligned}P_\mu &= -i\partial_\mu \\ L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ D &= -ix^\mu\partial_\mu \\ K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\end{aligned}$$

### 1.5.1 Some useful identities

Before tackling the full algebra, we establish several fundamental commutator identities involving the coordinate $x_\mu$ and the derivative $\partial_\nu$. These allow us to simplify the more complex interactions between $D$ and $K_\mu$.

**Basic Identity:**

$$\begin{aligned}[x_\alpha, \partial_\beta]f &= x_\alpha\partial_\beta f - \underbrace{\partial_\beta(x_\alpha f)}_{(\partial_\beta x_\alpha)f + x_\alpha\partial_\beta f} \\ &= x_\alpha\partial_\beta f - x_\alpha\partial_\beta f - (\partial_\beta x_\alpha)f \\ &= -(\partial_\beta x_\alpha)f\end{aligned}$$

Thus, we have the operator identity:

$$\begin{aligned}[x_\alpha, \partial_\beta] &= -\partial_\beta x_\alpha = -g_{\beta\alpha}\partial^\mu x_\alpha \\ &= g_{\beta\alpha}\end{aligned} \tag{1.5.1}$$

**Quadratic Identity:**

$$\begin{aligned}[x^2, \partial_\beta] &= [x^\alpha x_\alpha, \partial_\beta] \\ &= x^\alpha [x_\alpha, \partial_\beta] + [x^\alpha, \partial_\beta] x_\alpha \\ &= -x^\alpha g_{\beta\alpha} - \delta^\alpha_\beta x_\alpha \\ &= -x_\beta - x_\beta \\ &= -2x_\beta \end{aligned} \tag{1.5.2}$$

Similarly, for the product $x_\mu x^\nu$:

$$\begin{aligned}[x_\mu x^\nu, \partial_\beta] &= x_\mu [x^\nu, \partial_\beta] + [x_\mu, \partial_\beta] x^\nu \\ &= -x_\nu \delta^\nu_\beta - x^\nu g_{\beta\alpha} \end{aligned} \tag{1.5.3}$$

### 1.5.2 Derivation of the Conformal Algebra

We now consider the commutation relations of the generators. By definition, translations commute with one another:

$$\begin{aligned}[P_\mu, P_\nu] &= [-i\partial_\mu, -i\partial_\nu] \\ &= -[\partial_\mu, \partial_\nu] = 0 \end{aligned}$$

The non-trivial interactions are derived as follows.

**Commutations with Dilatation ($D$)**

$$\begin{aligned}[P_\mu, D] &= [-i\partial_\mu, -ix^\alpha \partial_\alpha] \\ &= -[\partial_\mu, x^\alpha \partial_\alpha] \\ &= -x^\alpha [\partial_\mu, \partial_\alpha] - [\partial_\mu, x^\alpha] \partial_\alpha \\ &= -\delta^\alpha_\mu \partial_\alpha = -\partial_\mu = -i(-i\partial_\mu) \\ &= -iP_\mu \end{aligned}$$

This shows that $P_\mu$ has scaling dimension 1. For Special Conformal Transformation:

$$\begin{aligned}[D, K_\mu] &= -[x^\alpha \partial_\alpha, 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu] \\ &= -2[x^\alpha \partial_\alpha, x_\mu x^\beta \partial_\beta] + [x^\alpha \partial_\alpha, x^2 \partial_\mu] \\ &= -2\left\{ x^\alpha [\partial_\alpha, x_\mu x^\beta] \partial_\beta + x_\mu x^\beta [x^\alpha, \partial_\beta] \partial_\alpha \right\} \\ &\quad + x^\alpha [\partial_\alpha, x^2] \partial_\mu + x^2 [x^\alpha, \partial_\mu] \partial_\alpha \\ &= -2\left\{ x^\alpha \left( g_{\alpha\mu} x^\beta + \delta^\beta_\alpha x_\mu \right) \partial_\beta + x_\mu x^\beta (-\delta^\alpha_\beta) \partial_\alpha \right\} \\ &\quad + 2x^2 \partial_\mu - \cancel{x^\alpha x^2 \partial_\alpha \partial_\mu} + \cancel{x^2 x^\alpha \partial_\alpha \partial_\mu} - x^2 \partial_\mu \\ &= -\cancel{2x_\mu x^\beta \partial_\beta} - 2x^\beta x_\mu \partial_\beta + \cancel{2x_\mu x^\beta \partial_\beta} + x^2 \partial_\mu \\ &= -(2x^\beta x_\mu \partial_\beta - x^2 \partial_\mu) \\ &= -iK_\mu \end{aligned}$$

### Commutations with SCT ($K_\mu$)

The most complex interaction involves the translation and special conformal generators, which do not commute but rather close onto the dilatation and Lorentz generators:

$$\begin{aligned}
[P_\mu, K_\nu] &= [-i\partial_\mu, -i(2x_\mu x^\alpha \partial_\alpha - x^2\partial_\nu)] \\
&= -[\partial_\mu, 2x_\nu x^\alpha \partial_\alpha - x^2\partial_\nu] \\
&= -2x_\nu x^\alpha [\partial_\mu, \partial_\alpha] - 2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + x^2[\partial_\mu, \partial_\nu] + [\partial_\mu, x^2]\partial_\nu \\
&= -2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + [\partial_\mu, x^2]\partial_\nu \\
&= -2(g_{\mu\nu}x^\alpha + \delta^\alpha_\mu x_\nu)\partial_\alpha + 2x_\mu\partial_\nu \\
&= -2g_{\mu\nu}x^\alpha\partial_\alpha - 2(x_\nu\partial_\mu - x_\mu\partial_\nu) \\
&= -2ig_{\mu\nu}D - 2iL_{\mu\nu} \\
&= -2i(g_{\mu\nu}D - L_{\mu\nu})
\end{aligned}$$

Finally, we verify that the SCT generators commute with each other. This is a crucial property for the consistency of the conformal group:

$$\begin{aligned}
[K_\mu, K_\nu] &= -[2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu, 2x_\nu x^\beta\partial_\beta - x^2\partial_\nu] \\
&= -4[x_\mu x^\alpha\partial_\alpha, x_\nu x^\beta\partial_\beta] + 2[x_\mu x^\alpha\partial_\alpha, x^2\partial_\nu] + 2[x^2\partial_\mu, x_\nu x^\beta\partial_\beta] - [x^2\partial_\mu, x^2\partial_\nu] \\
&= -4x_\nu x^\beta\left[x_\mu x^\alpha, \partial_\beta\right]\partial_\alpha - 4x_\mu x^\alpha\left[\partial_\alpha, x_\nu x^\beta\right]\partial_\beta + 2x_\mu x^\alpha\left[\partial_\alpha, x^2\right]\partial_\nu + 2x^2\left[x_\mu x^\alpha, \partial_\nu\right]\partial_\alpha \\
&\quad + 2x^2\left[\partial_\mu, x_\nu x^\beta\right]\partial_\beta + 2x_\nu x^\beta\left[x^2, \partial_\beta\right]\partial_\mu - x^2[\partial_\mu, x^2]\partial_\nu - x^2[x^2, \partial_\nu]\partial_\mu \\
&= \cancel{4x_\nu x^\beta(g_{\mu\beta}x^\alpha + \delta^\alpha_\beta x_\mu)\partial_\alpha} - \cancel{4x_\mu x^\alpha(g_{\alpha\nu}x^\beta + \delta^\beta_\alpha x_\nu)\partial_\beta} + 4x_\mu x^2\partial_\nu - 2x^2(\cancel{g_{\mu\nu}x^\alpha} + \delta^\alpha_\nu x_\mu)\partial_\alpha \\
&\quad + 2x^2(\cancel{g_{\mu\nu}x^\beta} + \delta^\beta_\mu x_\nu)\partial_\beta - 4x_\nu x^2\partial_\mu - 2x^2x_\mu\partial_\nu + 2x^2x_\nu\partial_\mu \\
&= \cancel{4x_\mu x^2\partial_\nu - 2x_\mu x^2\partial_\nu} + \cancel{2x_\nu x^2\partial_\mu - 4x_\nu x^2\partial_\mu} - \cancel{2x^2x_\mu\partial_\nu} + \cancel{2x^2x_\nu\partial_\mu} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[K_\mu, L_{\alpha\beta}] &= [-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu), i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= [2x_\mu x^\nu\partial_\nu - x^2\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\
&= 2[x_\mu x^\nu\partial_\nu, x_\alpha\partial_\beta] - [x^2\partial_\mu, x_\alpha\partial_\beta] + \underbrace{2[x_\mu x^\nu\partial_\nu, x_\beta\partial_\alpha] - [x^2\partial_\mu, x_\beta\partial_\alpha]}_{\alpha\leftrightarrow\beta} \\
&= 2\left\{x_\mu x^\nu[\partial_\nu, x_\alpha]\partial_\beta + x_\alpha[x_\mu x^\nu, \partial_\beta]\partial_\nu\right\} - x^2[\partial_\mu, x_\alpha]\partial_\beta - x_\alpha[x^2, \partial_\beta]\partial_\mu - (\alpha\leftrightarrow\beta) \\
&= \cancel{2x_\mu x^\nu(g_{\nu\alpha})\partial_\beta} - 2x_\alpha\left(g_{\mu\beta}x^\nu + \cancel{\delta^\nu_\beta x_\mu}\right)\partial_\nu - x^2g_{\mu\alpha}\partial_\beta + 2x_\alpha x_\beta\partial_\mu - (\alpha\leftrightarrow\beta) \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2g_{\mu\alpha}\partial_\beta + \cancel{2x_\alpha x_\beta\partial_\mu} + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2g_{\mu\beta}\partial_\alpha - \cancel{2x_\beta x_\alpha\partial_\mu} \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2g_{\mu\alpha}\partial_\beta + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2g_{\mu\beta}\partial_\alpha \\
&= -g_{\mu\beta}(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha) + g_{\mu\alpha}(2x_\beta x^\nu\partial_\nu - x^2\partial_\beta) \\
&= ig_{\mu\alpha}K_\beta - ig_{\mu\beta}K_\alpha = i(g_{\mu\alpha}K_\beta - g_{\mu\beta}K_\alpha)
\end{aligned}$$

### Commutation with Rotation

$$\begin{aligned}
[P_\mu, L_{\alpha\beta}] &= [-i\partial_\mu, -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= -[\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha]
\end{aligned}$$

$$
\begin{aligned}
&= -[\partial_\mu, x_\alpha]\partial_\beta + [\partial_\mu, x_\beta]\partial_\alpha \\
&= g_{\alpha\mu}\partial_\beta - g_{\beta\mu}\partial_\alpha \\
&= i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)
\end{aligned}
$$

### 1.5.3 Summary of the Conformal Algebra

Combining these results with the standard Poincaré commutators, the full conformal algebra is:

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})
\end{aligned}
$$

## 1.6 Conformal Generators on the Field

Finite form of conformal transformation ($x' = \Lambda x$)[3]

$$
\begin{aligned}
\Phi'_a(x') &= U(\Lambda)\Phi_a(x)U^{-1}(\Lambda) \\
\Phi'_a(\Lambda x) &= \sum_b \pi_{ab}(\Lambda)\Phi_b(x) \\
&= \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x') \\
&= \pi_{ab}(e^{i\omega_g c_g})\Phi_b(e^{-i\omega_g c_g}x')
\end{aligned} \tag{1.6.1}
$$

We have dropped the $\sum$ sign and summation over repeated indices are implied. Infinitesimal form of (1.6.1):

$$
\begin{aligned}
\Phi'_a(x') &= (1 - i\omega_g \overbrace{T_g}^{\text{generator only acting on field}})_{ab}\Phi_b(\Lambda^{-1}x') \\
&= (1 - i\omega_g T_g)_{ab}\underbrace{\Phi_b[(1 - i\omega_g \overbrace{c_g}^{\text{generator which only acts on } x'^\mu})x'^\mu]}_{\Phi_b(x') + \{(1 - i\omega_g c_g)x'^\mu - x'^\mu\}\partial_\mu\Phi_b(x')} \\
&= (1 - i\omega_g T_g)_{ab}[\Phi_b(x') - i\omega_g c_g x'^\mu \partial_\mu \Phi_b(x')] \\
\Phi'(x') &= \Phi(x') - i\omega_g \left[T_g + \underbrace{c_g x'^\mu \frac{\partial}{\partial x'^\mu}}_{\text{accounts for the change in argument of field}}\right]\Phi(x') + \mathcal{O}(\omega_g^2)
\end{aligned}
$$

However, we will not use this approach but rather we will consider the transformations at origin and then translate it to every other point. This approach is based on studying the stabilizer subgroup of the Conformal Symmetry.[4] So, if we study the same at origin:

$$
\begin{aligned}
\Phi'(0) &= \Phi(x') - i\omega_g \left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]\Phi(x')\Big|_{x'=0} \\
&= \Phi(0) - i\omega_g T_g \Phi(0)
\end{aligned}
$$

[3] tobias osborne's lecture notes pg 18

[4] pg 7 of "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications"

using translation operator

$$\begin{aligned} e^{ix^\lambda P_\lambda}\Phi'(0)e^{-ix^\alpha P_\alpha} &= e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} - e^{ix^\lambda P_\lambda} i\omega_g T_g \Phi(0) e^{-ix^\alpha P_\alpha} \\ \Phi'(x) &= \Phi(x) - e^{ix^\lambda P_\lambda} i\omega_g T_g e^{-ix^\sigma P_\sigma} e^{ix^\beta P_\beta}\Phi(0)e^{-ix^\alpha P_\alpha} \\ &= \Phi(x) - i\omega_g \underbrace{e^{ix^\lambda P_\lambda} T_g e^{-ix^\sigma P_\sigma}}_{\text{we will find these “translated operators" later}} \Phi(x) \end{aligned}$$

For translation

$$\begin{aligned} \Phi'(x+a) &= e^{ia^\lambda P_\lambda}\Phi(x)e^{-ia^\alpha P_\alpha} \\ &= e^{ia^\lambda [P_\lambda,\ ]}\Phi(x) \end{aligned}$$

using (1.6.3)

$$= e^{a\cdot\partial}\Phi(x)$$

For rotation, at $x'^\mu = 0. \implies x^\mu = 0$

$$\Phi'_a(0) = \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}0) = \pi_{ab}(\Lambda)\Phi_b(0)$$

Now, assuming the generator of rotation $T_g = L_{\mu\nu}$ acts like[5]

$$L_{\mu\nu}\Phi_a(0) = S_{\mu\nu}\Phi_a(0) \tag{1.6.2}$$

at origin. At any other point, it will behave as:

$$\begin{aligned} L_{\mu\nu}\Phi_a(x) &= e^{ix^\beta P_\beta} L_{\mu\nu}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ &= \underbrace{e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}}_{?} \underbrace{e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}}_{\Phi_a(x)} \end{aligned}$$

by taking the derivative of second term, we obtain the following commutator

$$\begin{aligned} \Phi_a(x) &= e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ \partial_\mu\Phi_a(x) &= (\partial_\mu e^{ix^\lambda P_\lambda})\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)(\partial_\mu e^{-ix^\alpha P_\alpha}) \\ &= iP^\mu e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}(-iP^\mu) \\ &= iP^\mu\Phi_a(x) - i\Phi_a(x)P^\mu \\ &= i[P^\mu, \Phi_a(x)] \end{aligned} \tag{1.6.3}$$

We will now derive the form of $e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}$[6]:

$$\begin{aligned} e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} &= L_{\mu\nu} + [L_{\mu\nu}, -ix^\alpha P_\alpha] + \frac{1}{2!}[[L_{\mu\nu}, -ix^\alpha P_\alpha], -ix^\alpha P_\alpha] + \dots \\ &= L_{\mu\nu} + ix^\alpha \underbrace{[P_\alpha, L_{\mu\nu}]}_{i(g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu)} + \dots \\ &= L_{\mu\nu} + i^2 x^\alpha (g_{\alpha\mu}P_\nu - g_{\alpha\nu}uP_\mu) \\ &= L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \end{aligned}$$

[5] pg 10, paragraph 2 of “Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions"

[6] using BCH lemma $e^A B e^{-A} = e^{[A,\ ]}B$

$$= L_{\mu\nu} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{we found in section 1.4}}$$

we know, at $x' = 0$ we have $L_{\mu\nu} = S_{\mu\nu}$, so for the sake of consistency we get

$$e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} = \underbrace{S_{\mu\nu}}_{\text{Spin Operator}} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{transforms the argument of field}}$$

The exponential map of above can be found in any textbook on QFT which describes rotation or Lorentz transformation.[7] If we ignore $S_{\mu\nu}$, then we can see how the last part acts on field:

$$\begin{aligned}
x'^\mu &= \left(\delta^\mu_\nu + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right)x^\mu \\
&= x^\mu + \frac{i}{2}\omega_{\alpha\beta} i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)x^\nu \\
\Phi'(x) &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha g^{\beta\sigma}\partial_\sigma - x^\beta g^{\alpha\sigma}\partial_\sigma)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)x^\sigma\partial_\sigma\Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x\cdot\partial\Phi(x) \\
&\approx \Phi\left(x^\mu - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x^\nu\right) \\
\Phi'(x') &= \Phi(x)
\end{aligned}$$

For dilatation, at $x'^\mu = 0$, $x'^\mu = (1+\lambda)x^\mu = 0 \implies x^\mu = 0$. We have $\omega_g = \lambda$ and $T_g = D$:

$$D\Phi_a(0) = \tilde{\Delta}\Phi_a(0) \tag{1.6.4}$$

corresponding commutator (by operating it on eigenstate of dilatation)

$$\begin{aligned}
D\left|\Delta\right\rangle &= \left[D, \Phi_\Delta(0)\right]\left|0\right\rangle + \Phi_\Delta(0)D\left|0\right\rangle \\
\tilde{\Delta}\left|\Delta\right\rangle &= \left[D, \Phi_\Delta(0)\right]\left|0\right\rangle + 0 \\
\tilde{\Delta}\Phi_\Delta(0)\left|0\right\rangle &= \left[D, \Phi_\Delta(0)\right]\left|0\right\rangle
\end{aligned}$$

Applying the same procedure, we consider:

$$\begin{aligned}
e^{ix^\beta P_\beta} D e^{-ix^\sigma P_\sigma} &= D + [D, -ix^\beta P_\beta] + \frac{1}{2!}[[D, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= D - ix^\alpha(iP_\alpha) \\
&= D + x^\alpha P_\alpha \\
&= D - ix^\alpha\partial_\alpha
\end{aligned} \tag{1.6.5}$$

for the sake of consistency at $x' = 0$

$$= \tilde{\Delta} - ix^\alpha\partial_\alpha$$

[7] check eqn 1.141 and 1.150 of "QFT in curved spacetime" by Leonard Parker

Now, we consider

$$D\Phi_a(x) = (\tilde{\Delta} - ix^\alpha \partial_\alpha)\Phi_a(x)$$

redefining $\tilde{\Delta} \equiv -i\Delta$, we get

$$D\Phi_a(x) = -i(\Delta + x^\alpha \partial_\alpha)\Phi_a(x)$$

Similarly,[8]

$$\begin{aligned}
[D, \Phi_a(x)] &= De^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma}D \\
&= e^{ix^\lambda P_\lambda}\underbrace{e^{ix^\alpha P_\alpha}De^{-ix^\beta P_\beta}}_{=D+x^\alpha P_\alpha}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)\underbrace{e^{-ix^\sigma P_\sigma}De^{-ix^\alpha P_\alpha}}_{=D+x^\alpha P_\alpha}e^{-ix^\lambda P_\lambda} \\
&= e^{ix^\beta P_\beta}[D + x^\alpha P_\alpha, \Phi_a(0)]e^{-ix^\sigma P_\sigma} \\
&= e^{ix^\beta P_\beta}\underbrace{[D, \Phi_a(0)]}_{\tilde{\Delta}\Phi_a(0)}e^{-ix^\sigma P_\sigma} + e^{ix^\beta P_\beta}\underbrace{[x^\alpha P_\alpha, \Phi_a(0)]}_{=x^\alpha[P_\alpha, \Phi_a(0)]}e^{-ix^\sigma P_\sigma} \\
&= \tilde{\Delta}\Phi_a(x) - ix\cdot\partial\Phi_a(x) \\
&= -i(\Delta + x\cdot\partial)\Phi_a(x)
\end{aligned}$$

For SCT, $x'^\mu = 0 \implies x^\mu = 0$. Hence, we will consider the same equations, but in this context:

$$K_\mu\Phi_a(0) = \kappa_\mu\Phi_a(0)$$

Again, applying the same procedure,

$$\begin{aligned}
e^{ix^\beta P_\beta}K_\mu e^{-ix^\sigma P_\sigma} &= K_\mu + [K_\mu, -ix^\beta P_\beta] + \frac{1}{2!}[[K_\mu, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu - ix^\beta[K_\mu, P_\beta] + \frac{1}{2}[-ix^\beta[K_\mu, P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu + 2x^\beta(g_{\mu\beta}D - L_{\mu\beta}) + \frac{1}{2}[2x^\beta(g_{\mu\beta}D - L_{\mu\beta}), -ix^\alpha P_\alpha] \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} - ix_\mu x^\alpha[D, P_\alpha] + ix^\beta x^\alpha \underbrace{[L_{\mu\beta}, P_\alpha]}_{-i(g_{\alpha\mu}P_\beta - g_{\alpha\beta}P_\mu)} \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + x_\mu x^\alpha P_\alpha + x_\mu x^\beta P_\beta - x_\alpha x^\alpha P_\mu \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + 2x_\mu x^\alpha P_\alpha - x_\alpha x^\alpha P_\mu
\end{aligned}$$

From the generator of dilatation and SCT, we have[9]

$$[D, K_\mu] = -iK_\mu \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu$$

and

$$[D, L_{\mu\nu}] = 0 \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, S_{\mu\nu}] = 0$$

For primary fields:

$$K_\mu\Phi_a(0) = 0$$

Since, for primary field $\tilde{\Delta}$ commutes with all other operators which belong to the stability subgroup. By Schur's lemma $\tilde{\Delta} \propto I$, where $I$ is an identity operator. The SCT and momentum generator acts as ladder operator for Dilatation.

$$\begin{aligned}
[D, [P_\mu, \Phi(0)]] &= [P_\mu, [D, \Phi(0)]] + [[D, P_\mu], \Phi(0)] = -i(\Delta + 1)[P_\mu, \Phi(0)] \\
[D, [K_\mu, \Phi(0)]] &= [K_\mu, [D, \Phi(0)]] + [[D, K_\mu], \Phi(0)] = -i(\Delta - 1)[K_\mu, \Phi(0)]
\end{aligned}$$

[8] from pg 31 of 2309.10107, and $x$ is not an operator here but a number

[9] same notes, look at eqn 65 to 70 (pg 18-19), all these commutators are for $T_g$

## 1.7 Conformal Invariance of Scalar Field

Start from the free, massless scalar action in $d$ dimensions:

$$S[\phi] = \frac{1}{2}\int d^d x\, (\partial_\mu \phi)(\partial^\mu \phi).$$

Under a scale (dilatation) transformation

$$x \mapsto x' = \lambda x, \qquad \lambda > 0,$$

a primary scalar field transforms as

$$\phi(x) \mapsto \phi'(x') = \lambda^{-\Delta}\phi(x),$$

where $\Delta$ is the scaling dimension to be determined. The measure scales as[10]

$$d^d x \mapsto d^d x' = \lambda^d d^d x,$$

and the derivative transforms as

$$\partial_\mu \phi(x) \mapsto \partial'_\mu \phi'(x') = \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu\big(\lambda^{-\Delta}\phi(x)\big) = \lambda^{-1}\lambda^{-\Delta}\partial_\mu\phi(x) = \lambda^{-(\Delta+1)}\partial_\mu\phi(x).$$

Hence the kinetic density transforms as

$$(\partial\phi)^2 \mapsto \lambda^{-2(\Delta+1)}(\partial\phi)^2,$$

and the full integrand scales by

$$d^d x\,(\partial\phi)^2 \mapsto \lambda^{d-2(\Delta+1)}\, d^d x\,(\partial\phi)^2.$$

Classical scale invariance of the action requires the exponent of $\lambda$ to vanish:

$$d - 2(\Delta + 1) = 0.$$

Solving for $\Delta$ gives

$$\Delta = \frac{d}{2} - 1. \tag{1.7.1}$$

**Remark.** *We can derive this condition by considering the it from the perspective of massless scalar field $\Phi(x)$ which satisfies the following Equation of Motion:*

$$\nabla^2\Phi(x) = 0 \tag{1}$$

*The infinitesimal generator of dilatation is $D = x^\mu\partial_\mu$. Under a dilatation, the field transforms with conformal weight $\Delta$ as:*

$$\delta_D\Phi = (D + \Delta)\Phi \tag{2}$$

*Using the commutator $[\nabla^2, D] = 2\nabla^2$, we calculate the Laplacian of the transformed field:*

$$\begin{aligned}\nabla^2(\delta_D\Phi) &= \nabla^2(D+\Delta)\Phi &(3)\\ &= (D\nabla^2 + [\nabla^2, D] + \Delta\nabla^2)\Phi &(4)\\ &= (D + 2 + \Delta)\nabla^2\Phi &(5)\end{aligned}$$

*If $\nabla^2\Phi = 0$, then $\nabla^2(\delta_D\Phi) = 0$ for any $\Delta$. The generator for SCT can be given as $K_\mu = 2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu$. The field transforms accordingly as:*

$$\delta_{K_\mu}\Phi = (K_\mu + 2\Delta x_\mu)\Phi \tag{6}$$

[10] the metric and metric determinant do not change due to Weyl scaling, resulting in measure being transformation dependent

*The relevant commutator in $d$ dimensions was found to be:*

$$[\nabla^2, K_\mu] = 4x_\mu \nabla^2 + 2(2-d)\partial_\mu \tag{7}$$

*Next, upon applying the Laplacian to the transformed field, we find:*

$$\nabla^2(\delta_{K_\mu}\Phi) = \nabla^2(K_\mu + 2\Delta x_\mu)\Phi \tag{8}$$
$$= (K_\mu \nabla^2 + [\nabla^2, K_\mu] + 2\Delta x_\mu \nabla^2 + 2\Delta[\nabla^2, x_\mu])\Phi \tag{9}$$

*Assuming the field is on-shell ($\nabla^2\Phi = 0$) and using $[\nabla^2, x_\mu] = 2\partial_\mu$, we have:*

$$\nabla^2(\delta_{K_\mu}\Phi) = 0 + [2(2-d)\partial_\mu]\Phi + 0 + [2\Delta(2\partial_\mu)]\Phi \tag{10}$$
$$= [2(2-d) + 4\Delta]\partial_\mu\Phi \tag{11}$$

*For the transformation to be a symmetry, the right-hand side must vanish:*

$$2(2-d) + 4\Delta = 0 \implies \Delta = \frac{d-2}{2} \tag{12}$$

For flat spacetime, the global scale transformation and the local Weyl transformation lead to equivalent conditions on $\Delta$, but in curved space the situation is more complicated: requiring invariance under local rescalings forces both the conformal weight and a specific non-minimal coupling $\xi R\phi^2$. Consider the action for a scalar field in $d$ dimensions:

$$S[\varphi, g] = \int d^d x \sqrt{-g}\left[\frac{1}{2}g^{ab}\nabla_a\varphi\nabla_b\varphi + \frac{1}{2}\xi R\varphi^2\right].$$

Under a local Weyl transformation,

$$g_{ab} \to e^{2\sigma(x)}g_{ab}, \qquad \varphi(x) \to e^{-\Delta\sigma(x)}\varphi(x),$$

with infinitesimal form $\sigma(x) \ll 1$. We require $\delta S = 0$ for arbitrary $\sigma(x)$. The infinitesimal variations are

$$\delta g_{ab} = 2\sigma g_{ab}, \quad \delta g^{ab} = -2\sigma g^{ab}, \quad \delta\sqrt{-g} = d\,\sigma\sqrt{-g}.$$

We compute $\delta R = \delta(g^{ab}R_{ab}) = \delta g^{ab}R_{ab} + g^{ab}\delta R_{ab}$. The first term gives $\delta g^{ab}R_{ab} = -2\sigma R$. For the second term we use the Palatini identity

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_a \delta\Gamma^c_{cb}.$$

Hence

$$g^{ab}\delta R_{ab} = \nabla_c\big(g^{ab}\delta\Gamma^c_{ab}\big) - \nabla_a\big(g^{ab}\delta\Gamma^c_{cb}\big) = \nabla_c V^c,$$

with $V^c = g^{ab}\delta\Gamma^c_{ab} - g^{cb}\delta\Gamma^a_{ab}$. The variation of the Christoffel symbol for $\delta g_{ab} = 2\sigma g_{ab}$ is

$$\delta\Gamma^c_{ab} = \tfrac{1}{2}g^{cd}\big(\nabla_a\delta g_{bd} + \nabla_b\delta g_{da} - \nabla_d\delta g_{ab}\big) = \delta^c_b\nabla_a\sigma + \delta^c_a\nabla_b\sigma - g^{cd}g_{ab}\nabla_d\sigma.$$

Substituting:

$$g^{ab}\delta\Gamma^c_{ab} = g^{ab}\big(\delta^c_b\nabla_a\sigma + \delta^c_a\nabla_b\sigma - g^{cd}g_{ab}\nabla_d\sigma\big) = \nabla^c\sigma + \nabla^c\sigma - d\nabla^c\sigma = (2-d)\nabla^c\sigma,$$

$$g^{cb}\delta\Gamma^a_{ab} = g^{cb}\big(\delta^a_b\nabla_a\sigma + \delta^a_a\nabla_b\sigma - g^{ad}g_{ab}\nabla_d\sigma\big) = \nabla^c\sigma + d\nabla^c\sigma - \nabla^c\sigma = d\nabla^c\sigma.$$

Thus $V^c = (2-d)\nabla^c\sigma - d\nabla^c\sigma = 2(1-d)\nabla^c\sigma$. Therefore

$$g^{ab}\delta R_{ab} = \nabla_c V^c = 2(1-d)\nabla^2\sigma = -2(d-1)\nabla^2\sigma.$$

Adding the two contributions,

$$\boxed{\delta R = -2\sigma R - 2(d-1)\nabla^2\sigma}.$$

The kinetic Lagrangian density is $\mathcal{L}_{\text{kin}} = \frac{1}{2}g^{ab}\partial_a\varphi\partial_b\varphi\sqrt{-g}$. Using $\delta(\partial_a\varphi) = \partial_a(\delta\varphi) = -\Delta\partial_a\sigma\,\varphi - \Delta\sigma\partial_a\varphi$ and keeping only first order in $\sigma$ gives

$$\delta\mathcal{L}_{\text{kin}} = \frac{1}{2}\sqrt{-g}\left[(d-2)\sigma(\nabla\varphi)^2 + 2\Delta\,\sigma\varphi\nabla^2\varphi\right] + \text{total derivative}.$$

Hence

$$\delta S_{\text{kin}} = \int d^dx\,\sqrt{-g}\,\sigma\left(\frac{d-2}{2}(\nabla\varphi)^2 + \Delta\,\varphi\nabla^2\varphi\right).$$

For $S_\xi = \frac{1}{2}\xi\int d^dx\,\sqrt{-g}\,R\varphi^2$, we have

$$\delta(\sqrt{-g}R\varphi^2) = \sqrt{-g}\big[d\sigma R\varphi^2 - 2\Delta\sigma R\varphi^2 + \varphi^2\delta R\big],$$

with $\delta R$ from above. Thus

$$\delta S_\xi = \frac{1}{2}\xi\int d^dx\,\sqrt{-g}\,\sigma\Big[(d-2\Delta-2)R\varphi^2 - 4(d-1)(\nabla\varphi)^2 - 4(d-1)\varphi\nabla^2\varphi\Big].$$

Adding $\delta S_{\text{kin}}$ and $\delta S_\xi$:

$$\delta S = \int d^dx\,\sqrt{-g}\,\sigma\Bigg[\underbrace{\left(\frac{d-2}{2} - 2\xi(d-1)\right)}_{A}(\nabla\varphi)^2 + \underbrace{\big(\Delta - 2\xi(d-1)\big)}_{B}\varphi\nabla^2\varphi + \underbrace{\frac{\xi}{2}\big(d-2\Delta-2\big)}_{C}R\varphi^2\Bigg].$$

For $\delta S = 0$ with arbitrary $\sigma(x)$, each coefficient must vanish:

$$\begin{cases} A = 0: & \dfrac{d-2}{2} - 2\xi(d-1) = 0, \\ B = 0: & \Delta - 2\xi(d-1) = 0, \\ C = 0: & \dfrac{\xi}{2}(d-2\Delta-2) = 0. \end{cases}$$

Solving the first two yields

$$\boxed{\xi = \frac{d-2}{4(d-1)}}, \qquad \boxed{\Delta = \frac{d-2}{2}}.$$

## 1.8 Finite Conformal Transformation of Fields

To find the finite transformation generated by a vector field $\xi = \xi^\mu\partial_\mu$, we exponentiate the generator. For a scalar field $\phi(x)$, the transformed field $\phi'(x')$ is defined via:

$$\begin{aligned}\Phi'(x') &= e^{-i\omega_g\left[T_g + c_g x'^\mu\frac{\partial}{\partial x'^\mu}\right]}\Phi(x') \\ &= e^{-i\omega_g T_g}e^{-i\omega_g c_g x'\cdot\frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-i\omega_g T_g}\Phi(e^{-i\omega_g c_g}x')\end{aligned}$$

This section is taken from “advanced mathematical methods - conformal field theory" by David Duffins.[11]

$$\begin{aligned}\Phi'_a(x') = U(\Lambda)\Phi_a(x')U^{-1}(\Lambda) &= e^{-i\omega_g T_g}\Phi_a e^{i\omega_g T_g} \\ &= e^{-i\omega_g[T_g,\ ]}\Phi_a(x')\end{aligned}$$

[11] Active coordinate transformation is given as: $\Phi(x') = U(\Lambda)\Phi(x)U^{-1}(\Lambda)$ whereas passive transformation is given as $\Phi(x') = U^{-1}(\Lambda)\Phi(x)U(\Lambda)$

### 1.8.1 Translations

The generator for a translation in the $i$-direction is given by the simple derivative operator:

$$\xi = \partial_i \tag{1.8.1}$$

To generate a finite translation by a distance $a$, we exponentiate the generator $\xi = a^i\partial_i$. Acting on a coordinate $x^j$, we use the Taylor expansion of the exponential operator:

$$\begin{aligned} x'^j = e^{a^i\partial_i}x^j &= \left(1 + a^i\partial_i + \frac{1}{2!}(a^i\partial_i)^2 + \dots\right)x^j \\ &= x^j + a^i\delta_i^j + 0 \\ &= x^j + a^j \end{aligned} \tag{1.8.2}$$

For a general field $\phi(x)$, the operator $e^{a\cdot\partial}$ acts as the translation operator $T_a$, such that $e^{a\cdot\partial}\phi(x) = \phi(x+a)$. This is the familiar statement that the momentum operator (the generator of translations) generates the shift in the argument of the field. For scalar fields, we can similarly check:

$$\Phi(x) = e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} = e^{x\partial}\Phi(0) \tag{1.8.3}$$

which is identical to $\Phi'(x') = \Phi(x)$ as:

$$\begin{aligned} \Phi'(x') &= \Phi(x' - a) \\ &= e^{-a\frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-iaP}\Phi(x') \end{aligned} \tag{1.8.4}$$

### 1.8.2 Rotations

The Lorentz group consists of rotations and boosts. Since the algebraic structure for each is identical (differing only by the metric signature), we demonstrate the finite transformation for a rotation in the $xy$-plane. The generator is:

$$\xi = -y\partial_x + x\partial_y \tag{1.8.5}$$

This can be written in a more suggestive form as $\xi = \omega_{\mu\nu}x^\nu\partial^\mu$, where $\omega_{12} = -\omega_{21} = 1$. To find the finite transformation, let us see how this acts on the vector $\mathbf{x} = (x, y)^T$. We define the generator matrix $J$:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{such that} \quad \delta\begin{pmatrix} x \\ y \end{pmatrix} = \theta J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y\theta \\ x\theta \end{pmatrix} \tag{1.8.6}$$

The finite transformation for a rotation by angle $\theta$ is given by the matrix exponential $\mathcal{R} = \exp(\theta J)$. Using the property that $J^2 = -I$, $J^3 = -J$, and $J^4 = I$, the expansion sums into trigonometric functions:

$$\begin{aligned} e^{\theta\xi}x &= \sum_{n=0}^{\infty}\frac{1}{n!}\theta^n\left(-y\partial_x + x\partial_y\right)^n x \\ &= x + \theta\left(-y\partial_x + x\partial_y\right)x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 x + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 x\dots \\ &= x - \theta y - \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)y - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)x\dots \\ &= \left(1 - \frac{\theta^2}{2!} + \dots\right)x - \left(\theta - \frac{\theta^3}{3!} + \dots\right)y \\ &= \cos(\theta)x - \sin(\theta)y \end{aligned}$$

Similarly,

$$\begin{aligned} e^{\theta\xi}y &= \sum_{n=0}^{\infty} \frac{1}{n!}\theta^n \left(-y\partial_x + x\partial_y\right)^n y \\ &= y + \theta\left(-y\partial_x + x\partial_y\right) y + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 y + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 y \dots \\ &= y + \theta x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right) x - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right) y \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \dots\right) y + \left(\theta - \frac{\theta^3}{3!} + \dots\right) x \\ &= \cos(\theta) y + \sin(\theta) x \end{aligned}$$

Substituting the matrices back in, we recover the standard rotation matrix:

$$\begin{bmatrix} \eta' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta \\ x \\ y \\ z \end{bmatrix}$$

This result defines the active transformation of the coordinates $x \to x'$. It is important to note that for a field $\phi(x)$, the transformation is defined by the inverse: $\phi'(x) = \phi(e^{-\theta\xi}x)$. This ensures that the value of the field at a physical point remains invariant under the coordinate shift. While we have focused on rotations, boosts follow an identical derivation by replacing the trigonometric series with hyperbolic ones (cosh and sinh), reflecting the Lorentzian signature of the spacetime metric.

For a field with spin, such as a vector field $V^\mu(x)$, a rotation affects both the coordinate argument and the field indices. The total generator is $\mathcal{J}_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$. While $L_{\mu\nu}$ generates the orbital flow $x \to x'$, the spin generator $S_{\mu\nu}$ is a $d \times d$ matrix that mixes the components of the vector given by:

$$(S^{\alpha\beta})^\mu{}_\nu = i(\eta^{\alpha\mu}\delta^\beta_\nu - \eta^{\beta\mu}\delta^\alpha_\nu) \tag{1.8.7}$$

In the case of a rotation in the $xy$-plane ($\alpha = 1, \beta = 2$) in Euclidean space, the matrix $S \equiv S^{12}$ acts on the vector components $(V^1, V^2)$ as:

$$S = i(\eta^{1\mu}\delta^2_\nu - \eta^{2\mu}\delta^1_\nu) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{1.8.8}$$

The finite transformation of the indices is found by exponentiating this matrix. Given a rotation angle $\theta$, the transformation matrix $\Lambda$ is:

$$\Lambda = \exp(-i\theta S) = \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.8.9}$$

The full transformation of the vector field is defined such that the "arrow" rotates along with the space. For an active rotation $x' = R(\theta)x$, the field transforms as:

$$V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x) \tag{1.8.10}$$

Substituting the finite operators, we have:

$$\begin{aligned} V'^\mu(x') &= [\exp(-i\theta S)]^\mu{}_\nu \, [e^{\theta(x\partial_y - y\partial_x)} V^\nu(x')] \\ &= [\exp(-i\theta S)]^\mu{}_\nu \, V^\nu(x) \end{aligned} \tag{1.8.11}$$

This ensures that if you have a purely radial vector field $V \propto \mathbf{x}$, it remains radial after the rotation. Without the $\Lambda$ matrix, the field would "lag behind" the coordinate rotation, violating the geometric definition of a vector field.

### 1.8.3 Dilatation:

While translations and rotations preserve lengths, **dilatations** rescale them. We define a global scale transformation by $x \to x' = e^{\lambda}x$, where $\lambda$ is a real parameter. To find how a field $\Phi(x)$ behaves under this transformation, we must account for both the change in the field's argument and its intrinsic scaling dimension, $\Delta$.

**Scaling of the Field Argument:** Before considering the field's internal indices, we examine how the operator $e^{-\lambda x\cdot\partial}$ acts on the spatial dependence of a function. As you noted, this can be understood through the limit definition of the exponential. Consider the operator acting in $N$ infinitesimal steps:

$$\begin{aligned} e^{-\lambda x\cdot\partial}\Phi(x) &= \lim_{N\to\infty}\left(1-\frac{\lambda}{N}x\cdot\partial\right)^N\Phi(x) \\ &= \lim_{N\to\infty}\left(1-\frac{\lambda}{N}x\cdot\partial\right)\ldots\underbrace{\left(1-\frac{\lambda}{N}x\cdot\partial\right)\Phi(x)}_{\Phi\left[\left(1-\frac{\lambda}{N}\right)x\right]} \\ &= \Phi\left[\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^N x\right] \\ &= \Phi(e^{-\lambda}x) \end{aligned} \tag{1.8.12}$$

Thus, the operator $e^{-\lambda x\cdot\partial}$ effectively "contracts" the coordinate argument. This is the geometric part of the dilatation.

**Transformation at the Origin:** To isolate the intrinsic scaling properties of a field $\Phi$, we examine its behavior at the origin, $x = 0$. At this point, the derivative term $x\cdot\partial$ vanishes, leaving only the scaling dimension. In the active transformation picture, we define the dilatation generator $D$ such that its action on a field at the origin is:

$$i[D,\Phi(0)] = \Delta\Phi(0) \tag{1.8.13}$$

where $\Delta$ is the scaling dimension (often related to the mass dimension in QFT). Exponentiating this infinitesimal law to find the finite transformation at the origin:

$$\begin{aligned} \Phi'(0) &= e^{-i\lambda D}\Phi(0)e^{i\lambda D} \\ &= \lim_{N\to\infty}\left(1-\frac{i\lambda}{N}D\right)^N\Phi(0)\left(1+\frac{i\lambda}{N}D\right)^N \\ &= e^{-\lambda\Delta}\Phi(0) \end{aligned} \tag{1.8.14}$$

Here, the factor $e^{-\lambda\Delta}$ represents the physical "stretching" or "shrinking" of the field's magnitude. Note that in a **passive transformation**, where we view the same physical state from a rescaled coordinate system, the sign of the exponent is reversed, yielding $e^{\lambda\Delta}\Phi(0)$.

For a primary field at an arbitrary point $x$, a finite dilatation combines the scaling of the argument with the scaling of the field's amplitude. The generator of dilatations acting on the space of fields is given by:

$$\mathcal{D} = -i(x^\mu\partial_\mu + \Delta) \tag{1.8.15}$$

Acting with the finite operator $U(\lambda) = e^{i\lambda\mathcal{D}}$, the transformed field $\Phi'(x')$ is related to the original field by:

$$\begin{aligned} \Phi'(x') &= e^{\lambda(x\cdot\partial+\Delta)}\Phi(x) \\ &= e^{\lambda\Delta}\left(e^{\lambda x\cdot\partial}\Phi(x)\right) \\ &= e^{\lambda\Delta}\Phi(e^{\lambda}x) \end{aligned} \tag{1.8.16}$$

Substituting the coordinate transformation $x' = e^{\lambda}x$ (which implies $x = e^{-\lambda}x'$), we arrive at the standard transformation law for a conformal primary field under dilatation:

$$\Phi'(x') = e^{-\lambda\Delta}\Phi(e^{-\lambda}x') \tag{1.8.17}$$

This confirms that under a scale transformation $x \to \Omega x$, a primary field of dimension $\Delta$ scales as $\Phi \to \Omega^{-\Delta}\Phi$. This relation is fundamental to the *Conformal Bootstrap*, as it constrains the two-point correlation function to take the form $\langle\Phi(x)\Phi(0)\rangle \sim |x|^{-2\Delta}$.

### 1.8.4 Special Conformal Transformation

The infinitesimal special conformal transformation on coordinates can be given as:

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b\cdot x) - x^2 b^{\mu}$$

let $b^{\mu} = te^{\mu}$

$$x'^{\mu}(t) \equiv x^{\mu}(t) = x^{\mu} + 2t(e\cdot x)x^{\mu} - x^2 te^{\mu}$$

To find the finite form of the transformation we have to recursively apply the above equation multiple times (Lie Algebra sence). The usual way is to integrate the infinitesimal form. The other way, and since we know that the transformations satisfy the conformal Killing equation, is to find the integral curve of the corresponding conformal Killing vector field as they are equivalent (Differential Geometry sense). Consider the $t-$derivative of the above[12].

$$\frac{\mathrm{d}x^{\mu}(t)}{\mathrm{d}t} = 2(e\cdot x)x^{\mu} - x^2 e^{\mu} \tag{1.8.18}$$

defining $y^{\mu}(t) = \frac{x^{\mu}(t)}{x^2(t)}$

$$\begin{aligned}
\dot{y}^{\mu}(t) &\overset{\text{quotient rule}}{=} \frac{x^2\dot{x}^{\mu} - 2(\dot{x}\cdot x)x^{\mu}}{(x^2)^2} \\
&= \frac{x^2[2(e\cdot x)x^{\mu} - x^2 e^{\mu}] - 2[2(e\cdot x)x^{\nu} - x^2 e^{\nu}]x_{\nu}x^{\mu}}{x^4} \\
&= \frac{x^2[2(e\cdot x)x^{\mu} - x^2 e^{\mu}] - 2[2(e\cdot x)x^2 - x^2(e\cdot x)]x^{\mu}}{x^4} \\
&= \frac{x^2[\cancel{2(e\cdot x)x^{\mu}} - x^2 e^{\mu}] - \cancel{2(e\cdot x)x^2 x^{\mu}}}{x^4} \\
\dot{y}^{\mu}(t) &= -e^{\mu}
\end{aligned}$$

Solving the above differential equation

$$\begin{aligned}
y^{\mu}(t) &= y^{\mu}(0) - te^{\mu} \\
\frac{x^{\mu}(t)}{x^2(t)} &= \frac{x^{\mu}(0)}{x^2(0)} - te^{\mu}
\end{aligned}$$

going back to the old notation $x'^{\mu} \equiv x^{\mu}(t)$

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - te^{\mu}$$

[12] when we consider the differential equation, we are no longer thinking of it as transformation but rather flow along a trajectory parameterized by $t$. This part was taken from pg 16 of "Four point function in momentum spaces and topological terms in gravity"

$$= \frac{x^\mu}{x^2} - b^\mu \tag{1.8.19}$$

Squaring both sides

$$\begin{aligned}
\left(\frac{x'^\mu}{x'^2}\right)^2 \equiv \frac{x'^\mu}{x'^2}\frac{x'_\mu}{x'^2} &= \left(\frac{x^\mu}{x^2} - b^\mu\right)^2 \\
\frac{x'^2}{x'^4} &= \left(\frac{x^\mu}{x^2}\right)^2 + b^2 - \frac{2(x\cdot b)}{x^2} \\
\frac{1}{x'^2} &= \frac{1 + b^2x^2 - 2(x\cdot b)}{x^2} \\
x'^2 &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned} \tag{1.8.20}$$

referring to (1.8.19)

$$x'^\mu = x'^2\left[\frac{x^\mu}{x^2} - b^\mu\right]$$

and substituting (1.8.20)

$$\begin{aligned}
x'^\mu &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}\left[\frac{x^\mu}{x^2} - b^\mu\right] \\
&= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned}$$

Above procedure also suggests that, finite SCT could be described as a sequence of inversion → translation → inversion. Where inversion is defined as:

$$I(x^\mu) = \frac{x^\mu}{x^2}$$

Under SCT metric transforms as:

$$g'_{\alpha\beta}(x') = \Lambda^2 g_{\alpha\beta}(x) \tag{1.8.21}$$

*Proof.* To deduce how metric transforms under SCT, we consider

$$\frac{\partial x'^\mu}{\partial x^\nu} = \left\{\frac{\delta^\mu_\nu - 2b^\mu x_\nu}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\nu + 2b^2x_\nu)}{\Lambda^2}\right\}$$

Using the coordinate transformation rule for metric:

$$\begin{aligned}
g_{\alpha\beta}(x) &= \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g'_{\mu\nu}(x')\Big|_{x'=x'(x)} \\
&= \left\{\frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2x_\alpha)}{\Lambda^2}\right\} \\
&\quad \times \left\{\frac{\delta^\nu_\beta - 2b^\nu x_\beta}{\Lambda} - \frac{(x^\nu - b^\nu x^2)(-2b_\beta + 2b^2x_\beta)}{\Lambda^2}\right\} g'_{\mu\nu}(x') \\
&= \left\{\frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2x'_\alpha)}{\Lambda^2}\right\} \\
&\quad \times \left\{\frac{g'_{\mu\beta} - 2b_\mu x_\beta}{\Lambda} - \frac{(x_\mu - b_\mu x^2)(-2b_\beta + 2b^2x_\beta)}{\Lambda^2}\right\} \\
&= \frac{g'_{\alpha\beta} - 2b_\beta x_\alpha - 2b_\alpha x_\beta + 4b^2x_\alpha x_\beta}{\Lambda^2}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(x_\beta - b_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&+ \frac{(2(b\cdot x)x_\beta - 2b^2 x_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&- \frac{(x_\alpha - 2b_\alpha x^2)(-2b_\alpha + 2b^2 x_\beta)}{\Lambda^3} \\
&+ \frac{(2(b\cdot x)x_\alpha - 2b^2 x^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^3} \\
&+ \frac{(x^\mu - b^\mu x^2)(x_\mu - b_\mu x^2)(-2b_\alpha + 2b^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{(-2b_\beta x_\alpha - 2_\alpha x_\beta + 4b^2 x_\alpha x_\beta)(1 - 2b\cdot x + b^2 x^2)}{\Lambda^3} \\
&+ \frac{1}{\Lambda^3}\left\{2b_\alpha x_\mu - 2b_\alpha b_\beta x^2 - 2b^2 x_\alpha x_\beta + 2b^2 x^2 x_\alpha b_\beta\right. \\
&\quad \left. -4(b\cdot x)b_\alpha x_\beta + 4b^2 x^2 b_\alpha x_\beta + 4b^2 (b\cdot x)x_\alpha x_\beta - 4b^\mu x^2 x_\alpha x_\beta\right\} + (\alpha \leftrightarrow \beta) \\
&+ \frac{\left\{x^2 - 2(b\cdot x)x^2 + b^2 x^4\right\}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\}}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3} \\
&\times \big(\cancel{-2b_\beta x_\alpha} \cancel{- 2b_\alpha x_\beta} \cancel{+ 4b^2 x_\alpha x_\beta} \cancel{+ 4(b\cdot x)b_\alpha x_\beta} \cancel{- 8b^2 (b\cdot x)x_\alpha x_\beta} \\
&\quad \cancel{-2b^2 x^2 b_\beta x_\alpha} \cancel{- 2b^2 x^2 b_\alpha x_\beta} + 4b^\mu x^2 x_\alpha x_\beta + \cancel{2b_\alpha x_\beta} + \cancel{2b_\beta x_\alpha} \\
&\quad -4b_\alpha b_\beta x^2 \cancel{- 4b^2 x_\alpha x_\beta} \cancel{+ 2b^2 x^2 x_\alpha b_\beta} \cancel{+ 2b^2 x^2 x_\beta x_\alpha} \cancel{- 4bxb_\alpha x_\beta} \\
&\quad \cancel{-4bxb_\beta x_\alpha} + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha \cancel{+ 8b^2 b^2 x_\alpha x_\beta} - 8b^4 x^2 x_\alpha x_\beta\big) \\
&+ x^2 \frac{\Lambda}{\Lambda^4}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3}\left\{-4b_\alpha b_\beta x^2 + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha - 4b^4 x^2 x_\alpha x_\beta\right\} \\
&\quad + x^2 \frac{1}{\Lambda^3}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\} \\
g'_{\alpha\beta}(x') &= \Lambda^2 g_{\alpha\beta}(x)
\end{aligned}
$$

QED. □

### 1.8.5 Inversion

First we note that the inversion is a global conformal transformation and since it is undefined at origin, it does not have an infinitesimal part i.e. we can not expect inversion to be obtained by exponentiating an element from the conformal Lie algebra. It is not the connected element of conformal group and in embedding space formalism, inversion is related to parity. Another interesting point to note is that there is no parameter associated with the transformation here such as $\lambda$ for dilatation or $b^\mu$ for SCT. Lastly, it is also closely related to the stereographic projection. To show this let us study the stereographic projection of plane onto a sphere. Consider $x \in \mathbb{R}^n$, and define stereographic projection from the **north pole** of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as:

$$X^i = \frac{2x^i}{1+|x|^2}, \quad X^{n+1} = \frac{|x|^2 - 1}{1+|x|^2}$$

where $x^i$ are coordinates on the plane and $X^i$ are the coordiantes on the sphere in embedding space. Now project this point on the sphere back to $\mathbb{R}^n$ via stereographic projection from the **south pole**:

$$x'^i = \frac{X^i}{1 + X^{n+1}}$$

Substituting:

$$x'^i = \frac{\frac{2x^i}{1+|x|^2}}{1 + \frac{|x|^2-1}{1+|x|^2}} = \frac{2x^i}{(1 + |x|^2) + (|x|^2 - 1)} = \frac{2x^i}{2|x|^2} = \frac{x^i}{|x|^2}$$

Hence, the composition ($\mathbb{R}^n \xrightarrow{\text{north}} \mathbb{S}^n \xrightarrow{\text{south}} \mathbb{R}^n$) gives:

$$x^i \mapsto \frac{x^i}{|x|^2}$$

which is the **inversion** in the unit sphere. Even though this inversion does not have a killing vector associated with it, but it is reasonable to look for the killing vector associated with stereographic projection. In general, we note that these two transformation would have the following form:

$$x'^\mu = \Omega(x) x^\mu$$

If $\partial_\mu \partial_\nu (\frac{1}{\Omega}) \propto g_{\mu\nu}$. The killing vector associated with it will have the form:

$$K^A{}_\mu = \frac{1}{\Omega^2} \frac{\partial x'^A}{\partial x^\mu}$$

# Chapter 2

# Embedding coordinates for Euclidean Space

We start in the embedding space $\mathbb{R}^{d+1,1}$ with coordinates

$$X^{-1}, X^0, \underbrace{X^1, X^2, \ldots, X^d}_{X^\mu}$$

To simplify the discussion, it is convenient to introduce null coordinates, $X^M = (X^+, X^-, X^\mu)$, defined as[1]

$$\left.\begin{aligned} X^+ &= X^{-1} + X^0 \\ X^- &= X^{-1} - X^0 \end{aligned}\right\} X^{-1} = \frac{X^+ + X^-}{2}; \quad X^0 = \frac{X^+ - X^-}{2}$$

With this choice, the mostly-plus metric of the embedding space becomes

$$ds^2 = -(dX^{-1})^2 + (dX^0)^2 + \sum_{\mu=1}^{d}(dX^\mu)^2 = \sum_{\mu=1}^{d}(dX^\mu)^2 - dX^+ dX^-$$

This can be written in matrix form as

$$\eta_{MN} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & \cdots \\ -1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}$$

The power of the embedding formalism is that the conformal generators in $d$-dimensional Euclidean space can be realized as Lorentz generators in $d+2$ dimensions. To make this connection explicit, we introduce new coordinates $(\rho, \eta, x_\mu)$ via the transformation

$$\begin{aligned} X_{-1} &= \frac{\rho(1 - \eta^2 + \vec{x}^2)}{2} \\ X_0 &= \frac{\rho(1 + \eta^2 - \vec{x}^2)}{2} \\ X_\mu &= \rho x_\mu \end{aligned}$$

This parametrization makes manifest an important redundancy: scaling all embedding coordinates by an overall factor $\lambda$ corresponds simply to $\rho \mapsto \lambda\rho$. Thus, points related by rescalings $X'^A = \lambda X^A$ correspond to the same

[1] the index with lowest numeric value has the same sign in both $X^\pm$. If we had considered, $X^{d+1}$ rather than $X^{-1}$ then the definition would have been something like

$$X^\pm = X^0 \pm X^{d+1}$$

physical point in the conformal compactification. To proceed, we invert the transformation to express $(\rho, \eta, x_\mu)$ in terms of $X_A$:

$$\begin{aligned}\rho &= X_{-1} + X_0 \\ \eta &= \frac{\sqrt{\eta_{MN} X^M X^N}}{X_{-1} + X_0} \leftarrow \rho\eta = \sqrt{\eta_{MN} X^M X^N} \\ x_\mu &= \frac{X_\mu}{X_{-1} + X_0}\end{aligned}$$

Next, we compute the change of basis for derivatives. Using the chain rule, one finds

$$\begin{aligned}\frac{\partial}{\partial X_{-1}} &= \frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\ \frac{\partial}{\partial X_0} &= \frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\ \frac{\partial}{\partial X_\mu} &= \frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\end{aligned}$$

The Lorentz generators in $d+2$ dimensions are

$$J_{MN} = X_M \partial_N - X_N \partial_M$$

By combining them appropriately, we recover the familiar conformal generators in $d$ dimensions.

- **Translation**

$$\begin{aligned}P_\mu = J_{-1,\mu} + J_{0,\mu} &= (X_{-1} + X_0)\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} - \frac{\partial}{\partial X_{-1}}\right) \\ &= \rho\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - \rho x_\mu\left(\frac{1}{\rho\eta}\frac{\partial}{\partial \eta}\right) \\ &= \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x^\mu}\end{aligned}$$

- **Special Conformal Transformation**

$$\begin{aligned}K_\mu = J_{0,\mu} - J_{-1,\mu} &= (X_0 - X_{-1})\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_{-1}}\right) \\ &= \rho(\eta^2 - \vec{x}^2)\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - 2\rho x_\mu\left[\frac{\partial}{\partial \rho} - \left(\frac{\eta^2+\vec{x}^2}{2\rho\eta}\right) - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\ &= \frac{\eta^2 - \vec{x}^2}{\eta} x_\mu \frac{\partial}{\partial \eta} + (\eta^2 - \vec{x}^2)\frac{\partial}{\partial x_\mu} - 2\rho x_\mu \frac{\partial}{\partial \rho} + x_\mu \frac{\eta^2+\vec{x}^2}{\eta}\frac{\partial}{\partial \eta} + 2x_\mu(x\cdot\partial) \\ &= 2x_\mu(x\cdot\partial) - \vec{x}^2\partial_\mu + \eta^2\partial_\mu - 2\rho x_\mu\frac{\partial}{\partial \rho} + 2x_\mu\eta\frac{\partial}{\partial \eta}\end{aligned}$$

- **Dilatation**

$$\begin{aligned}D = J_{-10} &= X_{-1}\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^{-1}} = X_{-1}\frac{\partial}{\partial X_0} + X_0\frac{\partial}{\partial X_{-1}} \\ &= \frac{\rho(1-\eta^2+\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\ &\quad + \frac{\rho(1+\eta^2-\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\ &= \rho\frac{\partial}{\partial \rho} - x^\mu\frac{\partial}{\partial x^\mu} - \eta\frac{\partial}{\partial \eta}\end{aligned}$$

If we now restrict to the null cone $\eta = 0$ and fix $\rho$ (using the scaling redundancy), the generators simplify to their standard flat-space form:

$$\begin{aligned} P_\mu &= \partial_\mu \\ K_\mu &= 2x_\mu(x\cdot\partial) - x^\nu x_\nu \partial_\mu \\ D &= -x^\mu \partial_\mu \end{aligned}$$

Note that conformal algebra is satisfied by both $\pm P_\mu$ and $\pm K_\mu$. The null cone corresponds to $\eta = 0$ but no condition imposed on $\rho$. Thus, there's a gauge redundancy: different values of $\rho$ acting as scale factor for the coordinates correspond to the same physical point.

Next important to understand now is how the tensor fields transform under conformal transformation. We can use the embedding space to deduce their transformation law which is often more illuminating than the algebra gymnastics.

## 2.1 Tensor field under conformal transformation

The transformation rule for tensor field under conformal transformation is very complicated, therefore we will rely on Embedding formalism to find a simpler form. In this approach, physical spacetime coordinates $x^\mu$ are understood as projections from a higher-dimensional embedding space with coordinates $X^A \in \mathbb{R}^{d,2}$, where the conformal group $SO(d,2)$ acts linearly. Tensors in physical space are then obtained by pulling back embedding space tensors via this projection. Specifically, a tensor in physical space is related to its embedding space counterpart as follows:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \frac{\partial x^{\mu_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\mu_n}}{\partial X^{A_n}}\frac{\partial X^{B_1}}{\partial x^{\nu_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\nu_m}}\, T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

The map from embedding to physical space is given by

$$x^\mu = \frac{X^\mu}{X^+} \equiv \frac{X^\mu}{X^0 + X^{-1}}$$

but due to the projective nature of this construction—i.e., physical points correspond to rays in the embedding space—we are free to rescale $X \sim \lambda X$, which is a manifestation of dilatation symmetry. Under this rescaling, the tensor should satisfy

$$T(\lambda X) = \lambda^{-\Delta} T(X)$$

which defines its conformal weight $\Delta$. Using this, the projected tensor can be written as

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^\Delta \frac{\partial x^{\mu_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\mu_n}}{\partial X^{A_n}}\frac{\partial X^{B_1}}{\partial x^{\nu_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\nu_m}}\, T^{A_1\cdots A_n}_{B_1\cdots B_m}\left(\frac{X}{X^+}\right)$$

This includes both the Jacobian factors from the change of variables and the prefactor from conformal weight.

If we choose the embedding slice $X^+ = 1$, then the projection simplifies significantly. In this gauge, we define the following object:

$$e^\mu_A = X^+ \frac{\partial x^\mu}{\partial X^A}$$

and the projected tensor becomes

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = e^{\mu_1}_{A_1}\cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1}\cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

Using the explicit form of the projection $x^\mu = X^\mu/X^+$, we compute

$$\frac{\partial x^\mu}{\partial X^A} = \frac{\delta^\mu_A(X^0 + X^{-1}) - X^\mu(\delta^0_A + \delta^{-1}_A)}{(X^0 + X^{-1})^2}$$

and setting $X^+ = 1$, we get

$$e^\mu_A = \delta^\mu_A, \qquad e^0_A = e^{-1}_A = -x^\mu$$

However, if we don't choose the slice $X^+ = 1$, then the projected tensor carries an additional scaling dependence:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^{\Delta+n-m} \underbrace{e^{\mu_1}_{A_1}\cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1}\cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)}_{\text{depends only on physical point}}$$

This shows that tensors projected from different embedding space sections—i.e., different choices of $X^+$—differ by a power of $X^+$. So if two representations $x$ and $\tilde{x}$ correspond to the same physical point but lie on different sections (i.e., with different values of $X^+$), then the corresponding tensors are related as

$$T(\tilde{x}) = \left(\frac{X^+}{\tilde{X}^+}\right)^{\Delta+n-m} T(x)$$

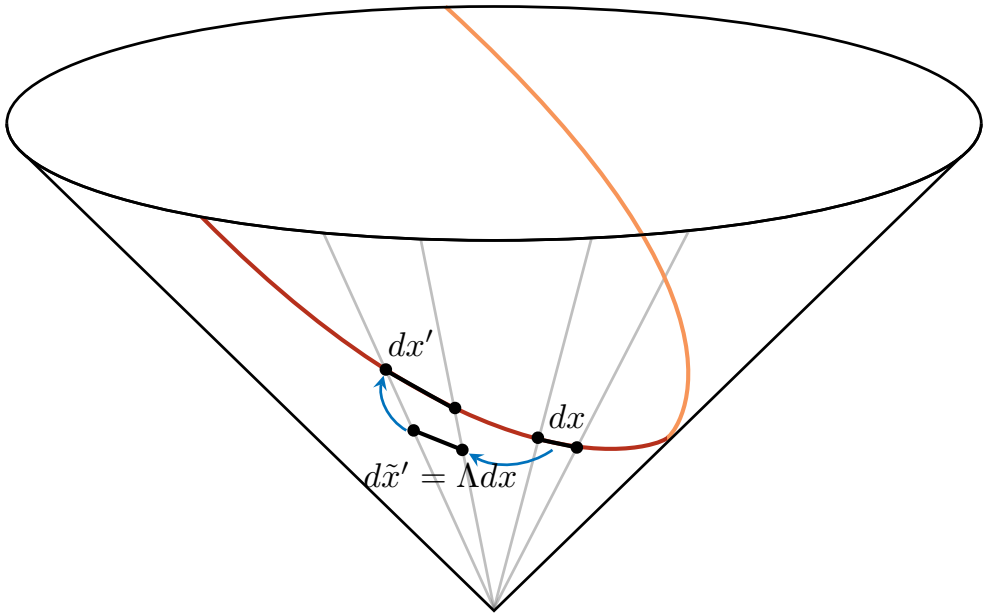



Figure 2.1: Upon Lorentz transformation, the points get mapped to different section however by utilizing the dilatation, we bring it back inside the original Euclidean section.

Next, consider how the tensor transforms under a conformal change of coordinates $x \mapsto x'$ which amounts to applying the corresponding Lorentz transformation $X \mapsto X' = \Lambda X$ in embedding space. Since the Lorentz transformation will map the tensor $T(X)$ to another tensor $T(X')$ living in different section, we will need to use above expression to map it back inside the original section labeled by $X^+$. The tensor at $\tilde{x}'$ is then:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(\tilde{x}') = \frac{\partial x'^{\mu_1}}{\partial X'^{A_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial X'^{A_n}}\frac{\partial X'^{B_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial X'^{B_m}}{\partial x'^{\nu_m}} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X')$$

Now applying the chain rule, we insert identities:

$$\frac{\partial x'^\mu}{\partial X'} = \frac{\partial x'^\mu}{\partial x^\alpha}\cdot\frac{\partial x^\alpha}{\partial X'}, \qquad \frac{\partial X'}{\partial x'} = \frac{\partial X'}{\partial x^\beta}\cdot\frac{\partial x^\beta}{\partial x'}$$

This yields:

$$\underbrace{T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(\tilde{x}')}_{\text{tensor field projected on section } X'^+} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right)$$

$$\times \underbrace{\left[\frac{\partial x^{\alpha_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\alpha_n}}{\partial X^{A_n}}\,\frac{\partial X^{B_1}}{\partial x^{\beta_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\beta_m}}\right] T^{A_1A_2\cdots A_n}_{B_1B_2\cdots B_m}(X)}_{\text{tensor field projected on section } X^+}$$

$$\left(\frac{X^+}{X'^+}\right)^{\Delta+n-m} T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(x') = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)$$

$$T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(x') = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right)\left(\frac{X'^+}{X^+}\right)^{\Delta+n-m} T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)$$

The factor $(X'^+/X^+)^{\Delta+n-m}$ arises because $x'$ and $\tilde{x}'$ are physically the same point, but obtained by projecting from different embedding sections. To express this ratio in terms of the coordinate Jacobian $|\partial x'/\partial x|$, note that from the projection $x^\mu = X^\mu/X^+$ and $x'^\mu = X'^\mu/X'^+$, one finds

$$\left|\frac{\partial x'}{\partial x}\right| = \left|\frac{\partial X}{\partial x}\cdot\frac{\partial x'}{\partial X}\right| = \left|\frac{\partial X}{\partial x}\cdot\frac{\partial x'}{\partial X'}\frac{\partial X'}{\partial X}\right| = \left|\left(\frac{X^+}{X'^+}\right)\frac{\partial X'}{\partial X}\right| = \left|\frac{X^+}{X'^+}\right|^d |\Lambda|$$

and therefore,

$$\left(\frac{X^+}{X'^+}\right)^{\Delta+n-m} = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta+n-m}{d}}$$

Finally, combining everything, the full transformation law for the projected tensor under a conformal coordinate transformation is:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta+n-m}{d}}\left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)$$

This expression makes it manifest that the projected tensor transforms as a tensor under general coordinate transformations, but with an additional *conformal weight* $\Delta + n - m$ that reflects both the homogeneity of the embedding space tensor and the number of upper and lower indices involved in the projection.

## 2.2 Finding Correlators from Embedding Space

Needless to say, it is significantly easier to construct Lorentz covariant expressions than conformally covariant ones. Therefore, the natural question arises: once we have constructed Lorentz covariant expressions in $d+2$ dimensions, **how do we descend to $d$ dimensions without breaking covariance?**

Since we have already fixed $\eta = 0$ in our derivation of the conformal generators, we now focus on the structure preserved by Lorentz transformations: the null light cone $X^2 = 0$ in embedding space. This cone, defined in $\mathbb{R}^{d+1,1}$ as the space of null rays through the origin, is given by:

$$\begin{aligned} X^2 &= -(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 \\ &= -X^+X^- + \sum_{\mu=1}^{d}(X^\mu)^2 = 0 \end{aligned}$$

Although correlators are initially constructed as Lorentz-invariant functions over the full $d+2$-dimensional ambient space, we now restrict them to the null cone. This constraint effectively reduces the support of such correlators to a $d+1$-dimensional submanifold, since one of the coordinate dependencies—say, $X^-$—can be eliminated using the condition $X^2 = 0$ (in our case it leads to $\eta = 0$).

Next, we reinterpret embedding space as a fiber bundle over the physical $d$-dimensional spacetime (where the CFT is defined). Each fiber consists of null lines in the $(d+2)$-dimensional space, and each point in the base space corresponds to an equivalence class of null vectors $X^A \sim \lambda X^A$, for any non-zero $\lambda$. This reflects the earlier observation regarding the arbitrariness of $\rho$: all such rescalings represent the same physical point in $d$ dimensions.

This identification has an important consequence: As mentioned earlier, this introduces gauge redundancy in our description. To eliminate another coordinate, say $X^+ = \rho$, we fix the gauge by selecting a section of the bundle with specific choice of the slice on the embedding space, typically the *Euclidean section*, defined by:

$$X^+ = \rho = f(X^\mu) \equiv f(x^\mu)$$

Although Lorentz transformations may take a null vector outside this section (as they mix time-like and space-like directions), theu can always be brought back by utilising the scaling equivalence $X^A \sim \lambda X^A$. This choice anchors us to physical $d$-dimensional spacetime, completing the descent from $d+2$ dimensions while maintaining conformal covariance inherited from Lorentz invariance in the higher-dimensional space. With these prescriptions in place we can now identify $X^\mu$ with the Euclidean space coordinates $x^\mu$ by stripping $\rho$ dependence.

$$X^\mu \equiv x^\mu$$

This leads to definition of $X^-$ based on null condition as:

$$X^- = \frac{\sum_{\mu=1}^{d}(X^\mu)^2}{X^+} = \frac{X^\mu X_\mu}{X^+} = \frac{x^2}{f(X^\mu)}$$

or, equivalently[2]

$$\rho(-\eta^2 + \vec{x}^2) = \frac{\rho^2\vec{x}^2}{\rho} \implies \eta = 0$$

The spacetime interval on this section is given as:

$$ds^2 = dx^2 - dX^+dX^-\Big|_{X^+=f(X^i),X^-=\frac{x^2}{X^+}}$$

This section satisfies two of the following conditions:

- section intersects each of the light rays at some point
- maps each point in $d$ dimensional Euclidean space to a point on the null cone in Embedding space.

We have shown how to get generators of conformal transformation from Lorentz generators by embedding a null cone in ambient space. Let us now analyze how Lorentz Transformation acts on a generic section on that null cone. The Lorentz transformation acting as rotation on the point $X^A$ in the null-cone will move it to another point on the null cone outside the the section $X^B = \Lambda^B{}_A X^A$.

[2] in our notation $x^\mu = \rho\vec{x}^\mu$

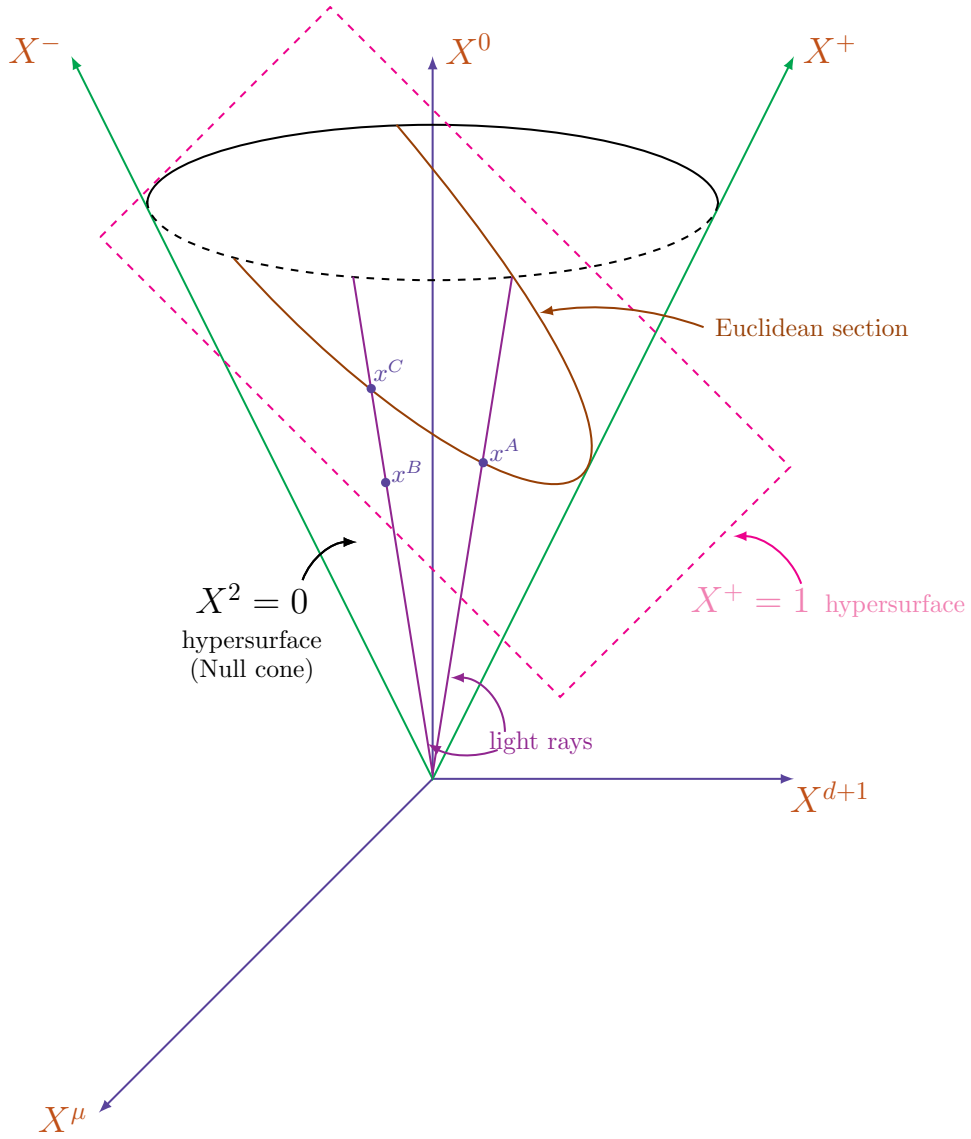



Figure 2.2: The hypersurface perpendicular to $X^+$ axis cutting at $X^+ = 1$ is shown as a plane and the null hypersurface is shown as the cone. The intersection of these two hypersurfaces describes the Euclidean Section. Dilatations are rotation in the $X^0X^{d+1}$ plane and SCT or momentum generators are rotations in $X^\mu X^{d+1}$ with $X^0X^{d+1}$ plane.

However, suppose via some conformal transformation (dilatation) in $d$ dimensional Euclidean Space, we can move $X^B$ to $X^C$ back into the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section.

$$\begin{aligned} ds_B^2 &= dX^M dX_M \\ &= d(\lambda(X)X^M)d(\lambda(X)X_M) \\ &= [\lambda dX^M + X^M(\boldsymbol{\nabla}\lambda\cdot dX)][\lambda dX_M + X_M(\boldsymbol{\nabla}\lambda\cdot dX)] \\ &= \lambda^2 dX^M dX_M + 2\lambda\underbrace{dX^M X_M}_{=0}(\boldsymbol{\nabla}\lambda\cdot dX) + \underbrace{X^M X_M}_{=0}(\boldsymbol{\nabla}\lambda\cdot dX)^2 \\ &= \lambda^2 dX^M dX_M = \lambda^2 ds_C^2 \end{aligned}$$

where we used, $X^2 = 0$ and $X^\mu dX_\mu = 0$ for restricting it to null cone.

Assuming the three conditions we used for simplification applies, the Lorentz Transformation in $d+2$-dimensional spacetime is equivalent to conformal transformation in $d-$dimensional spacetime iff metric in

$d-$dimensional space is Euclidean thus, $dX_+$ in $ds^2$ has to vanish. It gives us the condition for defining the Euclidean section as $X^+ = \rho = constant$ and thus, for the sake of simplificity, we take it as 1. Thus, we have two conditions which we can use to eliminate two extra degree of freedom.

In the embedding space formalism, choosing an Euclidean section corresponds to picking a specific way to embed the $d-$dimensional space in the $(d+2)-$dimensional space. We define the following map between $d$ dimensional Euclidean Space with conformal symmetry to null cone in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$(X^+, X^-, X^\mu) \equiv \left(1, x^2, x^\mu\right)$$

Here, we note that choosing a constant value for $X^+$ would give us a section on the cone on which the induced metric is Euclidean.

## 2.3 Tensors in Embedding Space

In this section, we will only concern ourselves with traceless and symmetric fields in $\mathbb{R}^d$ and leave the anti-symmetric tensors for future. Consider a symmetric and traceless tensor[3] $O_{M_1\ldots M_S}$ defined on the cone $X^2 = 0$ in $\mathbb{R}^{d+1,1}$. Under the rescaling $X \to \lambda X$, the tensor transforms as

$$O_{M_1\ldots M_S}(\lambda X) = \lambda^{-\Delta} O_{M_1\ldots M_S}(X)$$

i.e. it is a homogeneous function of degree $\Delta$. We expect $O_{M_1\ldots M_S}$ to get mapped to traceless and symmetric primary field in $\mathbb{R}^d$. Since, each index go from 0 to $d+1$, in $\mathbb{R}^{d+1,1}$ we find that, for $d+2-$dimensional fields other than scalar have 2 more degree of freedom per index than $d-$dimensional fields. In order to remove the extra degree of freedom, we consider the transversality condition.

$$X^{M_1} O_{M_1\ldots M_S} = 0$$

We define the physical field to be:

$$\phi_{\mu\nu\lambda\ldots}(x) = \frac{\partial X^{M_1}}{\partial x^\mu}\frac{\partial X^{M_2}}{\partial x^\nu}\frac{\partial X^{M_3}}{\partial x^\lambda}\ldots O_{M_1\ldots M_S}(X)\Big|_{X=X(x)}$$

Note that, this definition implies a redundancy. Indeed, anything proportional to $X^M$ gives zero since

$$X^2 = 0 \implies X_M \frac{\partial X^M}{\partial x^\mu} = 0$$

Therefore, $O_{M_1\ldots M_S}(X) \to O_{M_1\ldots M_S}(X) + X_{M_1} F_{M_2\ldots M_S}(X)$ gets mapped to the same physical field. This $SO(d+1,1)$ tensor is sometimes referred to as pure gauge in the literature. It is this gauge redundancy that reduces another degree of freedom per index by making it unphysical.

## 2.4 Examples: Two point and Three point correlator

In the previous section, we demonstrated how the embedding space formalism can be used to derive conformally invariant correlators. We now apply this framework to explicitly construct two- and three-point functions. From **??**, we know that ratios of distances are invariant under translations and dilatations, but not under special conformal transformations (SCTs). Our goal, therefore, is to build quantities from these ratios and the metric tensor that remain invariant under SCTs and are symmetric under the exchange $\mu \leftrightarrow \nu$.

We begin with the simplest case: the one-point function of a scalar primary operator $\hat{\mathcal{O}}_\Delta$ (which carries no Lorentz indices). Imposing conformal invariance on the correlator, the one-point function reduces to

$$\langle \hat{\mathcal{O}}_\Delta(x'^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_\Delta(x'^\mu)\rangle \tag{2.4.1}$$

[3] symmetric tensors with spin $s$ under $SO(d)$ form irreducible representations that correspond to integer spin particles (bosons). Anti-symmetric tensor fields have interpretation like they correspond to bivector of spinors etc.

This condition must be enforced for all four conformal transformations. We will begin by enforcing translation.

**Translation:** $\tilde{x}^\mu = x^\mu + a^\mu$

For translations, the Jacobian is unity, so the operator remains unchanged:

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \hat{\mathcal{O}}_\Delta(x^\mu).$$

Imposing this in (2.4.1) gives

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle.$$

Since a correlator is simply a function, this implies

$$f(\tilde{x}^\mu) = f(x^\mu).$$

Requiring invariance under arbitrary translations forces the function to be independent of its argument, hence a constant:

$$\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle = C.$$

We must still impose invariance under the remaining conformal transformations. We now turn to dilatations.

**Dilatation:** $\tilde{x}^\mu = \lambda x^\mu$

Under dilatations, the Jacobian gives the transformation

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right|^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu).$$

Imposing this in (2.4.1) yields

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu)\rangle = \lambda^{-\Delta}\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle.$$

From translation invariance, we already have

$$\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle = C,$$

so that

$$C = \lambda^{-\Delta} C.$$

Since this must hold for arbitrary $\lambda$, it follows that $C = 0$ unless $\Delta = 0$.

In a unitary CFT, the only operator with $\Delta = 0$ is the identity. Hence, all nontrivial scalar one-point functions vanish:

$$\boxed{\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle = 0 \quad \text{for } \Delta \neq 0.} \tag{2.4.2}$$

The remaining conformal transformations impose no further constraints, so this result is completely general for any CFT.

### 2.4.1 Two-point Scalar Primary

For two-point functions, we impose

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle. \tag{2.4.3}$$

We enforce this for each conformal transformation, beginning with translations.

**Translation**

Writing the correlator as a function,

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(x_1^\mu, x_2^\mu),$$

translation invariance implies

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu).$$

Using $\tilde{x}^\mu = x^\mu + a^\mu$, this becomes

$$f(x_1^\mu - a^\mu, x_2^\mu - a^\mu) = f(x_1^\mu, x_2^\mu), \quad \forall a^\mu.$$

This requires the function to depend only on the separation:

$$f(x_1^\mu, x_2^\mu) = f(x_1^\mu - x_2^\mu).$$

Hence,

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(x_1^\mu - x_2^\mu).$$

**Rotation:** $\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$

Rotations also have unit Jacobian, so

$$f(\tilde{x}_1^\mu - \tilde{x}_2^\mu) = f(x_1^\mu - x_2^\mu).$$

Substituting $\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$,

$$f\big(\Lambda^\mu{}_\nu(x_1^\nu - x_2^\nu)\big) = f(x_1^\mu - x_2^\mu).$$

Thus, the function is invariant under rotations and can only depend on the magnitude of the separation:

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(|x_1^\mu - x_2^\mu|).$$

We now proceed to impose dilatation invariance.

**Dilatation**

Under dilatations, scalar primaries transform as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu).$$

Substituting into (2.4.3) gives

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \lambda^{-(\Delta_1+\Delta_2)}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle.$$

Using the result from translation and rotation invariance,

$$f(|\tilde{x}_1^\mu - \tilde{x}_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1^\mu - x_2^\mu|).$$

Since $\tilde{x}^\mu = \lambda x^\mu$, this becomes

$$f(\lambda|x_1^\mu - x_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1^\mu - x_2^\mu|).$$

This scaling relation fixes the functional form uniquely to

$$f(|x_1^\mu - x_2^\mu|) = C\,|x_1^\mu - x_2^\mu|^{-(\Delta_1+\Delta_2)},$$

where $C$ is a constant. Therefore,

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = C\,|x_1^\mu - x_2^\mu|^{-(\Delta_1+\Delta_2)}. \tag{2.4.4}$$

We next impose invariance under special conformal transformations.

**Special Conformal Transformation**

Imposing special conformal invariance directly is cumbersome. Instead, we use the fact that an SCT can be written as an inversion, followed by a translation, followed by another inversion. Since translation invariance has already been enforced, it suffices to impose invariance under inversion:

$$x^\mu = \frac{\tilde{x}^\mu}{\tilde{x}^2}.$$

The Jacobian for inversion is

$$\left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \frac{1}{\tilde{x}^{2D}},$$

so scalar primaries transform as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \frac{1}{(\tilde{x}^2)^\Delta}\hat{\mathcal{O}}_\Delta(x^\mu).$$

Substituting into (2.4.3) gives

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2)\rangle = \frac{1}{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1)\hat{\mathcal{O}}_{\Delta_2}(x_2)\rangle.$$

Using the form fixed by dilatation invariance,

$$\frac{C}{|\tilde{x}_1-\tilde{x}_2|^{\Delta_1+\Delta_2}} = \frac{1}{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}\frac{C}{|x_1-x_2|^{\Delta_1+\Delta_2}}.$$

Rearranging,

$$\frac{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1-\tilde{x}_2|^{\Delta_1+\Delta_2}} = \frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}}. \tag{2.4.5}$$

Using the inversion identity

$$\frac{\tilde{x}_1^2\tilde{x}_2^2}{(\tilde{x}_1-\tilde{x}_2)^2} = \frac{1}{(x_1-x_2)^2},$$

this condition is satisfied only if $\Delta_1 = \Delta_2$.

Hence, the two-point function vanishes unless the operators have equal scaling dimension. The final result is

$$\boxed{\langle \hat{\mathcal{O}}_{\Delta_1}(x_1)\hat{\mathcal{O}}_{\Delta_2}(x_2)\rangle = \frac{C\,\delta_{\Delta_1\Delta_2}}{|x_1-x_2|^{\Delta_1+\Delta_2}}}. \tag{2.4.6}$$

By convention, one often normalizes operators such that $C = 1$.

**From Quantum Field Theory**

The Euclidean space correlator can be calculated by Fourier Transforming the momentum space propagator

$$G_E^{(0)}(x-x') = \int \frac{d^Dp}{(2\pi)^D}\,G_E^{(0)}(p)\,e^{ip_\mu(x_\mu-x'_\mu)}$$

where,

$$G_E^{(0)}(p) = \frac{1}{p^2}$$

Therefore, the correlation function in real (Euclidean) space is the integral

$$G_E^{(0)}(x-x') = \int \frac{d^Dp}{(2\pi)^D}\,\frac{e^{ip_\mu(x_\mu-x'_\mu)}}{p^2}$$

we can simplify the integrand by using Schwinger trick:

$$\frac{1}{p^2} = \frac{1}{2}\int_0^\infty d\alpha\, e^{-\frac{\alpha}{2}p^2}$$

Now, let's use this to re-express our correlator as double integral

$$G_E^{(0)}(x-x') = \frac{1}{2}\int_0^\infty d\alpha \int \frac{d^Dp}{(2\pi)^D}\, e^{-\frac{\alpha}{2}p^2 + ip_\mu(x_\mu - x'_\mu)}$$

The integrand is gaussian and integral can be calculated by considering shift in variable

$$\frac{\alpha}{2}p^2 - ip_\mu(x_\mu - x'_\mu) = \frac{1}{2}\Big(\sqrt{\alpha}p_\mu - i\tfrac{x_\mu - x'_\mu}{\sqrt{\alpha}}\Big)^2 + \frac{1}{2}\Big(\tfrac{x_\mu - x'_\mu}{\sqrt{\alpha}}\Big)^2$$

and by using the Gaussian integral,

$$\int \frac{d^Dp}{(2\pi)^D}\, e^{-\frac{1}{2}\left(\sqrt{\alpha}p_\mu - i\frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2} = (2\pi\alpha)^{-D/2}$$

We find the correlation function to be,

$$G_E^{(0)}(x-x') = \frac{1}{2(2\pi)^{D/2}}\int_0^\infty d\alpha\, \alpha^{-D/2}\, e^{-\frac{|x-x'|^2}{2\alpha}} = \frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{\frac{D}{2}}|x-x'|^{D-2}}$$

This is same as what we found previously by considering only symmetry constraint with $\Delta = \frac{D}{2} - 1$ as found in (1.7.1). Another approach to prove the same is to consider massless limit of massive propagator in position space. We do this by restricting ourselves to 4 dimension, We start from the scalar Feynman propagator in position space:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i\, e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}.$$

Let us denote

$$t = x^0 - y^0, \qquad \mathbf{r} = \mathbf{x} - \mathbf{y}, \qquad r = |\mathbf{r}|.$$

Then

$$D_F(t,\mathbf{r}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\int \frac{dp^0}{2\pi}\, \frac{i\, e^{-ip^0t}e^{i\mathbf{p}\cdot\mathbf{r}}}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon},$$

where

$$E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}.$$

The integrand has poles at

$$p^0 = \pm E_{\mathbf{p}} \mp i\epsilon.$$

Using the Feynman contour prescription, we get

$$\int \frac{dp^0}{2\pi}\, \frac{i\, e^{-ip^0t}}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon} = \frac{1}{2E_{\mathbf{p}}}\Big[\theta(t)e^{-iE_{\mathbf{p}}t} + \theta(-t)e^{iE_{\mathbf{p}}t}\Big].$$

Substituting back, we obtain

$$D_F(t,\mathbf{r}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{1}{2E_{\mathbf{p}}}\Big[\theta(t)e^{-iE_{\mathbf{p}}t} + \theta(-t)e^{iE_{\mathbf{p}}t}\Big]e^{i\mathbf{p}\cdot\mathbf{r}}.$$

In spherical coordinates, $d^3\mathbf{p} = p^2 dp\, d\Omega$, and

$$\int d\Omega\, e^{i\mathbf{p}\cdot\mathbf{r}} = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, e^{ipr\cos\theta} = 4\pi \frac{\sin(pr)}{pr}.$$

Hence,

$$\begin{aligned} D_F(t,r) &= \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \frac{4\pi \sin(pr)}{pr} \frac{1}{2E_p} \Big[\theta(t) e^{-iE_p t} + \theta(-t) e^{iE_p t}\Big] \\ &= \frac{1}{2\pi^2 r} \int_0^\infty dp\, \frac{p}{2E_p}\, \sin(pr) \Big[\theta(t) e^{-iE_p t} + \theta(-t) e^{iE_p t}\Big] \\ &= \frac{1}{4\pi^2 r} \int_0^\infty dp\, \frac{p}{E_p}\, \sin(pr) \Big[\theta(t) e^{-iE_p t} + \theta(-t) e^{iE_p t}\Big]. \end{aligned}$$

Let

$$E = \sqrt{p^2 + m^2} \quad \Longrightarrow \quad p = \sqrt{E^2 - m^2}, \qquad dp = \frac{E}{p} dE.$$

Therefore,

$$\frac{p}{E} dp = dE.$$

As $p$ runs from $0 \to \infty$, $E$ runs from $m \to \infty$. Thus the propagator becomes

$$D_F(t,r) = \frac{1}{4\pi^2 r} \int_m^\infty dE\, \sin\big(r\sqrt{E^2 - m^2}\big) \Big[\theta(t) e^{-iEt} + \theta(-t) e^{iEt}\Big].$$

For definiteness, take $t > 0$:

$$D_F(t,r) = \frac{\theta(t)}{4\pi^2 r} \int_m^\infty dE\, \sin\big(r\sqrt{E^2 - m^2}\big) e^{-iEt}.$$

Use the substitution $E = m\cosh u$, so that

$$\sqrt{E^2 - m^2} = m \sinh u, \qquad dE = m \sinh u\, du.$$

Then

$$D_F(t,r) = \frac{m\,\theta(t)}{4\pi^2 r} \int_0^\infty du\, \sinh u\, \sin(mr \sinh u)\, e^{-imt\cosh u}.$$

Now express the sine in exponential form:

$$\sin(mr\sinh u) = \frac{1}{2i}\Big(e^{imr\sinh u} - e^{-imr\sinh u}\Big),$$

so

$$D_F(t,r) = \frac{m\,\theta(t)}{8\pi^2 i r} \int_0^\infty du\, \sinh u \Big[e^{-im(t\cosh u - r\sinh u)} - e^{-im(t\cosh u + r\sinh u)}\Big].$$

The modified Bessel function of the second kind has the integral representation

$$K_\nu(z) = \int_0^\infty e^{-z\cosh u} \cosh(\nu u)\, du.$$

By appropriate analytic continuation $z \to im\sqrt{t^2 - r^2}$ and differentiation, one finds

$$\int_0^\infty du\, \sinh u\, e^{-im(t\cosh u - r\sinh u)} = \frac{\sqrt{t^2 - r^2}}{imt}\, K_1\big(im\sqrt{t^2 - r^2}\big),$$

and after restoring the correct $i\epsilon$-prescription, the final Lorentz-invariant result is

$$\boxed{D_F(x-y) = \frac{m}{4\pi^2\sqrt{-(x-y)^2 + i\epsilon}}\, K_1\Big(m\sqrt{-(x-y)^2 + i\epsilon}\Big).}$$

Taking the massless limit, we arrive at:

$$D_F(x-y) = \frac{1}{4\pi^2 |x - x'|^2}$$

### 2.4.2 Three-point Scalar Primary

For the three-point function, we impose

$$\langle\hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = \langle\hat{\tilde{\mathcal{O}}}_1(x_1^\mu)\hat{\tilde{\mathcal{O}}}_2(x_2^\mu)\hat{\tilde{\mathcal{O}}}_3(x_3^\mu)\rangle. \tag{2.4.7}$$

As in the two-point case, we enforce each symmetry in turn. The steps are analogous, so we streamline the presentation while highlighting the key results.

#### Poincaré

Translation invariance implies that the correlator depends only on coordinate differences, while rotational invariance further restricts this dependence to their magnitudes. With three points, there are three independent separations, so the most general form is

$$\langle\hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|),$$

where $x_{ij}^\mu = x_i^\mu - x_j^\mu$.

#### Dilatation

Under dilatations, scalar primaries transform as $\hat{\mathcal{O}}_\Delta \to \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta$, giving

$$\langle\hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(\tilde{x}_3^\mu)\rangle = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)}\langle\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle.$$

Using the Poincaré result, this implies

$$f(\lambda|x_{12}^\mu|, \lambda|x_{23}^\mu|, \lambda|x_{31}^\mu|) = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|).$$

To understand this constraint, we expand $f$ as

$$f = \sum_{n,m,p} c_{nmp}\, |x_{12}^\mu|^n |x_{23}^\mu|^m |x_{31}^\mu|^p.$$

Substituting into the scaling relation shows that all terms must vanish unless

$$n + m + p = -(\Delta_1 + \Delta_2 + \Delta_3).$$

Thus, dilatation and Poincaré invariance restrict the correlator to a homogeneous function of this degree.

#### Special Conformal Transformation

As before, it is sufficient to impose inversion invariance. Although the algebra is somewhat involved, the result is a further constraint that fixes the exponents uniquely:

$$\begin{aligned} n &= \Delta_1 + \Delta_2 - \Delta_3, \\ m &= \Delta_1 + \Delta_3 - \Delta_2, \\ p &= \Delta_2 + \Delta_3 - \Delta_1. \end{aligned} \tag{2.4.8}$$

Combining all constraints, the three-point function is fixed to be

$$\boxed{\langle\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle = \frac{C_{123}}{|x_{12}^\mu|^n\, |x_{23}^\mu|^m\, |x_{31}^\mu|^p}}. \tag{2.4.9}$$

Thus, conformal symmetry completely fixes the spacetime dependence of the three-point function, leaving only the constants $C_{123}$. These coefficients are theory-dependent and encode the interaction strength between operators. They are commonly referred to as the *three-point coefficients*, *OPE coefficients*, or *structure constants*.

### 2.4.3 Going beyond scalars

Moving on, we next consider the two point corrector of vector field. The ansatz for such a correlator is:[4]

$$\langle J_\mu(x_1)J_\nu(x_2)\rangle = C\underbrace{\frac{1}{|x_1-x_2|^{2\Delta}}}_{\text{same as scalar case}}\left[g_{\mu\nu}+\delta\frac{(x_1-x_2)_\mu(x_1-x_2)_\nu}{(x_1-x_2)^2}\right]$$

The correlation function is invariant under translation, therefore we will consider following redefinition:

$$\begin{aligned}\langle J_\mu(x_1)J_\nu(x_2)\rangle &= \langle J_\mu(x_1-x_2)J_\nu(0)\rangle \\ &= \langle J_\mu(x_{12})J_\nu(0)\rangle = \langle J_\mu(x)J_\nu(0)\rangle\end{aligned}$$

Since SCT is just inversion→translation→inversion, we can this property to our advantage. As the correlation function is already invariant under translations, it suffices to verify its invariance under inversions. If this property holds, then by extension, the correlation function will also be invariant under SCT. The inversion transformation is given as[5]:

$$x'_\mu = \frac{x_\mu}{x^2} \qquad\qquad |x'|^2 = \frac{1}{|x|^2}$$

and

$$\frac{\partial x'_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\frac{x_\nu}{x^2} = \frac{1}{x^2}\left[g_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2}\right] = x'^2\underbrace{\left[g_{\mu\nu}-2\frac{x'_\mu x'_\nu}{x'^2}\right]}_{I_{\mu\nu}}$$

The vector field would transform as

$$\left\langle J_\mu(x'_1)J_\nu(x'_2)\right\rangle = \underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d}\left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}}_{\text{this was used to derive the correlation function for scalar case}}\overbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{-1/d}\left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{-1/d}\frac{\partial x'_\alpha}{\partial x^\mu}\frac{\partial x'_\beta}{\partial x^\nu}}^{\text{transformation rule for indices}}\left\langle J^\alpha(x_1)J^\beta(x_2)\right\rangle$$

we see that

$$\underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d}_{x=x_1}}_{|x'_1|^{-2\Delta}}\left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}_{x=x_2}\frac{1}{|x_{12}|^{2\Delta}} = |x'_1|^{-2\Delta}|x'_2|^{-2\Delta}\frac{1}{|x_{12}|^{2\Delta}} = \frac{1}{|x'_{12}|^{2\Delta}}$$

where we used

$$\begin{aligned}|x'_{12}|^2 &= \left(\frac{x_1^\mu}{x_1^2}-\frac{x_2^\mu}{x_2^2}\right)^2 \\ &= \frac{|x_1|^2}{x_1^4}+\frac{|x_2|^2}{x_2^4}-2\frac{x_1^\mu}{x_1^2}\frac{x_{2\mu}}{x_2^2} \\ &= \frac{1}{x_1^2}-2\frac{x_1}{x_1^2}\cdot\frac{x_2}{x_2^2}+\frac{1}{x_2^2}\end{aligned}$$

[4] pg 24 of "CFT with boundary and defects" by Herzog

[5] pg 17-18 of "Quantum Gravity and Cosmology based on Conformal Field Theory" and section 4.5 of "A conformal field theory primer in $D \geq 3$" by Andrew Evans

$$= \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} = \frac{|x_{12}|^2}{|x_1|^2|x_2|^2} = \frac{|x_{12}|^2}{|x_1'|^{-2}|x_2'|^{-2}}$$

Then, we only have to ensure that $g^{\mu\nu} + \delta\frac{x_{12}'^\mu x_{12}'^\nu}{x_{12}'^2}$ is invariant under inversion.[6]

$$\begin{aligned}
g^{\mu\nu} + \delta\frac{(x_{12}')^\mu (x_{12}')^\nu}{(x_{12}')^2} &= \left(\delta_\alpha^\mu - 2\frac{x_1^\mu x_{1\alpha}}{x_1^2}\right)\left(\delta_\beta^\nu - 2\frac{x_2^\nu x_{2\beta}}{x_2^2}\right)\left[g^{\alpha\beta} + \delta\frac{x_{12}^\alpha x_{12}^\beta}{x_{12}^2}\right]\Bigg|_{x^\mu = \frac{x'^\mu}{|x'|^2}} \\
&= \left(\delta_\alpha^\mu - 2\frac{x_1'^\mu x_{1\alpha}'}{|x_1|'^2}\right)\left(\delta_\beta^\nu - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right) \\
&\quad \times \left[g^{\alpha\beta} + \delta\frac{(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\alpha (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2}\right] \\
&= \left[g^{\mu\beta} + \delta\frac{(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\mu (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2}\right. \\
&\quad \left. - 2\delta\frac{x_1'^\mu(\cancel{|x_1'|^2}|x_2'|^2 - x_1'\cdot x_2'\cancel{|x_1'|^2})(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2\cancel{|x_1'|^2}|x_2'|^2}\right]\left(\delta_\beta^\nu - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right) \\
&= \left[g^{\mu\beta} - \delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2}\right] \\
&\quad \times \left(\delta_\beta^\nu - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right) \\
&= \left[g^{\mu\nu} - \delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2}\right] \\
&\quad - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 2\delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu (x_2'\cdot x_1'\cancel{|x_2'|^2} - \cancel{|x_2'|^2}|x_1'|^2)x_2'^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2\cancel{|x_2'|^2}} \\
&\quad + 4\frac{x_1'^\mu x_2'^\nu (x_1'\cdot x_2')}{|x_1|'^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu (x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad - \delta\frac{\{x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2')\}^\mu \{x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2(x_1'\cdot x_2')x_2'\}^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu (x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad - \delta\frac{\{x_2'|x_1'|^2 + |x_1' - x_2'|^2 x_1' - |x_1'|^2 x_1'\}^\mu \{x_1'|x_2'|^2 + |x_1' - x_2'|^2 x_2' - |x_2'|^2 x_2'\}^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2}
\end{aligned}$$

[6] the conformal factor is there following eqn 55 of TASI Lectures on the Conformal Bootstrap. The tensor operator under inversion transforms as mentioned in eqn 3.18 of Conformal Field Theory with Boundaries and Defects or eqn 1.55 and 1.60 of EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions

$$O'^\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta+1}{d}} \frac{\partial x'^\mu}{\partial x^\nu} O^\nu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\Delta/d} I_\nu^\mu(x') O^\nu(x')$$

$$O'_\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta-1}{d}} \frac{\partial x^\nu}{\partial x'^\mu} O_\nu(x')$$

$$
\begin{aligned}
&= g^{\mu\nu} - 2\frac{x_1'^{\mu}x_1'^{\nu}}{|x_1|'^2} - 2\frac{x_2'^{\nu}x_2'^{\mu}}{|x_2'|^2} + 4\frac{x_1'^{\mu}x_2'^{\nu}(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad + \delta\frac{x_{12}'^{\mu}x_{12}'^{\nu}}{x_{12}'^2} - \delta\frac{x_1'^{\mu}}{|x_1'|^2}x_{12}'^{\nu} + \delta\frac{x_2'^{\nu}}{|x_2'|^2}x_{12}'^{\mu} - \delta|x_{12}'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - (\delta+2)\frac{x_1'^{\mu}x_1'^{\nu}}{|x_1|'^2} - (\delta+2)\frac{x_2'^{\nu}x_2'^{\mu}}{|x_2'|^2} + 2(\delta+2)\frac{x_1'^{\mu}x_2'^{\nu}(x_1'\cdot x_2')}{|x_1|'^2} + \cancel{\delta\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2}} \\
&\quad \cancel{+\delta\frac{x_1'^{\mu}x_2'^{\nu}}{|x_2'|^2}} \cancel{-\delta|x_1'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2}} \cancel{-\delta|x_2'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2}} + \delta\frac{x_{12}'^{\mu}x_{12}'^{\nu}}{x_{12}'^2}
\end{aligned}
$$

which implies $\delta = -2$. Hence, the two point function is given as

$$\langle J_\mu(x)J_\nu(0)\rangle = \frac{C}{|x|^{2\Delta}}\left[g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right]$$

The embedding space formalism gives the same answer[7]: Considering a tensor field of $SO(d+1,1)$ denoted as $O_{A_1\ldots A_n}(X)$, with the properties

- defined on the null-cone $X^2 = 0$,
- traceless and symmetric,
- homogeneous of degree $-\Delta$ in $X$, i.e., $O_{A_1\ldots A_n}(\lambda X) = \lambda^{-\Delta}O_{A_1\ldots A_n}(X)$,
- transverse $X^{A_i}O_{A_1\ldots A_n}(X) = 0$, with $i = 1, \ldots, n$

It is clear that those are conditions rendering $O_{A_1\ldots A_n}(X)$ manifestly invariant under $SO(d+1,1)$. In order to find the corresponding tensor in $\mathbb{R}^d$, one has to restrict $O_{A_1\ldots A_n}(X)$ to the Poincaré section and project the indices as

$$\langle O^\mu(x_1)O^\nu(x_2)\rangle = \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_1^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle$$

For example, the most general form of the two-point function of two operators with spin-1 and dimension $\Delta$ can be derived as:[8]

$$\langle O^A(X_1)O^B(X_2)\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB} + \alpha\frac{X_2^A X_1^B}{X_1\cdot X_2} + \beta\frac{X_1^A X_2^B}{X_1\cdot X_2}\right]$$

We will drop the last term as it projects to zero anyways.

$$\langle O^A(X_1)O^B(X_2)\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB} + \alpha\frac{X_2^A X_1^B}{X_1\cdot X_2}\right]$$

According to the transverse condition

$$X_{A1}\langle O^A(X_1)O^B(X_2)\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}[X_1^B + \alpha X_1^B] = 0 \implies \alpha = -1$$

we now use the projection to find the correlation function in $\mathbb{R}^d$:

[7] section 2.4 of "Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions" or section 5.2.2 of "Conformal Field Theory" by Loriano Bonora

[8] here the terms in bracket is chosen such that they are invariant under the replacement $x \to \lambda x$. We are not using the transformation law for any of them. Under which, even the metric will change to $\eta_{AB} \to \lambda^{-2}\eta_{AB}$.

$$\begin{aligned}\langle O_\mu(x_1)O_\nu(x_2)\rangle &= \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle \\ &= \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta_{AB}-\frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right] \\ &= \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB}-\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right] \\ &= \frac{C_{12}}{(x_1-x_2)^{2\Delta}}\left[g_{\mu\nu}-2\frac{(x_1-x_2)_\mu(x_1-x_2)_\nu}{(x_1-x_2)^2}\right]\end{aligned}$$

where we used $X^A=(X^a,X^+,X^-)=(x^a,1,x^2)$, $X_B=\left(x_a,-\frac{1}{2}x^2,-\frac{1}{2}\right)$ and $\eta_{ab}=I_{d\times d}$ with $\eta_{+-}=\eta_{-+}=-1/2$

$$\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB}=g_{\mu\nu}$$

$$\begin{aligned}\frac{\partial X_1^A}{\partial x_1^\mu}X_{A2} &= \eta_{ab}\frac{\partial x_1^a}{\partial x_1^\mu}x_2^b-\frac{1}{2}\frac{\cancelto{0}{\partial 1}}{\partial x_1^\mu}x_2^2-\frac{1}{2}\frac{\partial x_1^2}{\partial x_1^\mu}1 \\ &= \eta_{ab}\delta_\mu^a x_2^b-x_{1\mu}=(x_2-x_1)_\mu=-(x_1-x_2)_\mu\end{aligned}$$

$$\frac{\partial X_2^B}{\partial x_2^\nu}X_{B1}=\eta_{ab}\frac{\partial x_2^a}{\partial x_2^\mu}x_1^b-\frac{1}{2}\frac{\cancelto{0}{\partial 1}}{\partial x_2^\mu}x_1^2-\frac{1}{2}\frac{\partial x_2^2}{\partial x_2^\mu}1=(x_1-x_2)_\nu$$

$$(X_1-X_2)^A(X_1-X_2)_A=(x_1-x_2)^a(x_1-x_2)_a-\frac{1}{2}(1-1)(x_1^2-x_2^2)-(x_1^2-x_2^2)(\tfrac{1}{2}-\tfrac{1}{2})$$

$$\implies X_1\cdot X_2=-\frac{1}{2}(x_1-x_2)^2$$

Next, we boostrap three point correlator. [9] On the null cone we will have

$$\langle\phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle=\frac{W_M}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}}$$

where the powers $\alpha_{ijk}$ of the scalar factor are determined by the dilatation as in case of scalar operators and the tensor structure $W_M$ equals to

$$W_M=\frac{(-2X_2\cdot X_3)X_{1M}-(-2X_1\cdot X_3)X_{2M}-(-2X_1\cdot X_2)X_{3M}}{(-2X_1\cdot X_2)^{\frac{1}{2}}(-2X_1\cdot X_3)^{\frac{1}{2}}(-2X_2\cdot X_3)^{\frac{1}{2}}}.$$

Let us comment a few things on the tensor structure. The relative sign is, as before, fixed by transversality.

$$\begin{aligned}(X_1)^M W_M&=0\\(X_2)^M W_M&=0\\(X_3)^M W_M&=0\end{aligned}$$

We drop the term proportional to $X_{3M}$, since would project to zero anyway. The scaling behavior of correlation function under dilatation is completely determined in the scalar part so the tensor structure have scaling 0 in all variables ($X\to\lambda X\implies W_\mu\to\lambda^0 W_\mu$). Finally, it is immediate to check that the tensor structure is transverse, i.e. $(X_3)_M W_M=0$. Projecting to physical space as:

$$\langle\phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle=\frac{\partial X_3^M}{\partial x_3^\mu}\langle\phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle$$

[9] pg 30 of Masters Thesis on “Spinning Correlators at Finite Temperature" of Oscar Arandes Tejerina

we find, as explicitly computed before,

$$\frac{\partial X_3^M}{\partial x_3^\mu} X_{iM} = (x_i - x_3)_\mu, \quad i = 1, 2$$
$$-2X_i \cdot X_j = (x_i - x_j)^2, \quad i = 1, 2, 3 \ (i < j),$$

so that we end up with the tensor structure

$$W_\mu = \frac{|x_2 - x_3|^2 (x_1 - x_3)_\mu - |x_1 - x_3|^2 (x_2 - x_3)_\mu}{|x_1 - x_2||x_1 - x_3||x_2 - x_3|} = \frac{|x_{23}|^2 (x_{13})_\mu - |x_{13}|^2 (x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}$$

Therefore, the three-point function of two scalars and one spin 1 operators in physical space is given by

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle &= \frac{\frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \\ &= \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2+1}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1+1}}\end{aligned}$$

The three-point function of higher-spin operators $J_{\mu_1\ldots\mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that $W_\mu$ is the only indexed object for three points that is conformal invariant.

## 2.5 Fermions in Embedding Space

Following is taken from section 3.2 of Lectures on Conformal Field Theories by Hugh Osborn. To discuss spinor fields in the embedding formalism requires extending the usual $d-$dimensional gamma matrices to $d+2$ dimensions. For $d = 2n$, we define[10]

$$\begin{aligned} a_0^\pm &= \frac{1}{2}(\pm\gamma^0 + \gamma^1) \\ a_1^\pm &= \frac{1}{2}(\gamma^2 \pm i\gamma^3) \\ a_2^\pm &= \frac{1}{2}(\gamma^4 \pm i\gamma^5) \\ &\vdots \\ a_{\frac{d-2}{2}}^\pm &= \frac{1}{2}(\gamma^{d-2} \pm i\gamma^{d-1}) \end{aligned}$$

where gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

One can show that:

$$\begin{aligned} &\{a_i^-, a_j^-\} = \{a_i^+, a_j^+\} = 0 \\ &\{a_i^-, a_j^+\} = \delta_{ij} \qquad i, j = 0, 1, 2 \ldots {}^{d-2}\!/_2. \end{aligned} \tag{2.5.1}$$

In the literature ${}^{d-2}\!/_2$ is defined as another variable labeled by $k$, but for the sake for clarity we will keep it explicit. This is the algebra of raising and lowering operators for ${}^{d}\!/_2$ independent two-level systems. We ask how many basis vectors are there (including lowest weight state) which could be formed by operating ${}^{d}\!/_2$ raising $a_i^+$ on lowest weight state:[11]

$$\sum_{r=0}^{{}^{d}\!/_2} {}^{d/2}C_r = 2^{d/2}$$

[10] we have abused notation for the sake of avoiding cluttering of indices and $\pm$

[11] we would like to remind ourselves that number of linearly independent basis is defined as the dimension of space.

It implies that in $d-$dimensions, we have $2^{d/2} \times 2^{d/2}$ dimensional matrix representation for $\gamma-$matrices. We will use the highest weight representation to determine $a_j^{\pm}$ and then use them to construct $\gamma_\mu$. From (2.5.1), we quickly observe that

$$(a_i^-)^2 = 0 = (a_i^+)^2$$

It implies that we can only act $a_i$ or $a_i^\dagger$ once on a state, the second time it acts the state is annihilation. We will build off our intuition from harmonic oscillator (fermionic) and assume that there is a lowest weight state $|\xi\rangle$ such that

$$a_i^- |\xi\rangle = 0 \quad \text{for all } i$$

Similarly, acting on it once by each $a_i^\dagger$ for all $i$, we can construct states in the representation. The states can be labeled $s = (s_0, s_1, ..., s_{\frac{d-2}{2}})$, where each of the $s_a = \pm\frac{1}{2}$:

$$|\xi^{(s)}\rangle = (a^+_{\frac{d-2}{2}})^{s_{\frac{d-2}{2}}+\frac{1}{2}} \dots (a_0^+)^{s_0+\frac{1}{2}} |\xi\rangle \tag{2.5.2}$$

The lowest weight state $|\xi\rangle$ corresponds to all $s_a = -\frac{1}{2}$. Taking the $|\xi^s\rangle$ as a basis, we derive the matrix elements of $\gamma_\mu$ from the definitions and the anti-commutation relation. Starting with $d = 2$, we have a single two-level system:

$$|\xi^{(\frac{1}{2})}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |\xi^{(-\frac{1}{2})}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

we can construct the raising and lowering operator connecting these two matrices as:

$$a_0^- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \qquad a_0^+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

we find:

$$\gamma^0 = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \qquad \gamma^1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

For $d = 4$, we have 2 independent fermionic oscillator:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

we construct, the following $a_i^+$ and $a_i^-$ operators for $i = 0, 1$.

$$a_0^+ = \begin{pmatrix}0 & -1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix} \qquad a_0^- = \begin{pmatrix}0 & 0 & 0 & 0\\-1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix}$$

and

$$a_1^+ = \begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \qquad a_1^- = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}$$

From (2.5.2) we see that[12]

$$a_0^+ \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \qquad a_1^+ \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}; \qquad a_0^+ a_1^+ \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Thus, we conclude that the gamma matrices are gives as:

$$\gamma^0 = \begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix} \qquad \gamma^1 = \begin{pmatrix}0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} \qquad \gamma^3 = \begin{pmatrix}0 & 0 & -i & 0\\ 0 & 0 & 0 & -i\\ i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{pmatrix}$$

The above choice of gamma matrices satisfy the clifford algebra, however the chosen basis is not familiar from QFT textbooks. Given a representation $\gamma^\mu$ in $d$ dimensions, we can construct a representation $\Gamma^\mu$ in $d+2$ dimensions using the prescription,

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -\gamma^\mu \otimes \sigma^3, \quad \mu = 0, \ldots, d-3,$$

$$\Gamma^{d-2} = \mathbb{I} \otimes \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathbb{I} \otimes \sigma^1, \quad \Gamma^{d-1} = \mathbb{I} \otimes \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \mathbb{I} \otimes \sigma^2$$

where the $\sigma^i$ obey

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

The $2 \times 2$ matrices that we add act on the index $s_{d-2/2}$, which newly appears in going from $d = 2k$ to $2k+2$ dimensions. In odd dimensions the first $d-1$ gamma matrices can be constructed as above, and $\Gamma_d = \pm\Gamma_1\Gamma_2\ldots\Gamma_{d-1}$ completes the gamma matrix algebra. There are two independent representations of the gamma matrix algebra in odd dimensions, differing in the sign of $\Gamma_d$. These representations are exchanged by parity, and both representations appear in a parity-conserving theory.

We now move onto calculating the correlation function involving spinors in embedding space formalism. To define spinor fields on null cone in embedding space requires that the number of component in $\mathbb{R}^d$ is half the number of components in $\mathbb{R}^{d+1,1}$.

$$\psi(x) \to \Psi(X), \qquad \bar{\psi}(x) \to \bar{\Psi}(X)$$

which satisfies the following homogeneity condition:

$$\Psi(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\Psi(X), \qquad \bar{\Psi}(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\bar{\Psi}(X)$$

The degrees of freedom of $\Psi; \bar{\Psi}$ are reduced to those for $\psi; \bar{\psi}$ by imposing the transversality condition like before:

$$\bar{\Gamma}_A X^A \Psi(X) = 0 \qquad \bar{\Psi}(X)\Gamma_A X^A = 0$$

It introduces the gauge invariance and thus the degrees of freedom are now halved by imposing the equivalence relations

$$\Psi'(X) \sim \Psi' + \bar{\Gamma}_A X^A \zeta(X) \qquad \bar{\Psi}'(X) \sim \bar{\Psi}' + \bar{\zeta}(X)\Gamma_A X^A \tag{2.5.3}$$

[12] $a_0^\pm$ acts like raising and lowering operator in the same oscillator while $a_1^\pm$ changes the oscillator.

for arbitrary spinor $\zeta(X); \bar{\zeta}(X)$ of appropriate homogeneity. From standard QFT, we are familiar that

$$V_A = \bar{\Psi}\Gamma_A\Psi'$$

transforms like a vector and we also have

$$\Psi(X) = \Gamma_B X^B \Psi'(X),$$
$$\bar{\Psi}(X) = \bar{\Psi}'(X)\bar{\Gamma}_B X^B.$$

Now compute the contraction:

$$V_A = \bar{\Psi}(X)\Gamma_A\Psi'(X) = \left(\bar{\Psi}'(X)\bar{\Gamma}_B \overset{\text{B-th component of coordinate (number)}}{\overbrace{X^B}}\right)\Gamma_A\Psi'(X)$$
$$= \bar{\Psi}'(X)\bar{\Gamma}_B X^B \Gamma_A \Psi'(X)$$

Use the Clifford algebra identity:

$$\bar{\Gamma}_A\Gamma_B = -\bar{\Gamma}_B\Gamma_A + 2\eta_{AB}$$
$$\Gamma_A\bar{\Gamma}_B = -\Gamma_B\bar{\Gamma}_A + 2\eta_{AB},$$

from (2.5.3), and above we rewrite:

$$V_A = \bar{\Psi}'\bar{\Gamma}_B X^B \Gamma_A \Psi' = -\bar{\Psi}'\bar{\Gamma}_A\Gamma_B X^B\Psi' + 2X_A\bar{\Psi}'\Psi'$$

The second term, $2X_A\bar{\Psi}'\Psi'$, is proportional to $X_A$ and is hence pure gauge under the equivalence relation:

$$V_A \sim V_A + X_A f(X)$$

so it can be discarded in physical quantities. Therefore, we obtain:

$$\bar{\Psi}\Gamma_A\Psi' + X_A f(X) \sim -\bar{\Psi}'\bar{\Gamma}_A\Gamma_B X^B\Psi' + 2X_A\bar{\Psi}'\Psi'$$
$$\bar{\Psi}(X)\Gamma_A\Psi'(X) \sim -\bar{\Psi}'(X)\bar{\Gamma}_A\Psi(X)$$

whereas,

$$\bar{\Psi}\Psi$$

transforms like a scalar. However, the above is only under (2.5.3) in odd dimensions. So it does not correspond to a scalar on the projective null cone in even dimensions.

# Chapter 3

# 2D CFT

## 3.1 Conformal Symmetry in Two Dimensions

Conformal invariance takes on a fundamentally new character in two dimensions. As hinted at in the preceding analysis, the case $d = 2$ requires special attention because the constraints on the metric become significantly more relaxed. In higher dimensions ($d \geq 3$), the conformal group is finite-dimensional, consisting of the $(d + 1)(d + 2)/2$ generators of $SO(d + 1, 1)$. In two dimensions, however, there exists an infinite variety of coordinate transformations that are locally conformal, even if they are not globally well-defined.

The origin of this expansion lies in complex analysis. In $d = 2$, the condition that a transformation $x \to x'$ preserves the metric up to a local scale factor is precisely the condition that the mapping be a **holomorphic function** of a complex variable $z = x^1 + ix^2$. Consequently, any analytic function $f(z)$ defines a local conformal transformation. This leads to an infinite-dimensional symmetry algebra—the **Virasoro algebra**—which is the hallmark of 2D Conformal Field Theory.

Within this infinite set of mappings, we must distinguish between the **global conformal group** and the local conformal transformations. The global group consists of the 6-parameter set of one-to-one (bijective) mappings of the Riemann sphere onto itself, known as the Möbius transformations:

$$z \to \frac{az + b}{cz + d}, \quad ad - bc = 1 \tag{3.1.1}$$

These transformations are the only ones that are globally well-defined and invertible. They constitute the $SL(2, \mathbb{C})/\mathbb{Z}_2$ group, and the analysis of the previous chapters for general $d$ remains valid when restricted to this subgroup.

However, a local field theory is sensitive to local symmetries. Even if a holomorphic mapping has singularities and cannot be extended globally, the energy-momentum tensor must still respond to it locally. This *local* conformal invariance is what enables the exact solution of many 2D theories. While in higher dimensions conformal symmetry constrains the functional form of correlation functions, in 2D, the infinite number of generators imposes such stringent constraints—expressed through the **Conformal Ward Identities**—that the spectrum and correlation functions of the theory can often be determined entirely. This transition from "constraint" to "solution" is what makes $d = 2$ the primary laboratory for the study of critical phenomena and string theory.

## 3.2 Conformal Group in Two Dimensions

We begin by considering the flat two-dimensional Euclidean space with coordinates $x^0$ and $x^1$. A conformal transformation in this space takes the form

$$x^\mu \to x^\mu + \epsilon^\mu(x)$$

where the conformal Killing equation imposes the condition

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \partial^\rho \epsilon_\rho$$
$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = g_{\mu\nu}(\partial_0 \epsilon_0 + \partial_1 \epsilon_1)$$

Evaluating these relations component by component, we obtain:

- For $\mu = \nu = 0$

$$2\partial_0 \epsilon_0 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_0 \epsilon_0 = \partial_1 \epsilon_1$$

- For $\mu = \nu = 1$

$$2\partial_1 \epsilon_1 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_1 \epsilon_1 = \partial_0 \epsilon_0$$

- For $\mu = 0, \nu = 1$

$$\partial_0 \epsilon_1 + \partial_1 \epsilon_0 = 0 \implies \partial_0 \epsilon_1 = -\partial_1 \epsilon_0$$

Thus only two independent constraints remain:

$$\boxed{\begin{aligned} \partial_0 \epsilon_0 &= \partial_1 \epsilon_1 \\ \partial_0 \epsilon_1 &= -\partial_1 \epsilon_0 \end{aligned}}$$

To make further progress, it is convenient to introduce complex coordinates $(z_1, z_2) \equiv (z, \bar{z})$ by embedding the $\mathbb{R}^2$ plane in $\mathbb{C}^2$. This embedding allows us to treat the two real directions as components of a single complex variable on the hypersurface $z_1^* = z_2$, which greatly simplifies the form of the constraints.

$$\left.\begin{aligned} z &= x^0 + ix^1 \\ \bar{z} &= x^0 - ix^1 \end{aligned}\right\} \begin{aligned} x^0 &= \frac{z + \bar{z}}{2} \\ x^1 &= \frac{z - \bar{z}}{2i} \end{aligned}$$

Now let us see how the transformation acts in these coordinates. Under $x^\mu \to x^\mu + \epsilon^\mu$, we find

$$z \to x^0 + \epsilon^0 + i(x^1 + \epsilon^1) = (x^0 + ix^1) + (\epsilon^0 + i\epsilon^1) = z + \epsilon(z, \bar{z})$$

and similarly,

$$\bar{z} \to \bar{z} + \bar{\epsilon}(z, \bar{z})$$

where we define $\bar{\epsilon} = \epsilon^0 - i\epsilon^1$. To rewrite the constraints, we must also translate derivatives into complex coordinates. Using the chain rule,

$$\partial_0 = \frac{\partial}{\partial x^0} = \frac{\partial z}{\partial x^0}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^0}\frac{\partial}{\partial \bar{z}} = \partial + \bar{\partial}$$
$$\partial_1 = \frac{\partial}{\partial x^1} = \frac{\partial z}{\partial x^1}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^1}\frac{\partial}{\partial \bar{z}} = i(\partial - \bar{\partial})$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$ are the Wirtinger derivatives:

$$\partial = \frac{\partial}{\partial z} = \frac{\partial x^0}{\partial z}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial z}\frac{\partial}{\partial x^1} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial x^1}\right) \tag{3.2.1}$$
$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{\partial x^0}{\partial \bar{z}}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial}{\partial x^1} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x^1}\right) \tag{3.2.2}$$

We also note that

$$\epsilon_0 = \frac{\epsilon + \bar{\epsilon}}{2}$$
$$\epsilon_1 = \frac{\epsilon - \bar{\epsilon}}{2i}$$

Substituting into the first constraint $\partial_0\epsilon_0 = \partial_1\epsilon_1$, we obtain.

$$\begin{aligned}
\partial_0\epsilon_0 &= \partial_1\epsilon_1 \\
(\partial + \bar{\partial})\left(\frac{\epsilon + \bar{\epsilon}}{2}\right) &= i(\partial - \bar{\partial})\left(\frac{\epsilon - \bar{\epsilon}}{2i}\right) \\
\partial\epsilon + \partial\bar{\epsilon} + \bar{\partial}\epsilon + \bar{\partial}\bar{\epsilon} &= \partial\epsilon - \partial\bar{\epsilon} - \bar{\partial}\epsilon + \bar{\partial}\bar{\epsilon} \\
\bar{\partial}\epsilon &= -\partial\bar{\epsilon}
\end{aligned} \tag{3.2.3}$$

From the second constraint $\partial_0\epsilon_1 = -\partial_1\epsilon_0$, we similarly find

$$\begin{aligned}
\partial_0\epsilon_1 &= -\partial_1\epsilon_0 \\
(\partial + \bar{\partial})\left(\frac{\epsilon - \bar{\epsilon}}{2i}\right) &= -i(\partial - \bar{\partial})\left(\frac{\epsilon + \bar{\epsilon}}{2}\right) \\
\partial\epsilon - \partial\bar{\epsilon} + \bar{\partial}\epsilon - \bar{\partial}\bar{\epsilon} &= \partial\epsilon + \partial\bar{\epsilon} - \bar{\partial}\epsilon - \bar{\partial}\bar{\epsilon} \\
\bar{\partial}\epsilon &= \partial\bar{\epsilon}
\end{aligned} \tag{3.2.4}$$

Combining both (3.2.3) and (3.2.4), we conclude

$$\partial\bar{\epsilon} = 0 \implies \frac{\partial\bar{\epsilon}}{\partial z} = 0$$
$$\bar{\partial}\epsilon = 0 \implies \frac{\partial\epsilon}{\partial\bar{z}} = 0$$

This shows that $\epsilon(z)$ is holomorphic while $\bar{\epsilon}(\bar{z})$ is antiholomorphic. Therefore, the conformal transformation factorizes neatly as

$$z \to z + \epsilon(z)$$
$$\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$$

or equivalently

$$z \to f(z)$$

with

$$\frac{\partial f}{\partial\bar{z}} = 0$$

i.e. $f(z)$ is analytic or holomorphic. Since $\epsilon(z)$ is holormorphic, it admits a Laurent expansion.

$$\begin{aligned}
z' &= z + \epsilon(z) \\
&= z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \\
&= z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \partial z \\
&= \left(1 + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \partial\right) z
\end{aligned}$$

It is now easy to read off the generators of conformal transformation in 2d from above. We define,

$$r_n = z^n \partial$$

Similarly,

$$\begin{aligned}
\bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) \\
&= \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \\
&= \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial}\bar{z} \\
&= \left(1 + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial}\right)\bar{z}
\end{aligned}$$

and hence

$$\bar{r}_n = \bar{z}^n \bar{\partial}$$

To determine the structure of the conformal algebra, we now compute the commutator between two such generators. A straightforward calculation shows that the $r_n$ satisfy the following commutation relation:

$$\begin{aligned}
[r_m, r_n] f(z,\bar{z}) &= [z^m\partial, z^n\partial] f(z,\bar{z}) \\
&= z^m\partial(z^n\partial f) - z^n\partial(z^m\partial f) \\
&= z^{m+n}\partial^2 f + z^m(\partial z^n)\partial f - z^{m+n}\partial^2 f - z^n(\partial z^m)\partial f \\
&= (n-m) z^{m+n-1}\partial f \\
&= (n-m) r_{m+n-1} f
\end{aligned}$$

Thus,

$$\boxed{[r_m, r_n] = (n-m) r_{m+n-1}}$$

Following the same steps, we can also find

$$\boxed{[\bar{r}_m, \bar{r}_n] = (n-m) \bar{r}_{m+n-1}}$$

and

$$\boxed{[r_m, \bar{r}_n] = 0}$$

It is conventional to redefine the generators as $r_n = -l_{n-1}$. With this shift, the algebra becomes

$$\begin{aligned}
[r_m, r_n] &= (n-m) r_{m+n-1} \\
[-l_{m-1}, -l_{n-1}] &= -(n-m) l_{m+n-2}
\end{aligned}$$

and then shifting $m \to m+1$ and $n \to n+1$

$$[l_m, l_n] = (m-n) l_{m+n}$$

similarly,

$$\begin{aligned}
[\bar{l}_m, \bar{l}_n] &= (m-n) \bar{l}_{m+n} \\
[l_m, \bar{l}_n] &= 0
\end{aligned}$$

where

$$\begin{aligned}
l_n &= -r_{n+1} = -z^{n+1}\partial \\
\bar{l}_n &= -\bar{r}_{n+1} = -\bar{z}^{n+1}\bar{\partial}
\end{aligned}$$

This is recognized as **Witt algebra**. They generate the conformal transformation in 2d Euclidean plane. To find the subset of generators $l_n$ which generate global conformal transformation, we note that $z^n$ has two singularity, one at $z = 0$ for $n < 0$ and another at $z = \infty$ for $n > 0$. Therefore, we impose the restriction that the representations of Witt algebra does not blow up in the limit $z \to 0$ or $z \to \infty$. We start with $z \to 0$ limit, $l_n = -z^{n+1}\partial$ needs to converge, implying:

$$n + 1 \geq 0 \implies n \geq -1$$

Since $z \to \infty$ does not appear as a pole, we consider the conformal mapping $w = \frac{1}{z}$ which brings the infinity at origin. Then,

$$\begin{aligned}\partial = \frac{\partial}{\partial z} &= \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \\ &= -\frac{1}{z^2}\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}\end{aligned}$$

Then,

$$\begin{aligned}l_n = -z^{n+1}\partial &= w^{-n-1}w^2\partial_w \\ &= w^{-n+1}\partial_w\end{aligned}$$

Then,

$$1 - n \geq 0 \implies n \leq 1$$

Therefore, $l_{-1}, l_0$ and $l_1$ generate global conformal transformation on the 2d Euclidean space.

## 3.3 Global Conformal Transformation

Last section was devoted to finding the generators of global conformal transformation. In this part, we look at the infinitesimal transformation generated by $l_{-1}$ , $l_0$ and $l_1$ and then find their finite counterpart. For $n = -1$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_{-1}$ looks like:

$$\begin{aligned}l_{-1} &= z^{-1+1}\partial = \partial \\ \bar{l}_{-1} &= \bar{z}^{-1+1}\bar{\partial} = \bar{\partial}\end{aligned}$$

So, this generates translation. For $n = 0$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_0 z$ looks like:

$$\begin{aligned}l_0 &= -z^{0+1}\partial = -z\partial \\ \bar{l}_0 &= -\bar{z}\bar{\partial}\end{aligned}$$

To interpret how the works in complex plane, we will use the polar coordinates $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$ where $r = \sqrt{z\bar{z}}$ and $\theta = \frac{1}{2i}\ln\left(\frac{z}{\bar{z}}\right)$

$$\begin{aligned}\partial = \frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} \\ &= \frac{\bar{z}}{2\sqrt{z\bar{z}}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar{z}}{z}\frac{\partial(z/\bar{z})}{\partial z}\frac{\partial}{\partial \theta} \\ &= \frac{\bar{z}}{2r}\frac{\partial}{\partial r} + \frac{1}{2iz}\frac{\partial}{\partial \theta}\end{aligned}$$

$$= \frac{e^{-i\theta}}{2}\frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir}\frac{\partial}{\partial \theta}$$

and,

$$\begin{aligned}
\bar{\partial} = \frac{\partial}{\partial \bar{z}} &= \frac{\partial r}{\partial \bar{z}}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \bar{z}}\frac{\partial}{\partial \theta} \\
&= \frac{z}{2\sqrt{z\bar{z}}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar{z}}{z}\frac{\partial(z/\bar{z})}{\partial \bar{z}}\frac{\partial}{\partial \theta} \\
&= \frac{z}{2\sqrt{z\bar{z}}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar{z}}{z}\left(-\frac{z}{\bar{z}^2}\right)\frac{\partial}{\partial \theta} \\
&= \frac{z}{2r}\frac{\partial}{\partial r} - \frac{1}{2i\bar{z}}\frac{\partial}{\partial \theta} \\
&= \frac{e^{i\theta}}{2}\frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir}\frac{\partial}{\partial \theta}
\end{aligned}$$

we can now recast or generators in polar coordinates as following:

$$\begin{aligned}
l_0 = -z\partial &= -re^{i\theta}\left[\frac{e^{-i\theta}}{2}\frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir}\frac{\partial}{\partial \theta}\right] \\
&= -\frac{r}{2}\left[\partial_r - \frac{i}{r}\partial_\theta\right]
\end{aligned}$$

$$\begin{aligned}
\bar{l}_0 &= -\bar{z}\bar{\partial} \\
&= -re^{-i\theta}\left[\frac{e^{i\theta}}{2}\frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir}\frac{\partial}{\partial \theta}\right] \\
&= -\frac{r}{2}\left[\partial_r + \frac{i}{r}\partial_\theta\right]
\end{aligned}$$

It is now clear that the following combination

$$l_0 + \bar{l}_0 = -r\partial_r$$

is the generator of dilatation and

$$i(l_0 - \bar{l}_0) = \partial_\theta$$

is the generator of rotation. For $n = -1$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_1 z^2$ looks like:

$$l_1 = -z^2\partial$$

These are nothing but Mobius transformation.

$$z' = \frac{az + b}{cz + d}$$

For translation we have $a = 1, c = 0$ and $d = 1$ and we have $ad - bc = 1$. For dilatation and rotation, we have

$$\begin{aligned}
z' &= z - \epsilon_0 z \\
&= (1 - \epsilon_0)z
\end{aligned}$$

$$= \frac{\frac{1-\epsilon_0}{\sqrt{1-\epsilon_0}}z + 0}{0z + \frac{1}{\sqrt{1+\epsilon_0}}} = \frac{\sqrt{1+\epsilon_0}z + 0}{0z + \frac{1}{\sqrt{1+\epsilon_0}}}$$

with $a = \sqrt{1+\epsilon_0}, b = 0, c = 0$ and $d = \frac{1}{\sqrt{1+\epsilon_0}}$ so that $ad - bc = 1$. For SCT, we have

$$\begin{aligned} z' &= z + \epsilon_1 z^2 \\ &= z(1 - \epsilon_1 z)^{-1} \\ &= \frac{z}{1 - \epsilon_1 z} = \frac{z + 0}{-\epsilon_1 z + 1} \end{aligned}$$

Since $\epsilon_n$ is in general a complex number. These are the set of $2 \times 2$ complex matrices with unit determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{with } ad - bc = 1$$

These matrices form the group $SL(2, \mathbb{C})$. But under the mapping $a \to -a$, $b \to -b$, $c \to -c$ and $d \to -d$, the generate the same transformation. Therefore, the conformal transformation in 2d forms the group $SL(2, \mathbb{C})/\mathbb{Z}_2$. In the Lorentzian case the group is replaced by $SL(2, R) \times SL(2, R) = SO(2, 2)$ where one factor of $SL(2, R)$ pertains to left-movers and the other to right-movers.

## 3.4 Virasoro Algebra

In some cases, the regularization scheme used to quantize a classical theory does not preserve the original symmetry. The classical generators, which we obtain by solving the Killing equations, satisfy a certain algebra—for example, the Witt algebra. However, in the quantum system, the requirement of normal ordering to render observables finite modifies the algebraic structure. In the next chapter, we will learn how to construct the corresponding quantum operators systematically using the operator formalism. As a result, these quantum operators no longer satisfy the Witt algebra but instead form the Virasoro algebra. This is unavoidable; it's a reflection of the fact that the classical symmetry is realized projectively in the quantum theory.

The Virasoro algebra is the **central extension** of Witt algebra. To centrally extend the algebra, a new generator—called the center—is introduced, which commutes with all other generators. To distinguish from the generators of Witt algebra, we use the notation $L_n$ to denote the generators of Virasoro algebra. The algebra is then given as:

$$[L_n, L_m] = (n - m)L_{n+m} + \underbrace{c\rho(n, m)}_{\text{complex number}}$$

This algebra contains a finite-dimensional subalgebra generated by $L_{0,\pm1}$,$\bar{L}_{0,\pm1}$. These are the generators that are well defined all over the complex plane and form the global conformal group in two dimensions just as we saw in Witt algebra. The rest of the generators are local. The object $\rho(n, m)$ which we have added to centrally extend the algebra has certain properties: it is a number which depends on $n$ and $m$, it commutes with all $L_n$, it satisfies recursion relation and it vanishes for certain $n$ and $m$. To see this, let us first note that

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + c\rho(n, m) \\ [L_m, L_n] &= -(n - m)L_{n+m} + c\rho(m, n) \end{aligned}$$

adding above expressions

$$\begin{aligned} [L_n, L_m] + [L_m, L_n] &= (n - m)L_{n+m} + c\rho(n, m) - (n - m)L_{n+m} + c\rho(m, n) \\ \rho(n, m) + \rho(m, n) &= 0 \end{aligned}$$

We have shown that $\rho(n, m)$ is anti-symmetric under the exchange of $n \leftrightarrow m$. Now let us set $m = 0$,

$$[L_n^{\text{old}}, L_0] = nL_n^{\text{old}} + c\rho(n, 0)$$

$$= n\left[L_n^{\text{old}} + \frac{c}{n}\rho(n,0)\right]$$

Redefining the generators for $n \neq 0$ as:

$$L_n^{\text{new}} = L_n^{\text{old}} + \frac{c}{n}\rho(n,0)$$

and

$$L_0^{\text{new}} = L_0^{\text{old}}$$

The algebra of the translated generators should not changed:

$$[L_n^{\text{old}}, L_0] = [L_n^{\text{new}} - \frac{c}{n}\rho(n,0), L_0]$$
$$[L_n^{\text{new}}, L_0] - \left[\frac{c}{n}\rho(n,0), L_0\right] = nL_n^{\text{old}} + c\rho(n,0)$$
$$[L_n^{\text{new}}, L_0] = nL_n^{\text{new}}$$

This proves that $\rho(n,0) = 0$ for $n \neq 0$. For $n = 1, m = -1$ we have:

$$\begin{aligned}[L_1^{\text{old}}, L_{-1}^{\text{old}}] &= 2L_0^{\text{old}} + c\rho(1,-1)\\ &= 2\left[L_0^{\text{old}} + \frac{c}{2}\rho(1,-1)\right]\\ &= 2L_0^{\text{new}}\end{aligned}$$

This, proves that $\rho(1,-1) = 0$, to find other $n$ and $m$ values for which $\rho(n,m)$ vanishes, we will utilize the jacobi identity.

$$[L_n,[L_m,L_r]] + [L_r,[L_n,L_m]] + [L_m,[L_r,L_n]] = 0$$

For $r = 0$

$$[L_n, mL_m] + [L_0, (n-m)L_{n+m}] + [L_m, -nL_n] = 0$$
$$m\left[(n-m)L_{n+m} + c\rho(n,m)\right] + (n-m)\left[-(n+m)L_{n+m}\right] - n\left[(m-n)L_{m+n} + c\rho(m,n)\right] = 0$$
$$\underbrace{\left[m(n-m) - (n-m)(n+m) - n(m-n)\right]}_{=0} L_{m+n} - c(m+n)\rho(m,n) = 0$$

Hence,

$$(m+n)\rho(m,n) = 0$$

It means, $\rho(m,n)$ can only be non zero. if $m+n=0$. So, $\rho(m,n) = \rho(n,-n)\delta_{m+n,0}$. Next to find the recursion relation, we utilize,

$$[L_n,[L_1,L_{-1-n}]] + [L_{-1-n},[L_n,L_1]] + [L_1,[L_{-1-n},L_n]] = 0$$
$$(n+2)[L_n,L_{-n}] + (n-1)[L_{-1-n},L_{n+1}] - (2n+1)[L_1,L_{-1}] = 0$$
$$(n+2)\left[2nL_0 + c\rho(n,-n)\right] + (n-1)[-2(n+1)L_0 + c\rho(-1-n,1+n)] - (2n+1)2L_0 = 0$$
$$\underbrace{\left[2n(n+2) - 2(n-1)(n+1) - 2(2n+1)\right]}_{=0} L_0 + c[(n+2)\rho(n,-n) - (n-1)\rho(n+1,-n-1)] = 0$$

Hence the recursion relation,

$$\rho(n+1,-n-1) = \frac{n+2}{n-1}\rho(n,-n)$$

under $n \to n+1$,

$$\rho(n,-n) = \frac{n+1}{n-2}\rho(n-1,-n+1)$$

We can see that for $n = 1$

$$\rho(1,-1) = \frac{2}{-1}\rho(0,0) = 0$$

For $n = 2$

$$\rho(2,-2) = \frac{3}{0}\rho(1,-1) = \text{indeterminate}$$

So, we only need to fix $\rho(2,-2)$ and all the other $\rho(n,m)$ gets fixed from that via this recursion relation.

$$\begin{aligned}\rho(n,-n) &= \frac{n+1}{n-2}\rho(n-1,-n+1)\\ &= \frac{n+1}{n-2}\cdots\frac{4}{1}\rho(2,-2)\\ &= \frac{1}{3!}\frac{(n+1)!}{(n-2)!}\rho(2,-2) = {}^nC_3 \quad \rho(2,-2)\end{aligned}$$

For free boson we normally like to set $c = 1$ which forces the normalize of $\rho(2,-2) = 1/2$ since $c \times \rho(n,m) = \xi c \times \frac{\rho(n,m)}{\xi}$ needs to stay constant.

$$\begin{aligned}[L_n, L_m] &= (n-m)L_{n+m} + {}^nC_3\rho(2,-2)\\ &= (n-m)L_{n+m} + c\frac{(n+1)!}{6(n-2)!}\frac{1}{2}\\ &= (n-m)L_{n+m} + c\frac{(n+1)!}{12(n-2)!}\end{aligned}$$

A short remark about the central charge that sometimes $c$ and $\bar{c}$ aren't complex conjugate of each other which allows us to interpret $z$ and $\bar{z}$ as being independent variables even though they are related to each other via complex conjugation operation.

## 3.5 How Fields transform under $SL(2,\mathbb{C})$

There are four types of fields we will be concerned about in 2d CFT. They are chiral fields that only depend on $z$ ($\bar{h} = 0$) and anti-chiral fields which only depend on $\bar{z}$ ($h = 0$). It is also common to use the terminology holomorphic and anti-holomorphic in order to distinguish between chiral and anti-chiral quantities. There is a special kind of field which has the same transformation law for both Global conformal transformation as well as local conformal transformaion, we call them Primary fields. Under conformal transformation $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, these Primary fields transform as (passive):

$$\phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(z,\bar{z})\Big|_{z=f(z),\bar{z}=\bar{f}(\bar{z})} = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z})) \tag{3.5.1}$$

where $(h,\bar{h})$ are conformal dimensions/weight of the field $\phi(z,\bar{z})$. Sometimes this transformation law (active) is also expressed as:

**Remark.** *The transformation rule:*

$$\phi'(z,\bar{z}) = \left(\frac{d}{dz}f^{-1}(z)\right)^h \left(\frac{d}{d\bar{z}}\overline{f^{-1}}(\bar{z})\right)^{\bar{h}} \phi\big(f^{-1}(z),\overline{f^{-1}}(\bar{z})\big) \tag{1}$$

*can be written more clearly with:* $\zeta = f^{-1}(z), \qquad \bar{\zeta} = \overline{f^{-1}}(\bar{z})$

$$\phi'(z,\bar{z}) = \left(\frac{d\zeta}{dz}\right)^{h}\left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\bar{h}}\phi(\zeta,\bar{\zeta}) \qquad \text{with } \zeta = f^{-1}(z),\ \bar{\zeta} = \overline{f^{-1}}(\bar{z}). \tag{2}$$

*Now we can write the derivative of the inverse in terms of the derivative of* $f$*:*

$$\left.\frac{d\zeta}{dz}\right|_{\zeta=f^{-1}(z)} = \frac{1}{\left.\frac{d}{d\zeta}f(\zeta)\right|_{\zeta=f^{-1}(z)}}, \qquad \left.\frac{d\bar{\zeta}}{d\bar{z}}\right|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})} = \frac{1}{\left.\frac{d}{d\bar{\zeta}}\overline{f(\zeta)}\right|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})}}. \tag{3}$$

$$\therefore \quad \phi'(z,\bar{z}) = \left(\frac{1}{\left.\frac{d}{d\zeta}f(\zeta)\right|_{\zeta=f^{-1}(z)}}\right)^{h}\left(\frac{1}{\left.\frac{d}{d\bar{\zeta}}\overline{f(\zeta)}\right|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})}}\right)^{\bar{h}}\phi\big(f^{-1}(z),\overline{f^{-1}}(\bar{z})\big). \tag{4}$$

*Replacing the left-hand argument* $f^{-1}(z)$ *by* $z$ *i.e. relabel* $\zeta \mapsto z,\ \bar{\zeta} \mapsto \bar{z}$*:*

$$\boxed{\phi'(z,\bar{z}) = \left(\left.\frac{d}{dz}f(z)\right|_{z=f^{-1}(z)}\right)^{-h}\left(\left.\frac{d}{d\bar{z}}\overline{f(z)}\right|_{\bar{z}=\overline{f^{-1}}(\bar{z})}\right)^{-\bar{h}}\phi\big(f^{-1}(z),\overline{f^{-1}}(\bar{z})\big)\ .} \tag{5}$$

These are active transformation because the RHS has the form $\pi_{ab}\phi_b(\Lambda^{-1}x)$. To see how the transformation works, we consider the following example, $z' = \lambda z$

$$\begin{aligned}\phi'(z,\bar{z}) &= \left(\frac{\partial\lambda z}{\partial z}\right)^{h}\left(\frac{\partial\bar{\lambda z}}{\partial\bar{z}}\right)^{\bar{h}}\phi(\lambda z,\lambda\bar{z})\\ &= \lambda^{h}\lambda^{\bar{h}}\phi(\lambda z,\lambda\bar{z})\\ \phi(z',\bar{z}') &= \lambda^{-h}\lambda^{-\bar{h}}\phi'(z,\bar{z})\end{aligned}$$

under infinitesimal conformal transformation $z' = f(z) = z + \epsilon$,

$$\begin{aligned}\phi'(z,\bar{z}) &= [1+\partial\epsilon]^{h}[1+\bar{\partial}\bar{\epsilon}]^{\bar{h}}\phi(z+\epsilon,\bar{z}+\bar{\epsilon})\\ &= [1+\partial\epsilon]^{h}[1+\bar{\partial}\bar{\epsilon}]^{\bar{h}}[\phi(z,\bar{z})+\epsilon\partial\phi(z,\bar{z})+\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})+\mathcal{O}(\epsilon^2)]\\ &= \phi(z,\bar{z})+\epsilon\partial\phi(z,\bar{z})+\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})+h\phi(z,\bar{z})\partial\epsilon+\bar{h}\phi(z,\bar{z})\bar{\partial}\bar{\epsilon}+\mathcal{O}(\epsilon^2)\\ &= [1+(\epsilon\partial+h\partial\epsilon)+(\bar{\epsilon}\bar{\partial}+\bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})\end{aligned}$$

So, the variation $\delta\phi(z,\bar{z})$ can be given as:

$$\sum_{n=-\infty}^{\infty}\epsilon_n[L_n,\phi]+\bar{\epsilon}_n[\bar{L}_n,\phi] = \delta\phi(z,\bar{z}) = [(\epsilon\partial+h\partial\epsilon)+(\bar{\epsilon}\bar{\partial}+\bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})$$

In the active point of view,

$$\begin{aligned}\phi'(z,\bar{z}) &= [1+\partial\epsilon]^{-h}[1+\bar{\partial}\bar{\epsilon}]^{-\bar{h}}\phi(z-\epsilon,\bar{z}-\bar{\epsilon})\\ &= [1+\partial\epsilon]^{-h}[1+\bar{\partial}\bar{\epsilon}]^{-\bar{h}}[\phi(z,\bar{z})-\epsilon\partial\phi(z,\bar{z})-\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})+\mathcal{O}(\epsilon^2)]\\ &= \phi(z,\bar{z})-\epsilon\partial\phi(z,\bar{z})-\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})-h\phi(z,\bar{z})\partial\epsilon-\bar{h}\phi(z,\bar{z})\bar{\partial}\bar{\epsilon}+\mathcal{O}(\epsilon^2)\end{aligned}$$

$$= [1 - (\epsilon\partial + h\partial\epsilon) - (\bar{\epsilon}\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon})]\phi(z, \bar{z})$$

The other kind is **quasi-primary field**, they only transform as primary field under Global conformal transformation. Based on (3.5.1), we can write the conformally invariant physical field as

$$\Phi \equiv \phi(z, \bar{z})(dz)^h \otimes (d\bar{z})^{\bar{h}}$$

such that

$$\Phi \equiv \phi(z, \bar{z})(dz)^h \otimes (d\bar{z})^{\bar{h}} = \phi'(w, \bar{w})(dw)^h \otimes (d\bar{w})^{\bar{h}} \tag{3.5.2}$$

## 3.6 Energy Momentum Tensor

Usually, a Field Theory is defined in terms of a Lagrangian action from which one can derive various objects and properties of the theory. In particular, the energy–momentum tensor can be deduced from the variation of the action with respect to the metric and so it encodes the behaviour of the theory under infinitesimal transformations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$.

$$S = \int d^D x \mathcal{L}$$
$$\delta S = \int d^D x \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \int d^D x T_{\mu\nu} \delta g^{\mu\nu}$$

Under $x^\mu \to x^\mu + \epsilon^\mu(x)$, $\delta g^{\mu\nu} = \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu$.

$$\begin{aligned}\delta S &= \int d^D x T_{\mu\nu}(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) \\ &= \int d^D x T_{\mu\nu} \frac{2}{D}(\partial^\rho \epsilon_\rho) g^{\mu\nu} = \frac{2}{D} \int d^D x T^\mu_\mu (\partial^\rho \epsilon_\rho) = 0 \implies T^\mu_\mu = 0\end{aligned}$$

This tell us that for the conformal invarience to hold, the stress energy tensor has to be traceless.

$$\begin{aligned}\delta S &= \int d^D x T_{\mu\nu}(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) \\ &= 2 \int d^D x T_{\mu\nu} \partial^\mu \epsilon^\nu \\ &= 2 \int d^D x \partial^\mu (T_{\mu\nu} \epsilon^\nu) - 2 \int d^D x (\partial^\mu T_{\mu\nu}) \epsilon^\nu = 0 \implies \partial^\mu T_{\mu\nu} = 0\end{aligned}$$

Hence, we conclude that stress energy tensor is canonically conserved as well as traceless in the presence of conformal symmetry. Next we see how this condition gets translated in complex coordinates. The goal is to find $T_{zz}$, $T_{z\bar{z}}$, $T_{\bar{z}z}$ and $T_{\bar{z}\bar{z}}$.

$$\begin{aligned}T_{zz} &= \frac{\partial x^0}{\partial z}\frac{\partial x^0}{\partial z} T_{00} + 2\frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial z} T_{01} + \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial z} T_{11} \\ &= \frac{1}{4} T_{00} + 2\frac{1}{2}\frac{1}{2i} T_{01} - \frac{1}{4} T_{11} \\ &= \frac{1}{4}[T_{00} - 2iT_{01} - T_{11}]\end{aligned}$$

$$\begin{aligned}T_{z\bar{z}} &= \frac{\partial x^0}{\partial z}\frac{\partial x^0}{\partial \bar{z}} T_{00} + \frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial \bar{z}} T_{01} + \frac{\partial x^1}{\partial z}\frac{\partial x^0}{\partial \bar{z}} T_{10} + \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial \bar{z}} T_{11} \\ &= \frac{1}{2}\frac{1}{2} T_{00} + \frac{1}{2}\frac{-1}{2i} T_{01} + \frac{1}{2i}\frac{1}{2} T_{10} + \frac{1}{2i}\frac{-1}{2i} T_{11}\end{aligned}$$

$$= \frac{1}{4}[T_{00} + T_{11}] = \frac{1}{4}\delta^{\mu\nu}T_{\mu\nu} = \frac{1}{4}T^{\mu}_{\mu} = 0$$

$$\begin{aligned} T_{\bar{z}\bar{z}} &= \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^0}{\partial \bar{z}}T_{00} + 2\frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{01} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{11} \\ &= \frac{1}{4}T_{00} + 2\frac{1}{2}\frac{-1}{2i}T_{01} - \frac{1}{4}T_{11} \\ &= \frac{1}{4}[T_{00} + 2iT_{01} - T_{11}] \end{aligned}$$

Next we investigate the form of $\partial^{\mu}T_{\mu\nu}$ in complex coordinates.

$$\begin{aligned} \partial_0 T_{00} + \partial_1 T_{10} &= 0 \\ \partial_0 T_{01} + \partial_1 T_{11} &= 0 \end{aligned} \tag{3.6.1}$$

We can now calculate $\partial_{\bar{z}}T_{zz}$:

$$\bar{\partial}T_{zz} = \frac{1}{4}(\bar{\partial}T_{00} - 2i\bar{\partial}T_{01} - \bar{\partial}T_{11})$$

using (3.2.1) and $T^{\mu}_{\mu} = T_{00} + T_{11} = 0$

$$\begin{aligned} &= \frac{1}{8}[(\partial_0 + i\partial_1)T_{00} - 2i(\partial_0 + i\partial_1)T_{01} - (\partial_0 + i\partial_1)T_{11}] \\ &= \frac{1}{8}[(\partial_0 T_{00} + 2\partial_1 T_{01} - \partial_0 T_{11}) + i(\partial_1 T_{00} - 2\partial_0 T_{01} - \partial_1 T_{11})] \\ &= \frac{1}{8}[2(\partial_0 T_{00} + \partial_1 T_{01}) - 2i(\partial_1 T_{11} + \partial_0 T_{01})] = 0 \end{aligned}$$

In the last step we used (3.6.1). Similarly one can show

$$\partial T_{\bar{z}\bar{z}} = 0$$

Thus, $T_{zz}$ is the chiral field we discussed earlier and $T_{\bar{z}\bar{z}}$ is the anti-chiral field. So in complex coordinates, the stress energy tensor looks like:

$$T_{\mu\nu}(z, \bar{z}) = \begin{bmatrix} 0 & T(z) \\ \bar{T}(\bar{z}) & 0 \end{bmatrix}$$

## 3.7 Ward Identities

The consequence of a symmetry of the action and measure on correlation functions may also be expressed via the so-called Ward identities. An infinitesimal transformation may be written in terms of the generator as:

$$\phi'(x) = \phi(x) - i\omega_a G_a \phi(x)$$

where $\omega_a$ is a collection of infinitesimal, constant parameters. We will consider the above variation of fields in the correlation function. The action is not invariant under such local transformation and its variation is given by:

$$\delta S = \int d^D x (\partial_\mu T^{\mu\nu})\epsilon_\nu$$

where $j^{\mu}_{a}$ is the conserved current. The correlation function can be given as:

$$\langle \phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = \frac{1}{Z}\int D\phi\phi(x_1)\phi(x_2)\dots\phi(x_n)e^{-S[\phi]}$$

We use the invariance of correlators under the transformation to argue,

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle &= \langle\phi'(x_1)\phi'(x_2)\dots\phi'(x_n)\rangle \\ &= \frac{1}{Z}\int D\phi'\phi'(x_1)\phi'(x_2)\dots\phi'(x_n)e^{-S[\phi']} = \frac{1}{Z}\int D\phi'\phi'(x_1)\phi'(x_2)\dots\phi'(x_n)e^{-S[\phi]-\delta S[\phi]} \\ &= \frac{1}{Z}\int D\phi'[\phi(x_1)\phi(x_2)\dots\phi(x_n)+\delta(\phi(x_1)\phi(x_2)\dots\phi(x_n))]e^{-S[\phi]-\int d^Dx\partial_\mu T^{\mu\nu}\epsilon_\nu(x)} \\ &= \frac{1}{Z}\int D\phi[\phi(x_1)\phi(x_2)\dots\phi(x_n)+\delta(\phi(x_1)\phi(x_2)\dots\phi(x_n))]e^{-S[\phi]}\left(1-\int d^Dx\partial_\mu T^{\mu\nu}\epsilon_\nu(x)+\dots\right)\end{aligned}$$

When expanded to first order in $\omega_a(x)$, the above yields

$$\langle\delta(\phi(x_1)\phi(x_2)\dots\phi(x_n))\rangle = \int dx\,\langle\partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle\,\epsilon_\nu(x)$$

$$-i\sum_{i=1}^{N}[\phi(x_1)\dots G_i^\mu\phi(x_i)\dots\phi(x_n)]\epsilon_\mu(x_i) = \int dx\,\langle\partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle\,\epsilon_\nu(x)$$

$$-i\int d^Dx\epsilon_\mu(x)\delta^D(x-x_i)\sum_{i=1}^{N}[\phi(x_1)\dots G_i^\mu\phi(x_i)\dots\phi(x_n)] = \int dx\,\langle\partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle\,\epsilon_\nu(x)$$

Hence,

$$\boxed{\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots G_i^\mu\phi(x_i)\dots\phi(x_n)\rangle}$$

### Translation

The generator of Translation $P^\mu = -i\partial^\mu$ is given as:

$$\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = -\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots\partial_i^\mu\phi(x_i)\dots\phi(x_n)\rangle$$

### Rotation

The generator of Rotation is $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) + S^{\mu\nu}$ and the associated conserved current is $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$.

$$\frac{\partial}{\partial x^\mu}\langle j^{\mu\nu\rho}(x)\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots J_i^{\nu\rho}\phi(x_i)\dots\phi(x_n)\rangle$$

$$\frac{\partial}{\partial x^\mu}\langle(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots[i(x^\nu\partial^\rho - x^\rho\partial^\nu) + S^{\nu\rho}]_i\phi(x_i)\dots\phi(x_n)\rangle$$

$$x^\rho\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n)\rangle + \delta^\rho_\mu\langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n)\rangle$$

$$-x^\nu\frac{\partial}{\partial x^\mu}\langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n)\rangle - \delta^\nu_\mu\langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n)\rangle$$

$$= -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots[i(x^\nu\partial^\rho - x^\rho\partial^\nu) + S^{\nu\rho}]_i\phi(x_i)\dots\phi(x_n)\rangle$$

$$\langle(T^{\rho\nu} - T^{\nu\rho})\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots S_i^{\nu\rho}\phi(x_i)\dots\phi(x_n)\rangle$$

### Dilatation

The generator for Dilatation is $D = x \cdot \partial$ and the associated conserved current is $j^\mu = T^\mu_\nu x^\nu$.

$$S[\phi] = \int d^2x\, \mathcal{L}(\phi^A, \partial_\mu \phi^A)$$

$$x'^\mu = (1+\varepsilon)x^\mu, \qquad \delta x^\mu = \varepsilon x^\mu, \qquad \bar{\delta}\phi^A = -\varepsilon \Delta_A \phi^A$$

$$\delta_{\text{tot}}\phi^A = \bar{\delta}\phi^A + \delta x^\nu \partial_\nu \phi^A$$

From Noether Theorem

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \bar{\delta}\phi^A - T^\mu{}_\nu\, \delta x^\nu$$

For dilatation, ($\varepsilon$ factor suppressed)

$$j^\mu_D = -\sum_A \Delta_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \phi^A - x_\nu T^{\mu\nu}$$

Let's define $V^\mu \equiv -\sum_A \Delta_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \phi^A$, then

$$j^\mu_D = V^\mu - x_\nu T^{\mu\nu}$$

$$\begin{aligned} \partial_\mu j^\mu_D &= \partial_\mu V^\mu - \partial_\mu (x_\nu T^{\mu\nu}) \\ &= \partial_\mu V^\mu - T^\mu{}_\mu - x_\nu \partial_\mu T^{\mu\nu} \\ &= \partial_\mu V^\mu + T^\mu{}_\mu \quad (\partial_\mu T^{\mu\nu} = 0) \end{aligned}$$

$$\partial_\mu j^\mu_D = 0 \iff T^\mu{}_\mu + \partial_\mu V^\mu = 0$$

The related Ward identity becomes:

$$\partial_\mu \langle T^\mu_\nu x^\nu \phi(x_1) \dots \phi(x_n) \rangle = -\sum_i \delta(x - x_i) \left[ x_i^\nu \frac{\partial}{\partial x_i^\nu} \langle \phi(x_1) \dots \phi(x_n) \rangle + \Delta_i \langle \phi(x_1) \dots \phi(x_n) \rangle \right]$$

which becomes

$$\left\langle T^\mu_\mu \phi(x_1) \dots \phi(x_n) \right\rangle = -\sum_i \delta(x - x_i) \Delta_i \langle \phi(x_1) \dots \phi(x_n) \rangle$$

### 3.7.1 In complex coordinates

We wish to rewrite these identities in terms of complex coordinates and complex components. But, we will first derive the relevant identities for doing so.

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

The metric in Euclidean space could be given as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In complex coordinates, it becomes:

$$g^{\alpha\beta} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

The anti-symmetric levi civita tensor in cartesian coordinates has the following form:

$$\epsilon^{\mu\nu}_{\text{cartesian}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

using the tensorial transformation law for the above coordinate transformation,

$$\begin{aligned} \epsilon^{\alpha\beta} &= J^{\alpha}_{\mu}\epsilon^{\mu\nu}_{\text{cartesian}}J^{\beta}_{\nu} \\ &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \end{aligned}$$

Lowering the indices:

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{-i}{2} & 0 \end{pmatrix}$$

For the delta functions, we remind ourselves that it is not defined as:

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$

But by the expression

$$\int_M d^2x\delta(x)f(z) = f(0)$$

Any function which satisfies this property is a valid representation of delta function. The $\infty$ of $\delta(x)$ at $x = 0$ in the above mentioned representation is defined by the integral $\int_{-\infty}^{\infty}\delta(x)dx = 1$. For more detail refer to section 7.2 of the textbook **Green's function** by *G.F. Roach*. It just so happens to be, that

$$\delta(x) = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z}$$

satisfies the required property. This identity is justified as follows. We consider a vector $F^\mu$ whose divergence is integrated within a region $M$ of the complex plane bounded by the contour $\partial M$. Gauss's theorem may be applied:

$$\int_M d^2x\,\partial_\mu F^\mu = \int_{\partial M} d\xi_\mu\, F^\mu \tag{3.7.1}$$

where $d\xi_\mu$ is an outward-directed differential of circumference, orthogonal to the boundary $\partial M$ of the domain of integration. It is more convenient to use a counterclockwise differential $ds^\rho$, parallel to the contour $\partial M$: $d\xi_\mu = \epsilon_{\mu\rho}ds^\rho$. In terms of complex coordinates, the above surface integral is nothing but a contour integral, where the (anti)holomorphic component of $ds^\rho$ is $dz$ ($d\bar{z}$):

$$\begin{aligned} \int_M d^2x\,\partial_\mu F^\mu &= \int_{\partial M} ds^\sigma\epsilon_{\sigma\mu}F^\mu \qquad (3.7.2) \\ &= \int_{\partial M}\left(dz\,\epsilon_{z\bar{z}}F^{\bar{z}} + d\bar{z}\,\epsilon_{\bar{z}z}F^z\right) \\ &= \frac{i}{2}\int_{\partial M}\left(-dz\,F^{\bar{z}} + d\bar{z}\,F^z\right) \qquad (3.7.3) \end{aligned}$$

Here the contour $\partial M$ circles counterclockwise. If $F^{\bar{z}}$ ($F^z$) is holomorphic (antiholomorphic), then Cauchy's theorem may be applied; otherwise the contour $\partial M$ must stay fixed. We consider then a holomorphic function $f(z)$ and check the correctness of the first representation in Eq. (5.33) by integrating it against $f(z)$ within a neighborhood $M$ of the origin:

$$\begin{aligned}\int_M d^2x\,\delta(x)f(z) &= \frac{1}{\pi}\int_M d^2x\, f(z)\,\partial_{\bar{z}}\frac{1}{z}\\ &= \frac{1}{\pi}\int_M d^2x\,\partial_{\bar{z}}\left(\frac{f(z)}{z}\right)\\ &= \frac{1}{2\pi i}\int_{\partial M} dz\,\frac{f(z)}{z}\\ &= f(0)\end{aligned} \tag{3.7.4}$$

In the second equation we have used the assumption that $f(z)$ is analytic within $M$, and in the third equation we have used the form (3.7.2) of Gauss's theorem with $F^{\bar{z}} = f(z)/\pi z$ and $F^z = 0$, and in the last equation we used Cauchy's residue theorem. Since the original ward identity was covariant, we now only need the following object to write the Ward identities in complex coordinates:

$$\begin{aligned}\partial_\mu \langle T^\mu_z \phi(z_1)\dots\phi(z_n)\rangle &= g^{\alpha\beta}\partial_\alpha \langle T_{\beta z}\phi(z_1)\dots\phi(z_n)\rangle = 2\partial_z \langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle + 2\partial_{\bar{z}}\langle T_{zz}\phi(z_1)\dots\phi(z_n)\rangle\\ \partial_\mu \langle T^\mu_{\bar{z}} \phi(z_1)\dots\phi(z_n)\rangle &= g^{\alpha\beta}\partial_\alpha \langle T_{\beta \bar{z}}\phi(z_1)\dots\phi(z_n)\rangle = 2\partial_z \langle T_{\bar{z}\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\partial_{\bar{z}}\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle\\ \langle T^\mu_\mu \phi(z_1)\dots\phi(z_n)\rangle &= g^{\alpha\beta}\langle T_{\alpha\beta}\phi(z_1)\dots\phi(z_n)\rangle = 2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle\\ \epsilon_{\mu\nu}\langle T^{\mu\nu}\phi(z_1)\dots\phi(z_n)\rangle &= \epsilon^{\alpha\beta}\langle T_{\alpha\beta}\phi(z_1)\dots\phi(z_n)\rangle = -2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle\end{aligned}$$

The Ward identities are then explicitly written as:

$$\begin{aligned}2\pi\partial_z\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_i \partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle\phi(z_1)\dots\phi(w_i)\dots\phi(z_n)\rangle\\ 2\pi\partial_z\langle T_{\bar{z}\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_i \partial_{z}\frac{1}{\bar{z}-\bar{z}_i}\partial_{\bar{z}_i}\langle\phi(z_1)\dots\phi(w_i)\dots\phi(z_n)\rangle\\ 2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i)\Delta_i\langle\phi(z_1).....\phi(z_n)\rangle\\ -2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i)s_i\langle\phi(z_1).....\phi(z_n)\rangle\end{aligned}$$

Adding and subtracting the last two expressions,

$$\begin{aligned}2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i)\frac{\Delta_i+s_i}{2}\langle\phi(z_1).....\phi(z_n)\rangle\\ 2\pi\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_i \partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle\phi(z_1)\dots\phi(z_n)\rangle\end{aligned} \tag{3.7.5}$$

$$\begin{aligned}2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i)\frac{\Delta_i-s_i}{2}\langle\phi(z_1).....\phi(z_n)\rangle\\ 2\pi\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle &= -\sum_i \partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle\phi(z_1)\dots\phi(z_n)\rangle\end{aligned} \tag{3.7.6}$$

Inserting these relations in the first two Ward identities:

$$\partial_{\bar{z}}\left\{\langle T(z,\bar{z})\phi(z_1)\dots\phi(z_n)\rangle - \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle\phi(z_1)\dots\phi(z_n)\rangle + \frac{h_i}{(z-w_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right]\right\} = 0 \qquad (3.7.7)$$

$$\partial_{z}\left\{\langle \bar{T}(z,\bar{z})\phi(z_1)\dots\phi(z_n)\rangle - \sum_{i=1}^{n}\left[\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle\phi(z_1)\dots\phi(z_n)\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right]\right\} = 0 \qquad (3.7.8)$$

where we have introduced a renormalized energy-momentum tensor

$$T = -2\pi T_{zz}, \qquad \bar{T} = -2\pi T_{\bar{z}\bar{z}}. \qquad (3.7.9)$$

Thus the expressions between braces in (3.7.7) and ((3.7.8)) are respectively holomorphic and antiholomorphic; we may write

$$\langle T(z)\phi(z_1)\dots\phi(z_n)\rangle = \sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle\phi(z_1)\dots\phi(z_n)\rangle + \frac{h_i}{(z-w_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right\} + \text{reg.} \qquad (3.7.10)$$

where "reg." stands for a holomorphic function of $z$, regular at $z = w_i$. There is a similar expression for the antiholomorphic counterpart. The Ward identity shows that the correlator of the field $T(z)$ with primary fields $\phi(w_i, \bar{w}_i)$ becomes singular as $z$ approaches the points $w_i$. The OPE of the energy-momentum tensor with primary fields is written simply by removing the brackets $\langle\dots\rangle$, it being understood that OPE is meaningful only within correlation functions.

## 3.8 Free Fields and Operator Product Expansion

The operator product expansion, or OPE, is the representation of a product of operators (at positions $z$ and $w$, respectively) by a sum of terms, each being a single operator, well-defined as $z \to w$, multiplied by a c-number function of $z-w$, possibly diverging as $z \to w$, and which embodies the infinite fluctuations as the two positions tend toward each other. For a single primary field $\phi$ of conformal dimension $h$ and $\bar{h}$, we have the OPE from Ward identity as:

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \\ T(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{z})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) \end{aligned} \qquad (3.8.1)$$

whenever appearing in OPEs, the symbol $\sim$ will mean equality modulo expressions regular as $z \to w$. Of course, the OPE contains also an infinite number of regular terms which, for the Energy Momentum tensor, can not be obtained from the conformal ward identity. In general, we would write the OPE of two fields $A(z)$ and $B(w)$ as

$$A(z)B(w) = \sum_{n=-\infty}^{\infty}\frac{\{AB\}_n(w)}{(z-w)^n}$$

where the composite $\{AB\}_n(w)$ are non singular at $z = w$. For instance, $\{T\phi\}_1 = \partial_w\phi(w)$. We stress that so far, quantities appearing in (3.8.1) are not operators but simply fields occuring within correlation functions. We will now proceed with specific examples, in order to familiarize ourselves with basic techniques and with simple but important systems.

### The Free Boson

From the point of view of the canonical or path integral formalism, the simplest conformal field theory is that of a free massless boson $\phi$, with the following action:

$$S = \frac{1}{2} g \int d^2x \partial_\mu \phi \partial^\mu \phi$$

where $g$ is some normalization parameter that we leave unspecified at the moment. The two-point function, or propagator can be found by comparing with

$$S = \frac{1}{2} \int \mathrm{d}^d x \mathrm{d}^d y \, \phi(x) A(x,y) \phi(y)$$

we have

$$A(x,y) = -g\delta^{(2)}(x-y)\Box\,, \tag{3.8.2}$$

We can calculate the two-point function $K(x,y) \equiv \langle \varphi(x_1)\varphi(x_2)\rangle = A^{-1}$ by solving the following equation:

$$\begin{aligned} \int d^2u A(x,u)K(u,y) &= \delta^{(2)}(x-y) \\ -\int d^2u g\delta^{(2)}(x-u)\Box K(u,y) &= \delta^{(2)}(x-y) \\ -g\Box K(x,y) &= \delta^{(2)}(x-y)\,, \end{aligned}$$

**Remark.** *It is easier to show that the Green's equation is invariant under Rotation, Translation and Dilatation. Therefore, we will show quickly that under the inversion map the Green's equation is unchanged. The inversion can be given as $x^\mu = \frac{w^\mu}{w^2}$, with the inverse transformation given as $w^\nu = \frac{x^\nu}{x^2}$. The Jacobian matrix $J^\nu{}_\mu = \frac{\partial w^\nu}{\partial x^\mu}$ is:*

$$\frac{\partial w^\nu}{\partial x^\mu} = \frac{\delta^\nu_\mu}{x^2} - \frac{2x^\nu x_\mu}{(x^2)^2} = w^2\left(\delta^\nu_\mu - 2\frac{w^\nu w_\mu}{w^2}\right) = w^2 R^\nu{}_\mu \tag{1}$$

*where $R^\nu{}_\mu$ is an orthogonal reflection matrix ($R^2 = I$). The partial derivative transforms as:*

$$\partial_\mu = \frac{\partial w^\nu}{\partial x^\mu}\partial'_\nu = w^2 R^\nu{}_\mu \partial'_\nu \tag{2}$$

*The Laplacian $\nabla^2_x = \delta^{\mu\sigma}\partial_\mu\partial_\sigma$ is calculated by acting with the first derivative on the second:*

$$\nabla^2_x = (w^2 R^\nu{}_\mu \partial'_\nu)(w^2 R^{\mu\rho}\partial'_\rho) \tag{3}$$

*Expanding this using the product rule yields two distinct pieces:*

1. ***Piece A:*** *When $\partial'_\nu$ acts purely on the field:*

$$(w^2 R^\nu{}_\mu)(w^2 R^{\mu\rho})\partial'_\nu\partial'_\rho = w^4\delta^{\nu\rho}\partial'_\nu\partial'_\rho = w^4\nabla^2_w \tag{4}$$

2. ***Piece B:*** *When $\partial'_\nu$ acts on the $w^2 R^{\mu\rho}$ pre-factors:*

$$w^2 R^\nu{}_\mu \left[\partial'_\nu(w^2 R^{\mu\rho})\right]\partial'_\rho = -2(d-2)w^2 w^\rho \partial'_\rho \tag{5}$$

*Combining these, the operator identity in $d$ dimensions is:*

$$\nabla^2_x = w^4\nabla^2_w - 2(d-2)w^2(w\cdot\nabla_w) \tag{6}$$

*We now act with $\nabla^2_x$ on a field with primary weight $\Delta = d-2$, defined as $\Phi(x) = w^{d-2}\phi(w)$. Using the identity above and the product rule:*

$$\nabla^2_x[w^{d-2}\phi] = w^4\nabla^2_w(w^{d-2}\phi) - 2(d-2)w^2(w\cdot\nabla_w)(w^{d-2}\phi) \tag{7}$$

*Evaluating the derivatives:*

$$\nabla_x^2[w^{d-2}\phi] = w^4[w^{d-2}\nabla_w^2\phi+2(d-2)w^{d-3}\partial_w\phi+(d-2)(d-1)w^{d-4}\phi]-2(d-2)w^2[w^{d-2}w\cdot\nabla_w\phi+(d-2)w^{d-2}\phi] \tag{8}$$

*The terms proportional to $\phi$ and $\nabla_w\phi$ cancel such that the identity simplifies to:*

$$\nabla_x^2[w^{d-2}\phi(w)] = w^{d+2}\nabla_w^2\phi(w) \tag{9}$$

*The Green's equation $\nabla_x^2 G(x,y) = \delta^{(d)}(x-y)$ is transformed using the following weights:*

- ***Field:*** $G(x,y) = w^{d-2}v^{d-2}\tilde{G}(w,v)$
- ***Source:*** $\delta^{(d)}(x-y) = w^{2d}\delta^{(d)}(w-v)$

*Substituting into the equation:*

$$v^{d-2}\left(\nabla_x^2[w^{d-2}\tilde{G}(w,v)]\right) = w^{2d}\delta^{(d)}(w-v) \tag{10}$$

*Applying the weighted operator identity:*

$$v^{d-2}w^{d+2}\nabla_w^2\tilde{G}(w,v) = w^{2d}\delta^{(d)}(w-v) \tag{11}$$

*At the singularity $w=v$, the exponents sum as $(d-2)+(d+2)=2d$:*

$$w^{2d}\nabla_w^2\tilde{G}(w,v) = w^{2d}\delta^{(d)}(w-v) \tag{12}$$

*Dividing by $w^{2d}$ recovers the form-invariant Green's equation:*

$$\nabla_w^2\tilde{G}(w,v) = \delta^{(d)}(w-v) \tag{13}$$

Because of rotation and translation invariance, the propagator $K(x,y)$ should depend only on the distance separating the two points. Thus, we can write $K(x,y) \equiv K(\rho)$ with $\rho = |x-y|$, and integrate over $x$ within a disk of radius $\rho$ around y. We find

$$\begin{aligned} 1 &= 2\pi g\int_0^r \mathrm{d}\rho\rho\Big(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\big(\rho K'(\rho)\big)\Big) \\ &= 2\pi g(-rK'(r)) \end{aligned}$$

The solution of the two-point function for massless free boson can be obtained up to an additive constant,

$$\langle\phi(x)\phi(y)\rangle = -\frac{1}{4\pi g}\ln(x-y)^2 + \text{const} \tag{3.8.3}$$

In terms of complex coordinates, this is

$$\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{4\pi g}\{\ln(z-w)+\ln(\bar{z}-\bar{w})\} + \text{const}$$

The holomorphic and anti-holomorphic components can be separated by taking the derivatives $\partial\phi$ and $\bar{\partial}\phi$:

$$\begin{aligned} \langle\partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w})\rangle &= -\frac{1}{4\pi g}\partial_w\partial_z\{\ln(z-w)\} \\ &= -\frac{1}{4\pi g}\partial_w\frac{1}{z-w} = -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \\ \langle\partial_{\bar{z}}\phi(z,\bar{z})\partial_{\bar{w}}\phi(w,\bar{w})\rangle &= -\frac{1}{4\pi g}\partial_{\bar{w}}\partial_{\bar{z}}\{\ln(\bar{z}-\bar{w})\} \end{aligned}$$

$$= -\frac{1}{4\pi g}\partial_{\bar{w}}\frac{1}{\bar{z}-\bar{w}} = -\frac{1}{4\pi g}\frac{1}{(\bar{z}-\bar{w})^2}$$

In the following, we will focus on holomorphic field $\partial\phi \equiv \partial_z\phi$.

$$\begin{aligned} T_{\mu\nu} &= g\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right] \\ &= g\partial_\mu\phi\partial_\nu\phi - \frac{g}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\ &= g\partial_\mu\phi\partial_\nu\phi - \frac{g}{2}\eta_{\mu\nu}\times 2\eta^{z\bar{z}}\partial_z\phi\underbrace{\partial_{\bar{z}}\phi}_{=0} \\ &= g\partial_\mu\phi\partial_\nu\phi \end{aligned}$$

Then,

$$T(z) = -2\pi T_{zz} = -2\pi g : \partial\phi\partial\phi : \qquad \text{and} \qquad \bar{T}(\bar{z}) = 0$$

Like all composite field, the energy momentum tensor has to be normal ordered, in order to ensure the vanishing of its vacuum expectation value. More explicitly, the exact meaning of above expression is

$$T(z) = -2\pi g \lim_{w\to z}[\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle]$$

The OPE of $T(z)$ with $\partial\phi$ may be calculated from Wick's theorem:[1]

$$\begin{aligned} T(z)\partial\phi(w) &= -2\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \\ &\sim -2\pi g : \partial\phi(z)\overbrace{\partial\phi(z) : \partial}\phi(w) - 2\pi g : \overbrace{\partial\phi(z)\partial\phi(z) : \partial}\phi(w) \\ &\sim \frac{\partial\phi(z)}{(z-w)^2} \end{aligned}$$

By expanding $\phi(z)$ around $w$, we arrive at the OPE

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)}$$

This shows that $\partial\phi$ is a primary field with conformal dimension $h = 1$. Wick's theorem also allows us to calculate the OPE of energy-momentum tensor with itself:

$$\begin{aligned} T(z)T(w) &= 4\pi^2g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\ &\sim 4\pi^2g^2 : \overbrace{\partial\phi(z)\overbrace{\partial\phi(z) :: \partial\phi(w)}\partial\phi(w)} : \; + 4\pi^2g^2 : \overbrace{\partial\phi(z)\overbrace{\partial\phi(z) :: \partial\phi(w)}\partial\phi(w)} : \\ &\quad + 4\pi^2g^2 : \partial\phi(z)\overbrace{\partial\phi(z) :: \partial\phi(w)\partial\phi(w)} : \; + 4\pi^2g^2 : \overbrace{\partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w)} : \\ &\quad + 4\pi^2g^2 : \overbrace{\partial\phi(z)\partial\phi(z) :: \partial}\phi(w)\partial\phi(w) : \; + 4\pi^2g^2 : \partial\phi(z)\overbrace{\partial\phi(z) :: \partial}\phi(w)\partial\phi(w) : \\ &\sim 8\pi^2g^2 : \overbrace{\partial\phi(z)\overbrace{\partial\phi(z) :: \partial\phi(w)}\partial\phi(w)} : \; + 16\pi^2g^2 : \partial\phi(z)\overbrace{\partial\phi(z) :: \partial\phi(w)}\partial\phi(w) : \\ &\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial\phi(z)\partial\phi(w) :}{(z-w)^2} \end{aligned}$$

Expanding $\phi(z)$ around $w$, we arrive at the OPE

$$\begin{aligned} &\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial\phi(w)\partial\phi(w) :}{(z-w)^2} - \frac{4\pi g : \partial^2\phi(w)\partial\phi(w) :}{(z-w)} \\ &\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \end{aligned}$$

We immediately see that the energy momentum tensor is not strictly a primary field, because of the anomalous term $1/2(z-w)^4$ which does not appear in (3.8.1).

[1] Since we are Wick contracting the normal ordered product of operators, the only term that it will give rise to are cross-contractions, since the wick contraction of operators which are already normal ordered vanishes.

### Free Fermion

Now we consider another simple model: free fermion. In two dimensions, the action of a free Majorana fermion is

$$S = \frac{1}{2} g \int \mathrm{d}^2 x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \,, \tag{3.8.4}$$

where the gamma matrices $\gamma^\mu$ satisfy the so-called **Clifford algebra**:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \,, \tag{3.8.5}$$

and we impose the **Majorana condition** ($\Psi^* = \Psi$) to the fermionic field to remove a half of the degrees of freedom. In the Euclidean space $\eta^{\mu\nu} = \mathrm{diag}(1,1)$, we take a basis of Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.8.6}$$

and therefore,

$$\begin{aligned} \gamma^0 \gamma^\mu \partial_\mu &= \gamma^0 (\gamma^0 \partial_0 + \gamma^1 \partial_1) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial + i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{\partial} \right] \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \partial_0 - i\partial_1 \\ \partial_0 + i\partial_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2\partial_z \\ 2\partial_{\bar{z}} & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \end{aligned}$$

Using this basis, we can express the action as

$$S = g \int \mathrm{d}^2 x (\overline{\psi} \partial \overline{\psi} + \psi \overline{\partial} \psi) \,, \tag{3.8.7}$$

where we write the two-component spinor $\Psi$ as $(\psi, \overline{\psi})$. Since the equations of motion are $\partial \bar{\psi} = 0$ and $\overline{\partial} \psi = 0$, $\psi(z)$ and $\overline{\psi}(\overline{z})$ holomorphic and antiholomorphic field, respectively.

Now let us calculate the two-point function as in the free fermion

$$K_{ij} = \langle \Psi_i(x) \Psi_j(y) \rangle \quad (i,j = 1,2) \,. \tag{3.8.8}$$

The action can be expressed by

$$S = \frac{1}{2} \int \mathrm{d}^2 x \mathrm{d}^2 y \Psi_i(x) A_{ij}(x,y) \Psi_j(y) \,, \tag{3.8.9}$$

where the kernel is

$$A_{ij}(x,y) = g \delta(x-y) (\gamma^0 \gamma^\mu)_{ij} \partial_\mu \,. \tag{3.8.10}$$

Recalling that propagator $K_{ij}$ is the inverse of $A_{ij}$. Therefore, we can write the equation for $K$ as

$$\begin{aligned} \int d^2 u A(x,u) K(u,y) &= \delta^{(2)}(x-y) \\ g \int d^2 u \delta^{(2)}(x-u) (\gamma^0 \gamma^\mu)_{ij} \frac{\partial}{\partial x^\mu} K(u,y) &= \delta^{(2)}(x-y) \delta_{ij} \\ g (\gamma^0 \gamma^\mu)_{ik} \frac{\partial}{\partial x^\mu} K_{kj}(x,y) &= \delta(x-y) \delta_{ij} \,. \end{aligned}$$

In terms of complex coordinates, this becomes

$$2g\begin{pmatrix}\overline{\partial} & 0\\ 0 & \partial\end{pmatrix}\begin{pmatrix}\langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle & \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle\\ \langle\overline{\psi}(z,\overline{z})\psi(w,\overline{w})\rangle & \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle\end{pmatrix}=\frac{1}{\pi}\begin{pmatrix}\overline{\partial}\frac{1}{z-w} & 0\\ 0 & \partial\frac{1}{\overline{z}-\overline{w}}\end{pmatrix}, \tag{3.8.11}$$

where we have used the complex form of the $\delta$-function:

$$\delta(x)=\frac{1}{\pi}\overline{\partial}\frac{1}{z}=\frac{1}{\pi}\partial\frac{1}{\overline{z}}\,.$$

Therefore, we obtain the two-point functions for the fermionic fields

$$\begin{aligned}\langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle&=\frac{1}{2\pi g}\frac{1}{z-w}\,,\\ \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle&=\frac{1}{2\pi g}\frac{1}{\overline{z}-\overline{w}}\,,\\ \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle&=0\end{aligned}$$

These, after differentiation, imply

$$\begin{aligned}\langle\partial_z\psi(z,\bar{z})\psi(w,\bar{w})\rangle&=-\frac{1}{2\pi g}\frac{1}{(z-w)^2}\\ \langle\partial_z\psi(z,\bar{z})\partial_w\psi(w,\bar{w})\rangle&=-\frac{1}{\pi g}\frac{1}{(z-w)^3}\end{aligned}$$

Thus the OPE of two holomorphic fields can be written as:

$$\psi(z)\psi(w)\sim\frac{1}{2\pi g}\frac{1}{z-w} \tag{3.8.12}$$

In order to see whether the fermion field is a primary field or not, we can calculate its OPE with the energy-momentum tensor. By using (**??**) in complex-coordinate form, we can calculate all the components of the energy-momentum tensor.

$$\begin{aligned}T^{\overline{z}\overline{z}}&=\frac{\partial\mathcal{L}}{\partial(\partial_{\bar{z}}\Phi)}\partial^{\bar{z}}\Phi=g^{\bar{z}z}\frac{\partial\mathcal{L}}{\partial(\overline{\partial}\Phi)}\partial\Phi=2\frac{\partial\mathcal{L}}{\partial(\overline{\partial}\Phi)}\partial\Phi=2g\psi\partial\psi\,,\\ T^{zz}&=\frac{\partial\mathcal{L}}{\partial(\partial_z\Phi)}\partial^{z}\Phi=g^{z\bar{z}}\frac{\partial\mathcal{L}}{\partial(\partial\Phi)}\overline{\partial}\Phi=2\frac{\partial\mathcal{L}}{\partial(\partial\Phi)}\overline{\partial}\Phi=2g\overline{\psi\partial\psi}\,,\\ T^{z\overline{z}}&=\frac{\partial\mathcal{L}}{\partial(\partial_z\Phi)}\partial^{\bar{z}}\Phi-g^{z\bar{z}}\mathcal{L}=2\frac{\partial\mathcal{L}}{\partial(\partial\Phi)}\partial\Phi-2\mathcal{L}=-2g\psi\overline{\partial}\psi\,.\end{aligned}$$

The traceless condition $T^{z\overline{z}}$ is preserved when taking into account the equation of motion, as we have discussed. The holomorphic part is defined as:

$$T(z)=-\pi g:\psi(z)\partial\psi(z):\,. \tag{3.8.13}$$

The normal-ordering product can be written in an equivalent way for the free field as follow:

$$:\psi\partial\psi:(z)=\lim_{w\to z}(\psi(z)\partial\psi(w)-\langle\psi(z)\partial\psi(w)\rangle)\,, \tag{3.8.14}$$

which is the same expression as in bosonic field theory. Then we can calculate the OPE between the fermion field and energy-momentum tensor directly.

$$\begin{aligned}T(z)\psi(w)&=-\pi g:\psi(z)\partial\psi(z):\psi(w)\\ &=-\pi g:\psi(z)\overbrace{\partial\psi(z):\psi}(w)-\pi g:\overbrace{\psi(z)\partial\psi(z):\psi}(w)\\ &\sim\frac{1}{2}\frac{\psi(z)}{(z-w)^2}+\frac{1}{2}\frac{\partial\psi(z)}{(z-w)}\end{aligned}$$

$$\sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}$$

In contracting $\psi(z)$ with $\psi(w)$ we have carried $\psi(w)$ over $\partial\psi(z)$, thus introducing a $(-)$ sign by Pauli's principle. The OPE immediately tells us that in the free fermion model, $\psi$ is a primary field with conformal dimension $1/2$. $TT$ OPE in this model can also be obtained directly by calculation.

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,. \tag{3.8.15}$$

### The Ghost System

In string theory applications, there appears another simple system, with the following action:

$$S = \frac{1}{2}g\int d^2x b_{\mu\nu}\partial^\mu c^\nu$$

where $b_{\mu\nu}$ is a traceless symmetric tensor, and where both $c^\mu$ and $b_{\mu\nu}$ are fermions (anti-commuting fields). These fields are called ghosts because they are not fundamental dynamical fields, but rather represent a jacobian arising from a change of variables in some functional integrals. More precisely, they are known as *reparametrization ghosts*. The role of these ghost fields is to cancel the unphysical gauge degrees of freedom.

The equation of motion are

$$\partial^\alpha b_{\alpha\mu} = 0 \qquad \text{and} \qquad \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0$$

In holomorphic form we write $c = c^z$ and $\bar{c} = c^{\bar{z}}$. The only nonzero components of the traceless symmetric tensor $b_{\mu\nu}$ are $b = b_{zz}$ and $\bar{b} = b_{\bar{z}\bar{z}}$. The equations of motion are then

$$\begin{aligned} &\bar\partial b = 0 && \partial\bar b = 0 && \\ &\bar\partial c = 0 && \partial\bar c = 0 && \partial c = -\bar\partial\bar c \end{aligned}$$

The propagator is calculated in the usual way, by writing the action as:

$$S = \frac{1}{2}\int d^2x d^2y b_{\mu\nu}(x) A^{\mu\nu}_\alpha c^\alpha(y)$$

$$A^{\mu\nu}_\alpha = \frac{1}{2}g\delta^\nu_\alpha\delta(x-y)\partial^\mu$$

where we must consider $(\mu,\nu)$ as a single composite index, symmetric under the exchange of $\mu$ and $\nu$. The factor of $\frac{1}{2}$ in front of $A^{\mu\nu}_\alpha$ compensates the double counting of each pair $(\mu,\nu)$ in the sum, which should be avoided since $b^{\mu\nu}$ is the same degree of freedom as $b^{\nu\mu}$. Again the propagator is $K = A^{-1}$, satisfying

$$\frac{1}{2}g\delta^\mu_\alpha\partial^\nu K^\beta_{\mu\nu}(x,y) = \delta(x-y)\delta_{\alpha\beta}$$

or, in complex representation

$$g\partial_{\bar z}K^\beta_{zz} = \frac{1}{\pi}\partial_{\bar z}\frac{1}{z-w}\delta_{\beta z}$$

which implies

$$\langle b(z)c(w)\rangle = K^z_{zz}(z,w) = \frac{1}{\pi g}\frac{1}{z-w}$$

In OPE form, this is

$$\boxed{b(z)c(w) \sim \frac{1}{\pi g}\frac{1}{z-w}}$$

from which we immediately derive

$$\begin{aligned}
\langle c(z)b(w)\rangle &= -\langle b(w)c(z)\rangle = \frac{1}{\pi g}\frac{1}{z-w} \\
\langle b(z)\partial_w c(w)\rangle &= -\langle \partial_z c(z)b(w)\rangle = \partial_w\left(\frac{1}{\pi g}\frac{1}{z-w}\right) = \frac{1}{\pi g}\frac{1}{(z-w)^2} \\
\langle \partial_z b(z)c(w)\rangle &= -\frac{1}{\pi g}\frac{1}{(z-w)^2} \\
\langle \partial_z b(z)\partial_w c(w)\rangle &= -\frac{2}{\pi g}\frac{1}{(z-w)^3}
\end{aligned}$$

The canonical energy-momentum tensor for this system is

$$T_B^{\mu\nu} = \frac{1}{2}g\left[b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right]$$

The Belinfante tensor is

$$T_B^{\mu\nu} = \frac{1}{2}g\left[b^{\mu\alpha}\partial^\nu c_\alpha + b^{\nu\alpha}\partial^\mu c_\alpha + \partial_\alpha b^{\mu\nu}c^\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right]$$

The normal ordered holomorphic component is obtained from the above by setting $\mu = \nu = 1$, that is, by considering $T^{\bar{z}\bar{z}} = 4T_{zz}$:

$$T(z) = \pi g : (2\partial c\, b + c\partial b) :$$

The OPE for this stress energy tensor with $c$ is again calculated using Wick's theorem:

$$\begin{aligned}
T(z)c(w) &= \pi g : (2\partial c\, b + c\partial b) : c(w) \\
&= \pi g : 2\partial c\, \overbrace{b : c}(w) + \pi g : c\overbrace{\partial b : c}(w) \\
&\sim \frac{2\partial_z c(z)}{z-w} - \frac{c(z)}{(z-w)^2} \\
&\sim \frac{2\partial_w c(w)}{z-w} - \frac{c(w) + (z-w)\partial_w c(w)}{(z-w)^2} \\
&\sim \frac{\partial_w c(w)}{z-w} - \frac{c(w)}{(z-w)^2}
\end{aligned}$$

$$\begin{aligned}
T(z)b(w) &= \pi g : (2\partial c\, b + c\partial b) : b(w) \\
&= -\pi g : 2b\overbrace{\partial c : b}(w) - \pi g : \partial b\overbrace{c : b}(w) \\
&\sim \frac{2b(z)}{(z-w)^2} - \frac{\partial_z b(z)}{(z-w)} \\
&\sim \frac{2b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)}
\end{aligned}$$

One must be careful because $b$ and $c$ are anticommuting so that interchange of fields flips the sign: one should anticommute the fields being paired until they are next to each other before doing the Wick contraction. The OPE of $T$ with itself, may contain more terms, which add up to following:

$$\begin{aligned}
T(z)T(w) &= \pi^2 g^2 : (2\partial c(z)\, b(z) + c(z)\partial b(z)) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : \\
&= \pi^2 g^2 : 2\partial c(z)\, b(z) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : + \pi^2 g^2 : c(z)\partial b(z) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) :
\end{aligned}$$

$$
\begin{aligned}
&= 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : + 2\pi^2 g : \partial c(z)\, b(z) :: c(w)\partial b(w) : + 2\pi^2 g^2 : c(z)\partial b(z) :: \partial c(w)\, b(w) : \\
&\quad + \pi^2 g^2 : c(z)\partial b(z) :: c(w)\partial b(w) : \\
&= 4\pi^2 g^2 : \overbrace{\partial c(z)\, \overbrace{b(z) :: \partial c(w)}\, b(w)} : \; + 4\pi^2 g^2 : \partial c(z)\, \overbrace{b(z) :: \partial c(w)}\, b(w) : \; + 4\pi^2 g^2 : \overbrace{\partial c(z)\, b(z) :: \partial c(w)}\, b(w) : \\
&+ 2\pi^2 g : \overbrace{\partial c(z)\, \overbrace{b(z) :: c(w)}\partial b(w)} : \; + 2\pi^2 g : \partial c(z)\, \overbrace{b(z) :: c(w)}\partial b(w) : \; + 2\pi^2 g : \overbrace{\partial c(z)\, b(z) :: c(w)}\partial b(w) : \\
&+ 2\pi^2 g^2 : \overbrace{c(z)\partial b(z) :: \partial c(w)\, b(w)} : \; + 2\pi^2 g^2 : c(z)\overbrace{\partial b(z) :: \partial c(w)\, b(w)} : \; + 2\pi^2 g^2 : \overbrace{c(z)\partial b(z) :: \partial c(w)\, b(w)} : \\
&+ \pi^2 g^2 : \overbrace{c(z)\partial b(z) :: c(w)\partial b(w)} : \; + \pi^2 g^2 : c(z)\overbrace{\partial b(z) :: c(w)\partial b(w)} : \; + \pi^2 g^2 : \overbrace{c(z)\partial b(z) :: c(w)\partial b(w)} :
\end{aligned}
$$

The end result is

$$
T(z)T(w) = \frac{-4}{(z-w)^4} + \frac{4\pi g : \partial c(z) b(w) :}{(z-w)^2} - \frac{4\pi g : b(z)\partial c(w) :}{(z-w)^2} - \frac{4}{(z-w)^4} + \frac{2\pi g : \partial c(z)\partial b(w) :}{z-w} - \frac{4\pi g : b(z)c(w) :}{(z-w)^3}
$$

$$
- \frac{4}{(z-w)^4} - \frac{4\pi g : c(z)b(w) :}{(z-w)^3} + \frac{2\pi g : \partial b(z)\partial c(w) :}{z-w} - \frac{1}{(z-w)^4} - \frac{\pi g : c(z)\partial b(w) :}{(z-w)^2} + \frac{\pi g : \partial b(z) c(w) :}{(z-w)^2} + \dots
$$

After some Taylor expansions to turn $f(z)$ functions into $f(w)$ functions, together with a little collecting of terms, this can be written as,

$$
T(z)T(w) = \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots
$$

## 3.9 Central Charge

The specific models treated in the last section lead us naturally to the following general OPE of the energy-momentum tensor.

$$
T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}
$$

where the constant $c$, not to be confused with ghost field $c_\mu$, depends on the specific model under study: it is equal to 1 for the free boson, $1/2$ for the free fermion, $-26$ for the reparametrization ghosts, and $-2$ for the simple ghost system. This model dependent constant term is called the *central charge*. Except for this anomalous term, the OPE simply means that $T$ is a quasi-primary field with conformal dimension $h = 2$.

The central charge may not be determined from symmetry considerations: its value is determined by the short-distance behavior of the theory. For free fields, as seen in the previous section, it is determined by applying Wick's theorem on the normal-ordered energy-momentum tensor. When two decoupled systems (e.g., two free fields) are put together, the energy-momentum tensor of the total system is simply the sum of the energy-momentum tensors associated with each part, and the associated central charge is simply the sum of the central charges of the parts. Thus, the central charge is somehow an extensive measure of the number of degrees of freedom of the system.

### Transformation of the Energy-Momentum Tensor

The departure of OPE from the general form (3.8.1) means that the energy-momentum tensor does not exactly transform like a primary field of dimension 2, contrary to what we expect classically. This happens because the normal ordering is not invariant under conformal transformation. According to conformal ward identity

$$
\begin{aligned}
\delta_\epsilon T(w) &= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) T(z) T(w) \\
&= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \left[ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.} \right]
\end{aligned}
$$

$$= -\frac{c/2}{3!}\partial_w^3\epsilon(w) - 2T(w)\partial_w\epsilon(w) - \epsilon(w)\partial_w T(w)$$

In the third equation, we used Cauchy's integral formula for derivatives. The "exponentiation" of this infinitesimal variation to a finite transformation $z \to w(z)$ is:

$$T'(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right] \tag{3.9.1}$$

where we have introduced the **Schwarzian derivative**:

$$\{w;z\} = \frac{\frac{\mathrm{d}^3 w}{\mathrm{d}z^3}}{\frac{\mathrm{d}w}{\mathrm{d}z}} - \frac{3}{2}\left(\frac{\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}}{\frac{\mathrm{d}w}{\mathrm{d}z}}\right)^2$$

Instead of giving the long and technical proof of the last statement, we shall derive the above for free boson system. We write the free boson energy-momentum tensor as

$$T(z) = -2\pi g \lim_{\delta\to 0} : \partial\phi\partial\phi := -2\pi g \lim_{\delta\to 0}\left[\partial\phi\left(z+\frac{1}{2}\delta\right)\partial\phi\left(z-\frac{1}{2}\delta\right) + \frac{1}{4\pi\delta^2}\right]$$

Consider the transformation $z \to w(z)$. Since $\phi$ has conformal dimension zero $\partial\phi$ transforms as

$$\partial_z\phi = \frac{\partial w}{\partial z}\frac{\partial\phi'(z)}{\partial w} = w^{(1)}\partial_w\phi'(w)$$

(here we denote the n-th derivative of $w$ by $w^{(n)}$ in order to lighten the notation). Hence $T(z)$ transforms as:

$$T(z) = -2\pi g \lim_{\delta\to 0}\left[w^{(1)}\left(z+\frac{1}{2}\delta\right)w^{(1)}\left(z-\frac{1}{2}\delta\right)\partial_w\phi'\left(w\left(z-\frac{1}{2}\delta\right)\right)\partial_w\phi'\left(w\left(z+\frac{1}{2}\delta\right)\right) + \frac{1}{4\pi\delta^2}\right] \tag{3.9.2}$$

we will use the following to simplify the above:

$$\begin{aligned}
w(z+\delta/2) &\simeq w(z) + \frac{\delta}{2}\partial_z w(z) + \frac{1}{2!}\Big(\frac{\delta}{2}\Big)^2\partial_z^2 w(z) + \frac{1}{3!}\Big(\frac{\delta}{2}\Big)^3\partial_z^3 w(z) + \dots \\
\partial_z w(z+\delta/2) &\simeq \partial_z w(z) + \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!}\Big(\frac{\delta}{2}\Big)^2\partial_z^3 w(z) + \dots \\
\big[w(z+\delta/2) - w(z-\delta/2)\big]^2 &= \big(\partial_z w(z)\big)^2\delta^2 + \frac{1}{12}\big(\partial_z^3 w\,\partial_z w(z)\big)\delta^4 + \mathcal{O}(\delta^6), \\
&= (w^{(1)}\delta)^2\left[1 + \frac{1}{12}\frac{w^{(3)}}{w^{(1)}}\delta^2 + \dots\right]
\end{aligned}$$

so the inverse is then,

$$\frac{1}{[w(z+\delta/2) - w(z-\delta/2)]^2} = \frac{1}{\delta^2}\frac{1}{(\partial_z w(z))^2} - \frac{1}{12}\frac{\partial_z^3 w(z)}{(\partial_z w(z))^3} + \mathcal{O}(\delta^2) \tag{3.9.3}$$

$$\begin{aligned}
\partial_z w(z+\delta/2)\partial_z w(z-\delta/2) &= \left[\partial_z w(z) + \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!}\Big(\frac{\delta}{2}\Big)^2\partial_z^3 w(z) + \dots\right] \\
&\quad\times\left[\partial_z w(z) - \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!}\Big(\frac{\delta}{2}\Big)^2\partial_z^3 w(z) + \dots\right] \\
&= (w^{(1)})^2 - \left(\frac{\delta}{2}\right)^2(w^{(2)})^2 + \left(\frac{\delta}{2}\right)^2 w^{(1)}w^{(3)} + \dots
\end{aligned} \tag{3.9.4}$$

Then,

$$
\begin{aligned}
T(z) &= \lim_{\delta\to 0}\left[w^{(1)}\left(z+\frac{1}{2}\delta\right)w^{(1)}\left(z-\frac{1}{2}\delta\right)\left\{-2\pi g:\partial_w\phi'(w)\partial_w\phi'(w):+\frac{1}{2\left[w\left(z+\frac{\delta}{2}\right)-w\left(z-\frac{\delta}{2}\right)\right]^2}\right\}-\frac{1}{2\delta^2}\right]\\
&= (w^{(1)}(z))^2T'(w)+\lim_{\delta\to 0}\left[\frac{w^{(1)}\left(z+\frac{\delta}{2}\right)w^{(1)}\left(z-\frac{\delta}{2}\right)}{2\left[w\left(z+\frac{\delta}{2}\right)-w\left(z+\frac{\delta}{2}\right)\right]^2}-\frac{1}{2\delta^2}\right]\\
&= (w^{(1)}(z))^2T'(w)\\
&\quad+\lim_{\delta\to 0}\left[\left\{(w^{(1)})^2-\left(\frac{\delta}{2}\right)^2(w^{(2)})^2+\left(\frac{\delta}{2}\right)^2w^{(1)}w^{(3)}+\dots\right\}\left\{\frac{1}{2\delta^2}\frac{1}{(w^{(1)})^2}-\frac{1}{24}\frac{w^{(3)}}{(w^{(1)})^3}+\dots\right\}-\frac{1}{2\delta^2}\right]\\
&= (w^{(1)}(z))^2T'(w)+\lim_{\delta\to 0}\left[\cancel{\frac{1}{2\delta^2}}-\frac{1}{8}\left(\frac{w^{(2)}}{w^{(1)}}\right)^2-\frac{1}{24}\frac{w^{(3)}}{w^{(1)}}+\frac{1}{8}\frac{w^{(3)}}{w^{(1)}}+\dots-\cancel{\frac{1}{2\delta^2}}\right]\\
&= (w^{(1)}(z))^2T'(w)+\frac{1}{12}\left[\frac{w^{(3)}}{w^{(1)}}-\frac{3}{2}\left(\frac{w^{(2)}}{w^{(1)}}\right)^2\right]
\end{aligned}
$$

Here we use (3.9.3) and (3.9.4) in the second step for simplication. There is another way to look at this derivation which sheds light on the origin of schwarzian derivative term. Let us focus on how the propagator transforms under conformal transformation:

$$
\begin{aligned}
\langle\phi(w(z_1))\phi(w(z_2))\rangle &= -\ln\left|w(z_1)-w(z_2)\right|^2\\
&= -\ln\left|z_{12}\frac{w(z_1)-w(z_2)}{z_1-z_2}\right|^2\\
&= -\ln|z_{12}|^2-\ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right|^2\\
&= \langle\phi(z_1)\phi(z_2)\rangle-\ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right|^2
\end{aligned}
$$

Let us write $w_1\equiv w(z_1)$ and $w_2\equiv w(z_2)$. One can show that for $|z_{12}|=|z_1-z_2|$ small:

$$
\begin{aligned}
\partial_{z_1}\partial_{z_2}\ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right| &= \partial_{z_1}\partial_{z_2}\ln|w(z_1)-w(z_2)|^2-\partial_{z_1}\partial_{z_2}\ln|z_{12}|^2\\
&= \partial_{z_1}\left[\frac{-w^{(1)}(z_2)}{w(z_1)-w(z_2)}\right]-\frac{1}{z_{12}^2}\\
&= \frac{w^{(1)}(z_2)w^{(1)}(z_1)}{(w(z_1)-w(z_2))^2}-\frac{1}{z_{12}^2}\\
&= \frac{2}{12}\left[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2}-\frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\right]+\mathcal{O}(z_{12}).
\end{aligned}
$$

Since only the $z_{12}\to 0$ limit is of interest we can drop all terms on the right-hand side that vanish in this limit. Substituting the result of this into the above we learn that:

$$
\boxed{\lim_{z_1\to z_2}\partial_{z_1}\partial_{z_2}\langle\phi(w(z_1))\phi(w(z_2))\rangle=\lim_{z_1\to z_2}\partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle-\frac{2}{12}\left[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2}-\frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\right]}
$$

Note that Schwarzian derivative for global conformal transformation

$$
w(z)=\frac{az+b}{cz+d} \tag{3.9.5}
$$

vanishes. The non-invariance of the vacuum and thus correlator is really what opens up the possibility of a trace anomaly $\langle U^{-1}T(w)U\rangle \equiv \langle T'(w)\rangle \neq 0$ and it is not at all coincidental. The conformal transformations which do change the vacuum are those that have non-vanishing Schwarzian derivative, and thus an extra inhomogeneous central charge term. Using this we can derive the transformation law for energy-momentum tensor:

$$\begin{aligned}
T^{(w)}(z_2) &\equiv :\lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_w \\
&= \lim_{z_1\to z_2}\Big[-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(w(z_1))\phi(w(z_2))\rangle\Big)\Big] \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle + \frac{2}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big]\Big)\Big\} \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle\Big)\Big\} - \frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big] \\
&=: \lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_z - \frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big] \\
&= T^{(z)}(z_2) - \frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big]
\end{aligned} \tag{3.9.6}$$

where we noted in the last two lines that:

$$\begin{aligned}
T^{(z)}(z_2) &\equiv :\lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_z \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle\Big)\Big\}
\end{aligned} \tag{3.9.7}$$

as shown above. So we learn that a finite holomorphic change in normal ordering, $z \to w(z)$, with fixed coordinates, $z_2$, of the energy-momentum tensor is given by:

$$\boxed{T^{(w)}(z_2) = T^{(z)}(z_2) - \frac{1}{12}\left[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\right]} \tag{3.9.8}$$

# Chapter 4

# Operator Formalism

In the previous chapter, conformal symmetry was seen to impose constraints on correlation functions in the form of Ward identities. These identities were conveniently expressed using operator product expansions between the energy–momentum tensor and local fields, but the OPEs were understood only as a shorthand for singularities inside correlators. Nothing required a Hilbert space or an operator formalism: in principle, everything could have been computed directly in the path integral by evaluating Green's functions and extracting their short-distance behavior. Up to this point, all we really needed was a way to compute the two-point correlator, whether by solving the Schwinger–Dyson equations or by brute-force path integration. The OPEs then followed from the explicit Green's function.

From here onward the viewpoint changes. Instead of relying on an explicit propagator, we will systematically construct an operator formalism in which OPEs can be obtained directly from symmetry and representation theory. This is a qualitatively new approach: even when the Green's function is not accessible by solving the equations of motion or performing the path integral, the operator method still allows us to determine the structure of OPEs and, in turn, recover the correlators themselves.

The Hilbert space is what makes this possible. Once a Hilbert space is defined, local fields are no longer only insertions in correlation functions — they correspond to states, and operators act as maps on this space. This representation turns the OPE into an actual operator identity, rather than a mnemonic for propagator singularities. The OPE then expresses how the action of one operator near another decomposes into the basis of states, giving us algebraic control that was absent in the purely path-integral description.

To set this up, we must specify a notion of "time," since operators evolve with respect to it. In Euclidean space, the natural choice is to take the radial coordinate as time, leading to radial quantization. States are then defined on concentric circles, time evolution is dilation, and contour integrals of the stress tensor implement the Virasoro algebra. Within this framework, commutators appear as contour manipulations, and the OPE becomes a universal computational tool, independent of explicit Green's functions.

## 4.1 Radial quantization

In the operator formalism one must first distinguish a time direction from a space direction. In Minkowski spacetime this choice is canonical, but in Euclidean space it is arbitrary. A Hilbert space by itself is just a complete inner product space; what makes it physically meaningful is the specification of a **time direction**, which singles out a Hamiltonian as the generator of time translations. The Hamiltonian defines a **vacuum state** (its lowest-energy eigenstate) and organizes the remaining states as excitations built on top of it.

This structure underlies any quantum theory: begin with a vacuum, generate excitations, and classify states by the Hamiltonian spectrum. In two-dimensional conformal field theory we can choose a different notion of "time": the **radial direction** from the origin. To motivate this further, consider $\mathbb{R}^d$ in spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_{d-1} = r^2 \left[ \frac{dr^2}{r^2} + d\Omega_{d-1} \right]$$

Now let $t = \log(r)$ so that

$$ds^2 = e^{2t}[dt^2 + d\Omega_{d-1}]$$

which is conformally related to the metric on $\mathbb{R} \times S^{d-1}$. Now if we consider a CFT on $\mathbb{R}^d$, the theory should be invariant under rescaling of metric, so studying that theory on $\mathbb{R}^d$ should be equivalent to studying the theory on $\mathbb{R} \times S^{d-1}$.

From a Minkowski space point of view (in particular in the context of string theory), we will initially define our theory on an infinite space-time cylinder, with time $t$ going from $-\infty$ to $+\infty$ along the "flat" direction of the cylinder, and space being compactified with $x$ going from 0 to $L$, and the point $(0, t)$ and $(L, t)$ being identified. If we continue to Euclidian space, the cylinder is described by a single complex coordinate

$$w = t + ix, \qquad x \sim x + L,$$

with Hamiltonian $H = -i\partial_t$. We can then consider the **local** conformal transformation that maps this cylinder to the plane via

$$z = e^{\frac{2\pi}{L}w} = 1 + \frac{2\pi}{L}w + \frac{1}{2!}\left(\frac{2\pi}{L}w\right)^2 + \frac{1}{3!}\left(\frac{2\pi}{L}w\right)^3 + \dots. \tag{4.1.1}$$

Here $t \to -\infty$ maps to $z = 0$, $t \to +\infty$ to $z = \infty$, and the compact $x$ direction gives angular periodicity. A very interesting feature of this map is that it takes circles of constant radius in $\mathbb{R}^2$ to constant $t$ slices on $\mathbb{R} \times S$. As a consequence, the dilatation operator on $\mathbb{R}^2$ which maps circles onto circles with different radius, corresponds to time translation on $\mathbb{R} \times S$, so it behave as Hamiltonian.

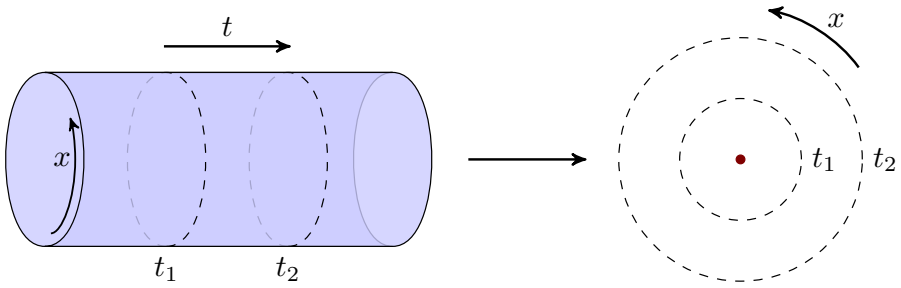



Figure 4.1: Conformal map from the cylinder to the complex plane.

Since periodicity in $x$ is reflected in the periodicity of $w$ and $\bar{w}$. All operators on cylinder admit the following expansion:

$$\phi(w, \bar{w}) = \sum_{m,n \in \mathbb{Z}} \phi_{n,m} e^{-nw} e^{-m\bar{w}} \tag{4.1.2}$$

where $\phi_{n,m}$ do not have any $(w, \bar{w})$ dependence. Consider the transformation of primary field living on cylinder to $z-$plane under the conformal mapping $z = e^w$:

$$\begin{aligned}
\Phi(z, \bar{z}) &= (\partial_z w)^h (\partial_{\bar{z}} \bar{w})^{\bar{h}} \Phi(w, \bar{w}) \\
&= z^{-h} \bar{z}^{-\bar{h}} \sum_{m,n \in \mathbb{Z}} \phi_{n,m} e^{-nw} e^{-m\bar{w}} \\
&= z^{-h} \bar{z}^{-\bar{h}} \sum_{m,n \in \mathbb{Z}} \phi_{m,m} z^{-n} \bar{z}^{-m} \\
&= \sum_{m,n} \phi_{n,m} z^{-n-h} \bar{z}^{-m-\bar{h}}
\end{aligned}$$

In free-field theories the vacuum is annihilated by positive-frequency modes; in interacting theories we assume asymptotic fields behave freely, e.g.

$$\phi_{\text{in}} \propto \lim_{t\to-\infty} \phi(x,t).$$

In radial quantization, this translates into

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle\,.$$

Expanding the operator in modes, we have

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \sum_{m,n} \phi_{n,m}\, z^{-n-h}\,\bar{z}^{-m-\bar{h}}\,|0\rangle\,.$$

For this expression to be non-singular as $z \to 0$, the modes with negative powers of $z$ and $\bar{z}$ must annihilate the vacuum. This requires

$$n+h>0 \;\;\Rightarrow\;\; n>-h, \qquad m+\bar{h}>0 \;\;\Rightarrow\;\; m>-\bar{h},$$

so that

$$\phi_{n,m}\,|0\rangle = 0 \qquad \text{for } n>-h \text{ and } m>-\bar{h}.$$

Thus, only the non-negative powers contribute. For $n+h<0$ and $m+\bar{h}<0$, we have

$$\lim_{z,\bar{z}\to 0} z^{-n-h}\bar{z}^{-m-\bar{h}}\,|0\rangle = 0$$

The only non-trivial contribution comes from the modes with

$$n+h=0, \qquad m+\bar{h}=0,$$

yielding

$$|\phi_{\text{in}}\rangle = \phi_{-h,-\bar{h}}\,|0\rangle\,.$$

Once the Hilbert space is tied to a vacuum and a Hamiltonian (here, the dilatation operator), operators can be classified by scaling dimensions, excitations arranged systematically, and operator product expansions formulated as exact operator identities—rather than being inferred indirectly from correlation functions.

### The Hermitian product

On this Hilbert space we must also define a bilinear product, which we do indirectly by defining an asymptotic "out" state, together with the action of Hermitian conjugation on conformal fields.

$$\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger = \left(\lim_{z,\bar{z}\to 0}\phi(z,\bar{z})\,|0\rangle\right)^\dagger$$

In Minkowski space, Hermitian conjugation does not affect the space-time coordinates. Things are different in Euclidian space, since the Euclidian time $\tau = it$ must be reversed ($\tau \to -\tau$) upon Hermitian conjugation if $t$ is to be left unchanged. In radial quantization this corresponds to the mapping:

$$\begin{aligned} w \equiv e^{\tau+ix} &\to e^{-\tau+ix} \\ &= e^{-(\tau-ix)} = \frac{1}{e^{\overline{\tau+ix}}} = \frac{1}{\bar{z}} \end{aligned}$$

Since $\phi$ is a primary field,

$$\phi(z,\bar{z}) = \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \phi(w,\bar{w})$$

$$
\begin{aligned}
&= \left(-\frac{1}{z^2}\right)^h \left(-\frac{1}{\bar{z}^2}\right)^{\bar{h}} \phi(w,\bar{w}) \\
&= (-1)^h (w^{2h})(-1)^{-\bar{h}} (\bar{w}^{2\bar{h}}) \phi(w,\bar{w}) \\
&= (-1)^{h+\bar{h}} w^{2h} \bar{w}^{2\bar{h}} \phi(w,\bar{w}) \\
\phi(w,\bar{w}) &= (-1)^{h+\bar{h}} w^{-2h} \bar{w}^{-2\bar{h}} \phi(z,\bar{z})
\end{aligned}
$$

Since $|\phi_{\text{in}}\rangle$ were defined using $\lim_{z\to 0}$, so $\lim_{w\to 0}$ could be used to define $|\phi_{\text{out}}\rangle$ state:

$$
\langle \phi_{\text{out}}| = \lim_{z,\bar{z}\to\infty} \langle 0|\, \phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = = \lim_{w,\bar{w}\to 0} (-1)^{h+\bar{h}} w^{-2h} \bar{w}^{-2\bar{h}} \langle 0|\, \phi\left(\frac{1}{w},\frac{1}{\bar{w}}\right)
$$

relabelling $w = z$,

$$
\langle \phi_{\text{out}}| = \lim_{z,\bar{z}\to 0} (-1)^{h+\bar{h}} z^{-2h} \bar{z}^{-2\bar{h}} \langle 0|\, \phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right)
$$

The factor $(-1)^{h+\bar{h}}$ is coming due to spin, if we ignore that:

$$
\langle \phi_{\text{out}}| = \lim_{z,\bar{z}\to 0} z^{-2h} \bar{z}^{-2\bar{h}} \langle 0|\, \phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = \lim_{z,\bar{z}\to 0} \langle 0|\, \phi(z,\bar{z})^\dagger
$$

Hence, we conclude

$$
\phi(z,\bar{z})^\dagger = z^{-2h} \bar{z}^{-2\bar{h}} \phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) \tag{4.1.3}
$$

Or we could alternatively do Fourier expansion on the radial plane, the adjoint property then reads

$$
\begin{aligned}
\Phi(z,\bar{z})^\dagger &= \bar{z}^{-2h} z^{-2\bar{h}} \Phi(1/\bar{z}, 1/z) \\
&= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} \left(\frac{1}{\bar{z}}\right)^{-n-h} \left(\frac{1}{z}\right)^{-m-\bar{h}} \\
&= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} (\bar{z})^{n+h} (z)^{m+\bar{h}} \\
&= \sum_{m,n} \phi_{n,m} z^{m-\bar{h}} \bar{z}^{n-h} \\
\sum_{m,n} \phi^\dagger_{n,m} \overline{(z^{-n-h} \bar{z}^{-m-\bar{h}})} &= \sum_{m,n} \phi_{-n,-m} z^{-m-\bar{h}} \bar{z}^{-n-h} \implies \boxed{\phi^\dagger_{n,m} = \phi_{-n,-m}}
\end{aligned}
$$

The inner product is well defined as well:

$$
\begin{aligned}
\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\, \phi(1/\bar{z}, 1/z) \phi(w,\bar{w})\, |0\rangle \\
&= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0|\, \phi(\bar{\xi},\xi) \phi(0,0)\, |0\rangle \\
&= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \frac{C_{12}}{\bar{\xi}^{2h} \xi^{2\bar{h}}} = C_{12}
\end{aligned}
$$

Unless the prefactors in (4.1.3) were missing the limit would have been ill defined.

### 4.1.1 Radial Ordering

Within radial quantization, the time ordering that appears in the definition of correlation functions becomes a radial ordering,

$$
\mathcal{R}(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w|\,, \\ \phi_2(w)\phi_1(z) & |z| < |w|\,. \end{cases} \tag{4.1.4}
$$

As usual, we will always omit the radial-ordered operator in the correlation function as well as in the OPE expansion. One consequence after specifying time direction is that we can relate OPE to commutation relations. For this, let us consider the contour integral around $w$ for two holomorphic fields $a(z)$ and $b(w)$. If the contour is not radially ordered over the whole path, we can decompose it into contributions that are radially ordered:

$$\oint_w dz\, a(z)b(w) = \underbrace{\oint_{|z|>|w|} dz\, a(z)b(w)}_{\text{radial ordering: } z \text{ outside } w} - \underbrace{\oint_{|z|<|w|} dz\, b(w)a(z)}_{\text{radial ordering: } z \text{ inside } w} = [A,\, b(w)]\,, \tag{4.1.5}$$

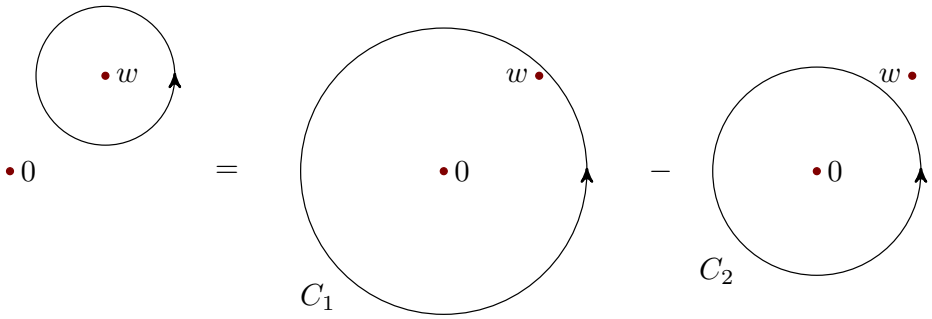
where the operator $A$ is the contour integral of $a(z)$ at a fixed time

$$A = \oint \mathrm{d}z\, a(z)\,. \tag{4.1.6}$$

Here we take the contours $C_1$ and $C_2$ at fixed-radius $|w| + \epsilon$ and $|w| - \epsilon$ with a small positive number $\epsilon$ as illustrated in Figure 4.2 . Then, with $B = \oint \mathrm{d}z\, b(z)$, we can generalize the relation (**??**) to

$$[A, B] = \oint_0 \mathrm{d}w [A, b(w)] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z a(z)b(w)\,. \tag{4.1.7}$$

Figure 4.2: Subtraction of contours




## 4.2 Virasoro Algebra

### 4.2.1 Conformal Generators

The ward identity

$$\delta_\epsilon \langle \phi_1 \dots \phi_n \rangle = \int_M d^2x \partial_\mu \langle T^{\mu\nu} \epsilon_\nu \phi_1 \dots \phi_n \rangle$$

could be expressed in complex coordinates using (3.7.2):

$$\begin{aligned}\delta_{\epsilon,\bar\epsilon} \langle \phi_1 \dots \phi_n \rangle &= \frac{i}{2} \int_C \left[ -dz \left\langle T^{\bar z\bar z} \epsilon_{\bar z} \phi_1 \dots \phi_n \right\rangle + d\bar z \left\langle T^{zz} \epsilon_z \phi_1 \dots \phi_n \right\rangle \right] \\ &= -\frac{1}{2\pi i} \int_C dz \left\langle T(z) \epsilon \phi_1 \dots \phi_n \right\rangle + \frac{1}{2\pi i} \int_C d\bar z \left\langle \bar T(\bar z) \bar\epsilon \phi_1 \dots \phi_n \right\rangle\end{aligned}$$

where we used,

$$\begin{aligned} T &= -2\pi T_{zz} = -2\pi g_{z\mu} g_{z\nu} T^{\mu\nu} = -2\pi \frac{1}{4} T^{\bar z\bar z} = -\frac{\pi}{2} T^{\bar z\bar z} \\ \bar T &= -2\pi T_{\bar z\bar z} = -2\pi g_{\bar z\mu} g_{\bar z\nu} T^{\mu\nu} = -2\pi \frac{1}{4} T^{zz} = -\frac{\pi}{2} T^{zz} \end{aligned}$$

and

$$\epsilon = \epsilon^z = g^{z\mu}\epsilon_\mu = 2\epsilon_{\bar{z}}$$
$$\bar{\epsilon} = \epsilon^{\bar{z}} = g^{\bar{z}\mu}\epsilon_\mu = 2\epsilon_z$$

If we apply, (4.1.5) and (4.1.7) to the conformal ward identity. Let $\epsilon(z)$ be holomorphic component of an infinitesimal conformal change of coordinates. We then define the conformal charge

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z) \tag{4.2.1}$$

with the help of (4.1.5), the conformal ward identity translates into

$$\delta_\epsilon\phi = -[Q_\epsilon, \phi]$$

which means that the operator $Q_\epsilon$ is the generator of conformal transformation. We may expand energy-momentum tensor according to (4.1.2):

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \qquad L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z)$$
$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+1}\bar{T}(\bar{z})$$

we may also expand the infinitesimal conformal change $\epsilon(z)$ as:

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n$$

Then the expression, (4.2.1) becomes:

$$\begin{aligned} Q_\epsilon &= \frac{1}{2\pi i}\oint dz\left(\sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n\right)T(z) \\ &= \sum_{n\in\mathbb{Z}}\epsilon_n\frac{1}{2\pi i}\oint dz z^{n+1}T(z) \\ &= \sum_{n\in\mathbb{Z}}\epsilon_n L_n \end{aligned}$$

The mode operators $L_n$ and $L_m$ of the energy-momentum tensor are the generators of the local conformal transformations on the Hilbert space, exactly like $l_n$ and $l_m$ of Witt algebra. The next part is to find the algebra obeyed by $L_n$ and $L_m$ which we described in previous chapter as the central extension of Witt algebra. We will now prove that here:

$$\begin{aligned} [L_m, L_n] &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}z^{m+1}w^{n+1}[T(z), T(w)] \\ &= \oint\frac{dw}{2\pi i}w^{n+1}\oint_{C(w)}\frac{dz}{2\pi i}z^{m+1}\mathcal{R}\big(T(z)T(w)\big) \\ &= \oint\frac{dw}{2\pi i}w^{n+1}\oint_{C(w)}\frac{dz}{2\pi i}z^{m+1}\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}\right] \\ &= \oint\frac{dw}{2\pi i}w^{n+1}\left[(m+1)m(m-1)w^{m-2}\frac{c}{2\cdot 3!} + 2(m+1)w^m T(w) + w^{m+1}\partial_w T(w)\right] \end{aligned}$$

$$= \oint \frac{dw}{2\pi i}\left[\frac{c}{12}(m^3-m)\,w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + w^{m+n+2}\partial_w T(w)\right]$$

$$= \oint \frac{dw}{2\pi i}\left[\frac{c}{12}(m^3-m)\,w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + \partial_w\{w^{m+n+2}T(w)\} - \partial_w w^{m+n+2}T(w)\right]$$

$$= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + 2(m+1)L_{m+n} + 0 - \oint \frac{dw}{2\pi i}\,(m+n+2)T(w)w^{m+n+1}$$

$$= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n}$$

$$= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}.$$

Collectively,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$$
$$[L_m, \bar{L}_n] = 0$$
$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$$

Notice that central charge term vanishes for $m = 0, \pm 1$. Therefore, the generator of global conformal transformation in 2D does not have central charge.

## 4.3 The Hilbert Space

The vacuum state $|0\rangle$ in CFT must be invariant under global conformal transformation. This means the generator of transformation

$$\frac{az+b}{cz+d}$$

which are $L_{\pm 1}$ and $L_0$ must annihilate the vacuum. This, in turn is recovered from the condition that $T(z)\,|0\rangle$ and $\bar{T}(\bar{z})\,|0\rangle$ are well defined as $z, \bar{z} \to 0$,

$$\lim_{z\to 0} T(z)\,|0\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}} z^{-n-2}L_n\,|0\rangle$$
$$\lim_{\bar{z}\to 0} \bar{T}(\bar{z})\,|0\rangle = \lim_{\bar{z}\to 0}\sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n\,|0\rangle$$

Here, $z^{-(n+2)}$ diverges at $z=0$ if $n+2>0$, i.e. if $n>-2$, or equivalently $n \geq -1$. Which implies, we need

$$L_n\,|0\rangle = 0$$
$$\bar{L}_n\,|0\rangle = 0 \qquad (\text{for } n \geq -1)$$

This includes a subcondition regarding invariance of $|0\rangle$ under global conformal group. It also implies the vanishing of the vacuum expectation value of the energy-momentum tensor:[1]

$$\langle 0|\,T(z)\,|0\rangle = \left(\sum_{n=-\infty}^{-1} z^{-n-2}\,\langle 0|\,L_n\right)|0\rangle + \sum_{n=0}^{\infty} z^{-n-2}\,\langle 0|\,(L_n\,|0\rangle) = 0$$

[1] If the vacuum is not invariant under local conformal transformation i.e. if vacuum is not annihilated by $L_n$ for $n>1$, then we can not ensure the vanishing of vev of energy-momentum tensor.

$$\langle 0|\,\bar{T}(\bar{z})\,|0\rangle = \left(\sum_{n=-\infty}^{-1} z^{-n-2}\,\langle 0|\,\bar{L}_n\right)|0\rangle + \sum_{n=0}^{\infty} z^{-n-2}\,\langle 0|\,\left(\bar{L}_n\,|0\rangle\right) = 0$$

Primary fields, when acting on vacuum, create asymptotic states, eigenstates of the Hamiltonian. A simple demonstration follows from the OPE (3.8.1) between $T(z)$ and a primary field $\phi(z,\bar{z})$ of dimension $(h,\bar{h})$, translated into operator language:

$$\begin{aligned}
[L_n,\phi(w,\bar{w})] &= \frac{1}{2\pi i}\oint_w dz\, z^{n+1}T(z)\phi(w,\bar{w}) \\
&= \frac{1}{2\pi i}\oint_w dz\, z^{n+1}\left[\frac{h\phi(w,\bar{w})}{(z-w)^2}+\frac{\partial\phi(w,\bar{w})}{z-w}+\text{reg.}\right] \\
&= h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \qquad (n\geq -1)
\end{aligned}$$

The antiholomorphic counterpart of this relation is

$$[\bar{L}_n,\phi(w,\bar{w})] = \bar{h}(n+1)\bar{w}^n\phi(w,\bar{w}) + \bar{w}^{n+1}\bar{\partial}\phi(w,\bar{w}) \qquad (n\geq -1)$$

After applying these relations to the asymptotic state

$$|h,\bar{h}\rangle \equiv \phi(0,0)|0\rangle,$$

we conclude that

$$\begin{aligned}
L_0\left|h,\bar{h}\right\rangle &= L_0\phi(0,0)\,|0\rangle \\
&= \phi(0,0)L_0\,|0\rangle + [L_0,\phi(0,0)]\,|0\rangle \\
&= 0 + h\phi(0,0)\,|0\rangle + 0 = h\left|h,\bar{h}\right\rangle
\end{aligned}$$

compactly,

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle, \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle.$$

Thus $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian. Likewise, we have

$$L_n|h,\bar{h}\rangle = 0, \qquad \bar{L}_n|h,\bar{h}\rangle = 0, \qquad \text{if } n>0.$$

Excited states above the asymptotic state $\left|h,\bar{h}\right\rangle$ may be obtained by applying ladder operators. The generators $L_{-m}(m>0)$ increases the eigenstate of $L_0$ a.k.a conformal weight by virtue of virasoro algebra

$$[L_0,L_{-m}] = (0+m)L_{0-m} = mL_{-m}$$

This means the excited states may be obtained by successive applications of these operators on the asymptotic state $|h\rangle$:

$$L_{-k_1}L_{-k_2}\ldots L_{-k_n}\,|0\rangle \qquad (1\geq k_1\geq\cdots\geq k_n)$$

This state is the eigenstate of $L_0$ with eigenvalue

$$h' = h+k_1+k_2+\cdots+k_n = h+N$$

and these are called descendants of the asymptotic state $|h\rangle$ and the integer $N$ is called the level of the descendant.

## 4.4 Free Boson

This section gives a detailed account of the canonical quantization of the free boson on the cylinder. The mode expansions are obtained, after imposing the appropriate boundary conditions. The mapping from the cylinder to the complex plane is used to define the conformal generators and, in particular, the vacuum energies. Free-field theories are of special importance not only because they can be solved explicitly, but also because they are the building blocks of more complicated models, or can be shown to be equivalent to interesting statistical models.

**Mode Expansion**

We let $\varphi(x,t)$ be a free Bose field defined on a cylinder of circumference $L$:

$$\varphi(x+L,t) \equiv \varphi(x,t)$$

This field may be Fourier expanded as follows:

$$\varphi(x,t) = \sum_n e^{2\pi i n x/L}\,\varphi_n(t),$$

with Fourier coefficients

$$\varphi_n(t) = \frac{1}{L}\int_0^L dx\, e^{-2\pi i n x/L}\,\varphi(x,t).$$

Next, we try to find the Fourier coefficients of kinetic part.

$$\partial_t\varphi(x,t) = \sum_n e^{2\pi i n x/L}\,\dot\varphi_n(t),$$
$$(\partial_t\varphi)^2 = \sum_{n,m} e^{2\pi i (n+m) x/L}\,\dot\varphi_n\dot\varphi_m.$$

Only the $m=-n$ term survives the integral over $x$, so

$$\boxed{\int_0^L dx\,(\partial_t\varphi)^2 = L\sum_n \dot\varphi_n\dot\varphi_{-n}} \tag{4.4.1}$$

Similarly,

$$\partial_x\varphi(x,t) = \sum_n \frac{2\pi i n}{L} e^{2\pi i n x/L}\,\varphi_n(t),$$
$$(\partial_x\varphi)^2 = \sum_{n,m} \frac{2\pi i n}{L}\frac{2\pi i m}{L} e^{2\pi i (n+m) x/L}\,\varphi_n\varphi_m.$$

$$\boxed{\int_0^L dx\,(\partial_x\varphi)^2 = L\sum_n \left(\frac{2\pi n}{L}\right)^2 \varphi_n\varphi_{-n}} \tag{4.4.2}$$

Putting them in the free field Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}g\int_0^L dx\,\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right] \\ &= \frac{gL}{2}\sum_n\left(\dot\varphi_n\dot\varphi_{-n} - \left(\frac{2\pi n}{L}\right)^2\varphi_n\varphi_{-n}\right).\end{aligned}$$

The momentum conjugate to $\varphi_n$ is

$$\pi_n = \frac{\partial\mathcal{L}}{\partial\dot\varphi_n} = gL\,\dot\varphi_{-n} \qquad\qquad [\varphi_n,\pi_m] = i\delta_{nm}.$$

The Hamiltonian is

$$\begin{aligned}H &= \sum_n \pi_n\dot\varphi_n - \mathcal{L} \\ &= \frac{1}{2gL}\sum_n\left(\pi_n\pi_{-n} + (2\pi n g)^2\,\varphi_n\varphi_{-n}\right).\end{aligned}$$

Thus the system reduces to a sum of decoupled harmonic oscillators with frequencies $\omega_n = \frac{2\pi|n|}{L}$. The usual procedure is to define creation and annihilation operators $\tilde{a}_n$ and $\tilde{a}_n^\dagger$:

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi g|n|}}(2\pi g|n|\phi_n + i\pi_{-n}) \tag{4.4.3}$$

such that $\left[\tilde{a}_n, \tilde{a}_m^\dagger\right] = \delta_{n,m}$ and $[\tilde{a}_n, \tilde{a}_m] = 0$. This ofcourse does not work for zero mode $\phi_0$. Instead we shall use the following operators:

$$a_n = \begin{cases} -i\sqrt{n}\,\tilde{a}_n & (n>0) \\ i\sqrt{-n}\,\tilde{a}_{-n}^\dagger & (n<0) \end{cases}, \qquad \bar{a}_n = \begin{cases} -i\sqrt{n}\tilde{a}_{-n} & (n>0) \\ i\sqrt{-n}\tilde{a}_n^\dagger & (n<0) \end{cases}$$

and we treat the zero mode $\phi_0$ separately. The associated commutation relations are

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_n, \tilde{a}_m] = 0, \quad [\tilde{a}_n, \tilde{a}_m] = n\delta_{n+m,0} \tag{4.4.4}$$

The Hamiltonian is then expressible as

$$H = \frac{1}{2gL}\sum_n \left(\pi_n\pi_{-n} + (2\pi gn)^2\varphi_n\varphi_{-n}\right) = \frac{\pi_0^2}{2gL} + \frac{(2\pi g\,0)^2}{2gL}\phi_0^2 + \frac{1}{2gL}\sum_{n\neq 0}\left(\pi_n\pi_{-n} + (2\pi gn)^2\varphi_n\varphi_{-n}\right) \tag{4.4.5}$$

We start with the oscillator operators:

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_n + i\pi_{-n}), \quad \tilde{a}_n^\dagger = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_{-n} - i\pi_n).$$

$$\tilde{a}_{-n} = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_{-n} + i\pi_n), \quad \tilde{a}_{-n}^\dagger = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_n - i\pi_{-n}).$$

$$4\pi gn\,\tilde{a}_n^\dagger\tilde{a}_n = (2\pi gn)^2\varphi_{-n}\varphi_n + \pi_n\pi_{-n} + i[\text{cross terms}],$$

$$4\pi gn\,\tilde{a}_{-n}^\dagger\tilde{a}_{-n} = (2\pi gn)^2\varphi_n\varphi_{-n} + \pi_{-n}\pi_n - i[\text{same cross terms}],$$

adding above equations,

$$4\pi gn(\tilde{a}_n^\dagger\tilde{a}_n + \tilde{a}_{-n}^\dagger\tilde{a}_{-n}) = 2[(2\pi gn)^2\varphi_n\varphi_{-n} + \pi_n\pi_{-n}] \implies 2\pi gn(\tilde{a}_n^\dagger\tilde{a}_n + \tilde{a}_{-n}^\dagger\tilde{a}_{-n}) = \pi_n\pi_{-n} + (2\pi gn)^2\varphi_n\varphi_{-n}$$

Substituting it in (4.4.5),

$$H = \frac{1}{2gL}\pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0} n(\tilde{a}_n^\dagger\tilde{a}_n + \tilde{a}_{-n}^\dagger\tilde{a}_{-n}).$$

$$a_n = -i\sqrt{n}\tilde{a}_n, \quad a_{-n} = a_n^\dagger = i\sqrt{n}\tilde{a}_n^\dagger, \qquad \bar{a}_n = -i\sqrt{n}\tilde{a}_{-n}, \quad \bar{a}_{-n} = \bar{a}_n^\dagger = i\sqrt{n}\tilde{a}_{-n}^\dagger \qquad (n>0).$$

So

$$n\tilde{a}_n^\dagger\tilde{a}_n = (i\sqrt{n}\tilde{a}_n^\dagger)(-i\sqrt{n}\tilde{a}_n) = a_{-n}a_n, \quad n\tilde{a}_{-n}^\dagger\tilde{a}_{-n} = (i\sqrt{n}\tilde{a}_{-n}^\dagger)(-i\sqrt{n}\tilde{a}_{-n}) = \tilde{a}_{-n}\tilde{a}_n.$$

$$\boxed{H = \frac{1}{2gL}\pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n)}. \tag{4.4.6}$$

The commutation relation (4.4.4), leads to

$$[H, a_{-m}] = \frac{\pi}{L}\sum_{n\neq 0}[a_{-n}a_n,\, a_{-m}]$$

$$
\begin{aligned}
&= \frac{\pi}{L}\sum_{n\neq 0}\Big(a_{-n}[a_n, a_{-m}] + [a_{-n}, a_{-m}]a_n\Big)\\
&= \frac{\pi}{L}\sum_{n\neq 0}\Big(a_{-n}\, n\, \delta_{n-m,0} + (-n)\, \delta_{-n-m,0}\, a_n\Big)\\
&= \frac{2\pi}{L} m\, a_{-m} = -i\dot{a}
\end{aligned}
$$

which means for $a_{-m}(m > 0)$, when applied to an eigenstate of $H$ of energy $E$, produces another eigenstate with energy $E + 2m\pi/L$. Since the fourier modes are

$$\varphi_n = \frac{i}{n\sqrt{4\pi g}}\,(a_n - \bar{a}_{-n})$$

The mode expansion at $t = 0$ may be written as:

$$\varphi(x) = \varphi_0 + \sum_{n\neq 0}\varphi_n e^{\frac{2\pi i n x}{L}} = \varphi_0 + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\,(a_n - \tilde{a}_{-n})e^{2\pi i n x/L}$$

The time evolution of the operators $\phi_0$, $a_n$ and $\bar{a}_n$ in the Heisenberg picture follow from the above Hamiltonian:

$$[H, \phi_0] = \frac{1}{2gL}[\pi_0^2, \phi_0] = -\frac{\pi_0[\phi_0, \pi_0] + [\phi_0, \pi_0]\pi_0}{2gL} = -\frac{i}{gL}\pi_0 \qquad \text{(using } [\varphi_0, \pi_0] = i\text{)}$$

Hence,

$$
\begin{aligned}
\varphi_0(t) &= \varphi_0(0) + \frac{1}{gL}\pi_0 t,\\
a_n(t) &= a_n(0)\, e^{-2\pi i n t/L},\\
\tilde{a}_n(t) &= \tilde{a}_n(0)\, e^{-2\pi i n t/L}
\end{aligned}
$$

In terms of constant operators, the mode expansion of the field at arbitrary time is then,

$$\varphi_{\text{cyl}} = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{-2\pi i n(t-x)/L} + \bar{a}_n e^{-2\pi i n(t+x)/L}\right) \tag{4.4.7}$$

where 'cyl' means the field defined on the cylinder.

**Relation between Cylinder and Plane**

Now, we move to Euclidean space-time ($t = -i\tau$) by taking $w = \tau - ix$ and $\overline{w} = \tau + ix$. The mode expansion of boson field on the cylinder now is then

$$\varphi_{\text{cyl}}(w, \overline{w}) = \varphi_0 - i\frac{1}{2gL}\pi_0(w + \overline{w}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{-2\pi n w/L} + \bar{a}_n e^{-2\pi n\overline{w})/L}\right), \tag{4.4.8}$$

with now $w \sim w + iL$. Using a conformal transformation as in Figure 4.1, we map all the operators from the cylinder to the complex plane:

$$z = e^{2\pi w/L}, \qquad \overline{z} = e^{2\pi\overline{w}/L}$$

we finally obtain the expansion by simply replacing $w$ by $z$.

$$\varphi_{\text{pl}}(z, \overline{z}) = \varphi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\overline{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}(a_n z^{-n} + \bar{a}_n \overline{z}^{-n})\,. \tag{4.4.9}$$

The propagator could be calculated by assuming $|z| > |z'|$ and considering the product

$$\varphi_{\rm pl}(w,\bar w)\,\varphi_{\rm pl}(w',\bar w') \;=\; \Big[\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z|^2 + i\frac{1}{\sqrt{4\pi g}}\sum_{m\neq 0}\frac{1}{m}(a_m z^{-m} + \overline{a}_m \bar z^{-m})\Big] \\ \times\Big[\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z'|^2 + i\frac{1}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}(a_n z'^{-n} + \overline{a}_n \bar z'^{-n})\Big].$$

We will move annihilation operators ($a_k, \overline{a}_k$ with $k>0$) and $\pi_0$ to the right to reach normal ordering; non-vanishing c-number contributions come from $[\varphi_0,\pi_0]=i$ and $[a_m,a_n]=m\delta_{m+n,0}$, $[\overline{a}_m,\overline{a}_n]=m\delta_{m+n,0}$. Let's first focus on these terms

$$\begin{aligned}
&\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0\ln|z|^2\Big)\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0\ln|z'|^2\Big) \\
&\qquad = \varphi_0^2 - i\frac{1}{4\pi g}\big(\varphi_0\pi_0\ln|z'|^2 + \pi_0\varphi_0\ln|z|^2\big) - \Big(\frac{1}{4\pi g}\Big)^2\pi_0^2\ln|z|^2\ln|z'|^2 \\
&\qquad = \varphi_0^2 - i\frac{1}{4\pi g}\big(\varphi_0\pi_0\ln|z'|^2 + (\varphi_0\pi_0 - i)\ln|z|^2\big) - \Big(\frac{1}{4\pi g}\Big)^2\pi_0^2\ln|z|^2\ln|z'|^2 \\
&\qquad = \varphi_0^2 - i\frac{1}{4\pi g}\varphi_0\pi_0\big(\ln|z'|^2 + \ln|z|^2\big) - \frac{1}{4\pi g}\ln|z|^2 - \Big(\frac{1}{4\pi g}\Big)^2\pi_0^2\ln|z|^2\ln|z'|^2 \\
&\qquad = \underbrace{\varphi_0^2 - i\frac{1}{4\pi g}\varphi_0\pi_0\big(\ln|z'|^2 + \ln|z|^2\big) - \Big(\frac{1}{4\pi g}\Big)^2\pi_0^2\ln|z|^2\ln|z'|^2}_{\text{normal ordered}} - \frac{1}{4\pi g}\ln|z|^2 \\
&\qquad = :\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0\ln|z|^2\Big)\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0\ln|z'|^2\Big): \;-\frac{1}{4\pi g}\ln|z|^2,
\end{aligned}$$

The zero-mode reordering produces the c-number $-\frac{1}{4\pi g}\ln|z|^2$. Next, we'll consider holomorphic oscillator pieces

$$i\frac{1}{\sqrt{4\pi g}}\sum_{m\neq 0}\frac{1}{m}a_m z^{-m} \quad\text{and}\quad i\frac{1}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}a_n z'^{-n}.$$

We can split the sum in two parts,

$$\sum_{m\neq 0}\frac{1}{m}a_m z^{-m} = \sum_{m=-\infty}^{-1}\frac{1}{m}a_m z^{-m} + \sum_{m=1}^{\infty}\frac{1}{m}a_m z^{-m}.$$

The only non-normal-ordered combination producing a c-number is the product of the $m>0$ part from the first bracket with the $n<0$ part from the second bracket. That term (including prefactors) is

$$\Big(i\frac{1}{\sqrt{4\pi g}}\Big)^2\Big(\sum_{m=1}^{\infty}\frac{1}{m}a_m z^{-m}\Big)\Big(\sum_{n=-\infty}^{-1}\frac{1}{n}a_n z'^{-n}\Big) = -\frac{1}{4\pi g}\sum_{m=1}^{\infty}\sum_{n=-\infty}^{-1}\frac{1}{mn}\,a_m a_n\, z^{-m}z'^{-n}.$$

Using $a_m a_n =: a_n a_m : + [a_m,a_n] =: a_n a_m : + m\delta_{m+n,0}$. The only contributing commutator in the sum over $n$ occurs when $n=-m$. Extracting that:

$$-\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m(-m)}\,m\,z^{-m}z'^{m} = -\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{-1}{m}\Big(\frac{z'}{z}\Big)^m = +\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m.$$

Thus the holomorphic oscillators give the c-number

$$\boxed{\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m}.$$

By identical steps for $\overline{a}_m$ modes we obtain

$$\boxed{\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{\bar{z}'}{\bar{z}}\Big)^m}\,.$$

Collecting the zero-mode, holomorphic and anti-holomorphic c-number contributions:

$$-\frac{1}{4\pi g}\ln|z|^2 \;+\; \frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m \;+\; \frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{\bar{z}'}{\bar{z}}\Big)^m.$$

We can use $-\ln(1-x)=\sum_{m=1}^{\infty}x^m/m$ to simplify the series expansion (valid for $|z'|<|z|$):

$$\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m=-\ln\Big(1-\frac{z'}{z}\Big),\qquad \sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{\bar{z}'}{\bar{z}}\Big)^m=-\ln\Big(1-\frac{\bar{z}'}{\bar{z}}\Big).$$

So the total c-number becomes

$$-\frac{1}{4\pi g}\ln|z|^2\;-\;\frac{1}{4\pi g}\ln\Big|1-\frac{z'}{z}\Big|^2=-\frac{1}{4\pi g}\ln\Big|z\Big(1-\frac{z'}{z}\Big)\Big|^2=-\frac{1}{4\pi g}\ln|z-z'|^2.$$

Therefore

$$\boxed{\varphi_{\rm pl}(w,\bar{w})\varphi_{\rm pl}(w',\bar{w}')=:\varphi_{\rm pl}(w,\bar{w})\varphi_{\rm pl}(w',\bar{w}'):-\frac{1}{4\pi g}\ln|z-z'|^2}$$

Since we know that the normal ordered product vanishes inside correlator, the propagator could be simply read off as:

$$\boxed{\big\langle\varphi_{\rm pl}(w,\bar{w})\,\varphi_{\rm pl}(w',\bar{w}')\big\rangle=-\frac{1}{4\pi g}\ln|z-z'|^2}$$

This is the same as (3.8.3). From now on, we will drop the subscript "pl." We know that $\varphi$ is **not** itself a primary field,[2] but its derivatives $\partial\phi$ and $\bar{\partial}\bar{\phi}$ are. Now we concentrate on the holomorphic field $\partial\varphi$

$$i\partial\varphi(z)=\frac{1}{4\pi g}\frac{\pi_0}{z}+\frac{1}{4\pi g}\sum_{n\neq0}a_nz^{-n-1}=\frac{1}{4\pi g}a_0z^{0-1}+\frac{1}{4\pi g}\sum_{n\neq0}a_nz^{-n-1}\,.$$

We may write the zero modes by $a_0$ and $\overline{a}_0$:

$$a_0\equiv\overline{a}_0\equiv\pi_0$$

Now the commutation relations of (4.4.4) can be extended to include the zero mode operator without changing the form of the algebra. The mode expansion of $\partial\varphi$ is consistent with the expansion of a primary field with $h=1$ then becomes:

$$i\partial\varphi(z)=\sum_n a_nz^{-n-1}\,. \tag{4.4.10}$$

Often the normalization of $g=\frac{1}{4\pi}$ is used in this kind of work. Using this mode expansion, we can also compute the two-point function of $\partial\varphi$. For ($|z|>|w|$):

$$\langle\varphi(z)\partial\varphi(w)\rangle=\sum_{m,n\neq0}\frac{1}{n}\langle a_na_m\rangle\,z^{-n}w^{-m-1}\,.$$

According to the commutation relation (4.4.4) and the fact that $a_m$ and $a_{-m}$ are annihilation and creation operators respectively, it follows that[3]

$$\langle\varphi(z)\partial\varphi(w)\rangle=\sum_{n>0}\sum_{m\neq0}\frac{n\delta_{n,-m}z^{-n}w^{-m-1}}{n}=\frac{1}{w}\sum_{n>0}(w/z)^n=\frac{1}{w}\frac{w/z}{1-w/z}=\frac{1}{z-w}\,. \tag{4.4.11}$$

[2] Refer to equation (5.25) in Sénéchal's CFT with $h=0$ as shown in Polchinski's String Theory, Vol. 1 (eq. 2.4.17), the corresponding two-point function should be constant rather than logarithmic.

[3] $\langle a_na_m\rangle=\langle0|\,a_ma_n\,|0\rangle+n\delta_{n,-m}\,\langle0|0\rangle=n\delta_{n,-m}$ for $n>0$

Its differentiation with respect to $z$ provides the two-point function.

$$\langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{(z-w)^2}\,. \tag{4.4.12}$$

The holomorphic energy-momentum tensor is given by

$$\begin{aligned} T(z) &= -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) : \\ &= \frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-n-m-2} : a_n a_m := \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}} : a_{k-m}a_m :\right) z^{-k-2} \\ &= \left(\frac{1}{2}a_0^2 + \sum_{m>0} : a_{-m}a_m :\right) z^{-2} + \sum_{k\neq 0} z^{-k-2}\frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{k-m}a_m : \\ &= \sum_{k\in\mathbb{Z}} z^{-k-2} L_k \end{aligned}$$

which implies

$$\begin{aligned} L_k &= \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{k-m}a_m : \qquad (k\neq 0) \\ L_0 &= \sum_{m>0} a_{-m}a_m + \frac{1}{2}a_0^2\,. \end{aligned}$$

The Hamiltonian (4.4.6) now can be written as,

$$H = \frac{2\pi}{L}(L_0 + \overline{L}_0)\,. \tag{4.4.13}$$

This confirms the role of $L_0$ and $\overline{L}_0$ as a Hamiltonian. Since we place the free boson on a cylinder of size $L$, energy is proportional to $\frac{2\pi}{L}$, which is called the **finite-size scaling**. The mode operator $a_m$ plays a similar role to $L_m$ with respect to $L_0$, because of the commutation relation $[L_0, a_{-m}] = ma_{-m}$. Therefore its effect on the conformal dimension is the same as that of $L_m$.

By definition, the normal ordering prescription implies that $\langle T(z)\rangle = 0$.[4] We can always define the normal ordering product by subtracting all the singular terms from the OPE. In fact the general expectation value for energy-momentum tensor (3.9.2) can be written as

$$\langle T(z)\rangle = -\frac{1}{2}\lim_{\epsilon\to 0}\left(\langle\partial\varphi(z+\epsilon)\partial\varphi(z)\rangle + \frac{1}{\epsilon^2}\right). \qquad (\text{with } g=\tfrac{1}{4\pi})$$

We will see that this relation is useful when we consider the anti-periodic boundary condition later. By plugging in the two-point function from (4.4.19), we see that $\langle T(z)\rangle = 0$, which implies that $(L_0)_{\rm pl}$ vanishes on the vacuum. We now map the theory back to a cylinder $z \to w = \frac{L}{2\pi}\ln z$, by using (3.9.8):

$$T_{\rm cyl}(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left[T_{\rm pl}(z) - \frac{1}{12}\{w,z\}\right] = \left(\frac{L}{2\pi z}\right)^{-2}\left[T_{\rm pl}(z) - \frac{1}{12}\frac{1}{2z^2}\right] = \left(\frac{2\pi}{L}\right)^2\left[T_{\rm pl}(z)z^2 - \frac{1}{24}\right] \tag{4.4.14}$$

where we use the central charge $c=1$ for the free boson. Taking the expectation value on both sides, we have

$$\langle T_{\rm cyl}\rangle = -\frac{1}{24}\left(\frac{2\pi}{L}\right)^2 \tag{4.4.15}$$

[4] However, as we will see in the next section, this is not the case for the fermionic theory with anti-periodic boundary condition.

On cylinder, we do Fourier expansion according to (4.1.2)

$$T(w) = \sum_{n\in\mathbb{Z}} e^{nw} L_k$$

The mode expansion of the energy-momentum tensor tell us that this nonzero expectation value implies a constant term in the expression for $L_0$ in terms of modes.

$$L_{0,\text{cyl}} = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n - \frac{1}{24}\,. \tag{4.4.16}$$

The Hamiltonian is now written as

$$H = \frac{2\pi}{L}(L_{0,\text{cyl}} + \overline{L}_{0,\text{cyl}})\,. \tag{4.4.17}$$

Actually, in the general case, Hamiltonian of a theory defined on a cylinder with central charge $c$ can be written as

$$H = \frac{2\pi}{L}(L_{0,\text{pl}} + \overline{L}_{0,\text{pl}} - \frac{c}{12})\,. \tag{4.4.18}$$

Thus, we can infer that the central charge $c$ shows up as the vacuum energy of a theory on a cylinder, and this is one instance of finite-size effects.

In the antiperiodic case, $\varphi(x+L,t) = -\varphi(x,t)$, this boundary condition implies that the field $\varphi$ is double-valued on the cylinder. Once the cylinder is mapped onto the plane, this amounts to defining the theory on a pair of Riemann sheets. The summation index starts at $n = \frac{1}{2}$ and takes half-integer values thereafter. The vacuum expectation values is taken in one of the two ground states and

$$\begin{aligned}\langle\varphi(z)\partial\varphi(w)\rangle &= \sum_{m,n\neq 0}\frac{1}{n}\langle a_n a_m\rangle z^{-\frac{2n+1}{2}} w^{-\frac{2m+1}{2}-1} = \sum_{n>0}\sum_{m\neq 0}\frac{n\delta_{n,-m} z^{-\frac{2n+1}{2}} w^{-\frac{2m+1}{2}-1}}{n}\\ &= \frac{1}{w}\sum_{n\geq 0}(w/z)^{\frac{2n+1}{2}} = \frac{1}{w}\frac{\sqrt{w/z}}{1-w/z} = \sqrt{\frac{z}{w}}\frac{1}{z-w}\,.\end{aligned}$$

Its differentiation with respect to $z$ provides the two-point function.

$$\langle\partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{2}\frac{\sqrt{z/w}+\sqrt{w/z}}{(z-w)^2}\,. \tag{4.4.19}$$

This expression has branch cut at $z = 0,\infty$ and $w = 0,\infty$; the antiperiodic boundary condition on $\phi$ as $z$ circles around the origin is incorporated in the square roots. The periodic and antiperiodic two point function coincide in the limit $z \to w$, meaning the short distance physics is independent of the choice of boundary conditions. The vacuum energy density obtained from the normal ordering prescription is then,

$$\langle T(z)_{\text{pl}}\rangle = \frac{1}{16z^2}$$

Since $L_0$ is the coefficient of $1/z^2$ in the mode expansion of the energy-momentum tensor, this nonzero expectation value implies a constant term in the expression for $L_0$.

$$L_{0,\text{pl}} = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n + \frac{1}{16}$$

## 4.5 Vertex Operator

Since canonical scaling dimension $(h,\bar{h})$ of the bosonic field $\varphi$ vanishes, it is possible to construct an infinite variety of local fields related to $\varphi$ without introducing a scale, namely the so called vertex operators. These operators become relevant in string theory because their action on absolute vacuum creates the conformal vacuum state of the theory. They are indeed primary fields and defined as:

$$\mathcal{V}_\alpha = :e^{i\alpha\varphi(z,\bar{z})}:$$

The normal ordering has the following meaning, in terms of the operators appearing in the mode expansion

# Chapter 5

# de Sitter

## 5.1 Introduction

Anti–de Sitter (AdS) and de Sitter (dS) are two of the three maximally symmetric solutions of Einstein's equations with constant curvature. Unlike flat Minkowski space, they feature genuine gravitational effects: the dynamics of spacetime itself influences particle motion. AdS has constant negative curvature, while dS has constant positive curvature. Their isometry algebras can be described linearly by embedding them in higher-dimensional flat (zero-curvature) spaces, where $\mathrm{AdS}_d$ is realized as a hyperboloid in $\mathbb{R}^{d-1,2}$, while $\mathrm{dS}_d$ sits inside $\mathbb{R}^{d,1}$. In both cases the isometry group is simply the Lorentz group of the embedding space:

$$\mathrm{Isom}(AdS_d) = SO(d-1,2), \qquad \mathrm{Isom}(dS_d) = SO(d,1).$$

To compare with the Poincaré algebra, one splits the generators $J_{AB}$ of the embedding group into "Lorentz" generators $iM_{\mu\nu} = J_{\mu\nu}$ and "translation-like" generators $iP_\mu = \frac{1}{R}J_{\mu d}$, where $R$ is the (A)dS radius and $\mu, \nu = 0, 1 \dots d-1$. The algebra of the embedding-space isometry generators is independent of the choice of metric signature and can be written as:

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} - \eta_{AD}J_{BC} + \eta_{BD}J_{AC} - \eta_{BC}J_{AD}$$

The commutators then organize as:

- The Lorentz subalgebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}),$$

  valid for both AdS and dS.

- Lorentz action on "translations":

$$[M_{\mu\nu}, P_\rho] = -[J_{\mu\nu}, J_{\rho d}] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu).$$

  valid for both AdS and dS.

- Commutator of translations:

$$[P_\mu, P_\nu] = \frac{1}{R^2}[J_{\mu d}, J_{\nu d}] = \frac{\eta_{dd}}{R^2}J_{\mu\nu} = \eta_{dd}\frac{i}{R^2}M_{\mu\nu},$$

where the **sign distinguishes the cases**: minus for AdS, plus for dS (mostly plus signature).[1] This is the only algebraic difference, and it encodes the curvature sign of the spacetime. The sign flip of a single commutator,

[1] refer to Newton-Hooke Algebras, Non-relativistic Branes and Generalized pp-wave Metrics but flip the metric sign to account for signature convention change.

while all others remain unchanged, is what ultimately alters the underlying physics. Locally, de Sitter (dS) and Anti-de Sitter (AdS) spaces can be related by analytic continuation, but this correspondence does not extend globally. The obstruction lies in their fundamentally different global structures: AdS has topology $\mathbb{R} \times S^{d-1}$ with a timelike conformal boundary, while dS has topology $S^d$ with spacelike conformal boundaries at future and past infinity. These differences in causal structure, boundary conditions, and topology prevent a global analytic continuation between the two spacetimes. (verify?)

The flat-space limit is obtained by letting $R \to \infty$. In this contraction, the $[P_\mu, P_\nu]$ commutator vanishes, recovering the familiar Poincaré algebra $ISO(d-1,1)$ with commuting translations. Thus, both AdS and dS reduce to flat Minkowski symmetry in the infinite-radius limit, but neither contains Poincaré as a subgroup at finite radius. The distinct signatures have important consequences for holography. In AdS, the boundary is timelike, and the isometry group $SO(d-1,2)$ acts as the Lorentzian conformal group of a $(d-1)$-dimensional Minkowski boundary, giving the basis for AdS/CFT. In dS, the boundary is spacelike, and the same structure $SO(d,1)$ acts as the Euclidean conformal group of a $(d-1)$-dimensional space, motivating the dS/CFT correspondence. This is a key principle of holography, where the isometry group of the higher-dimensional "bulk" spacetime matches the conformal group of the lower-dimensional boundary which can be seen from the table below.

| **Geometry Type** | **Dimensions** | **Isometry Group** | **Conformal Group** |
|---|---|---|---|
| Euclidean | $d$ | $ISO(d)$ | $SO(d+1,1)$ |
| de Sitter | $d+1$ | $SO(d+1,1)$ | $SO(d+1,2)$ |
| Minkowski | $d$ | $ISO(d-1,1)$ | $SO(d,2)$ |
| Anti-de Sitter | $d+1$ | $SO(d,2)$ | $SO(d,3)$ |

From a geometric perspective, $\Lambda = 0$ corresponds to flat Minkowski space, $\Lambda > 0$ gives dS as the maximally symmetric spacetime of constant positive curvature, and $\Lambda < 0$ gives AdS with constant negative curvature. For example, $dS_4$ can be realized as the hyperboloid in $\mathbb{R}^{4,1}$:[2]

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = l^2$$

It describes a time-like hyperboloid with topology $\mathbb{R} \times S^3$, the $S^3$ arising from the slicing at fixed $X^0$,

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \underbrace{l^2 + (X^0)^2}_{\text{constant}} > 0 \tag{5.1.1}$$

The spacetime interval in $\mathbb{R}^{4,1}$ is

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2.$$

Since de Sitter space is embedded in $\mathbb{R}^{(4,1)}$, whose isometry group is ISO$(4,1)$, the induced metric on de Sitter space must be invariant under a certain subgroup of ISO$(4,1)$ restricted to the hypersurface. However, the way these ambient-space isometries act on $\mathbb{R}^{4,1}$ differs from how they act on the de Sitter hypersurface $\mathbb{M}^{3,1}$. For this reason, we describe them in terms that reflect their action on de Sitter space itself:

- 3 Rotation + 3 Translation
$$x_i \to a_i + R_{ij} x_j$$
- 1 Dilatation
$$x_\mu \to \lambda x_\mu$$
- 3 Special Conformal Transformation ($\eta \to 0$ or $b^\mu = (0, \vec{b})$)
$$x_i \to \frac{x_i - b_i(-\eta^2 + \vec{x}^2)}{1 - 2\vec{b}\cdot\vec{x} + b^2(-\eta^2 + \vec{x}^2)}$$

[2] we define it like that because it helps us realize the de Sitter isometries in linear fashion.

We will derive these in the next section. For now, note that four-dimensional de Sitter space has a total of

$$\frac{5(5-1)}{2} = 10$$

independent isometries, forming the group $\mathrm{SO}(4,1)$. This happens because the choice of de Sitter hypersurface breaks the translation invariance of the ambient space.

### Coordinatizing de Sitter

There are many ways to introduce coordinate system on de Sitter space. We will start by setting a cartesian like coordinate system in ambient space and then introduce $d-2$ dimensional hypersurfaces which will slice the de Sitter space into space+time. The family of curve orthogonal to this hypersurface will become our timelike direction on de Sitter and points on this $d-2$ dimensional hypersurface will be spacelike. This will lead to natural introduction of inducing embedding space coordinate to label the points on de Sitter hypersurface. Let us first define $n^a$ as normal vectors on 3−sphere ($S^3$). We can parameterize (5.1.1) as following

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = l^2 + (X^0)^2$$
$$l^2 \underbrace{\sum_{a,b=1}^{4} \delta_{ab} n^a n^b}_{1} \cosh\left(\frac{\tau}{l}\right) = l^2 + l^2 \sinh\left(\frac{\tau}{l}\right)$$

i.e. we can introduce following global coordinates on de Sitter space as[3]:

$$\left.\begin{aligned} X^0 &= l^2 \sinh\left(\tfrac{\tau}{l}\right) \\ X^a &= l^2 n^a \cosh\left(\tfrac{\tau}{l}\right); \qquad a = 1,2,3,4 \end{aligned}\right\} \text{closed slicing}$$

we also define following notation:

$$\sum_{a,b=1}^{d-1} \delta_{ab} dn^a dn^b = d\Omega_{d-1}^2$$

For $\mathbb{S}^3$, the normal vector $n^a$ looks like

$$\begin{aligned} n^1 &= \cos\chi \\ n^2 &= \sin\chi \cos\theta \\ n^3 &= \sin\chi \sin\theta \cos\phi \\ n^4 &= \sin\chi \sin\theta \sin\phi \end{aligned}$$

where

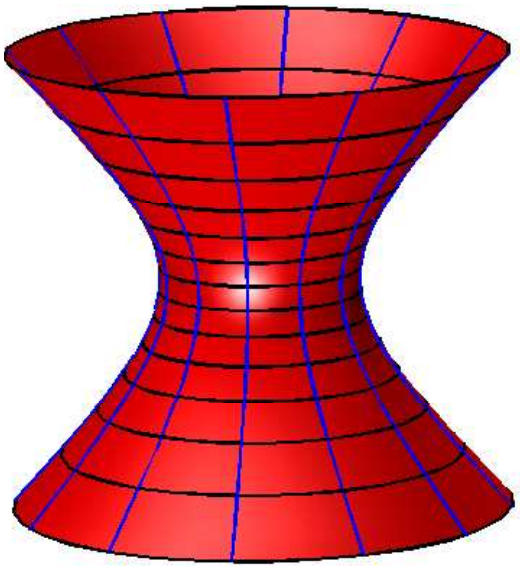
$$\chi \in [0,\pi], \quad \theta \in [0,\pi], \quad \phi \in [0,2\pi]$$

We quickly note that the above definition of coordinates gives rise to following induced metric on the hyperboloid.

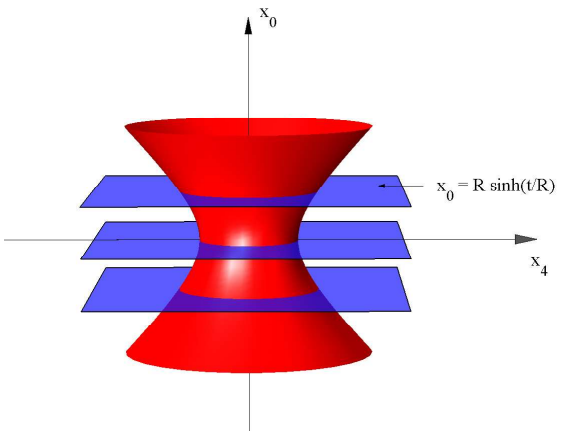
$$ds^2 = -d\tau^2 + l^2 \cosh^2\left(\frac{\tau}{l}\right) d\Omega_3^2$$

where $d\Omega_3^2$ is metric on $S^3$. Constant time slices are then compact. For $\tau > 0$, this is the typical picture of a closed Universe whose size is expanding exponentially as time evolves forward. The minimal size of the sphere is at $\tau = 0$, where the radius of the sphere is one (in units of the dS radius). In these coordinates $dS_4$ looks like a 3−sphere which starts out infinitely large at $\tau = -\infty$, then shrinks to a minimal finite size at $\tau = 0$, then grows again to infinite size as $\tau = \infty$. Note that this metric depends explicitly on the global time; dS does not have a **global** timelike Killing vector.

[3] chapter 4 of "Jerry B. Griffiths, Jirí Podolský - Exact Space-Times in Einstein's General Relativity"

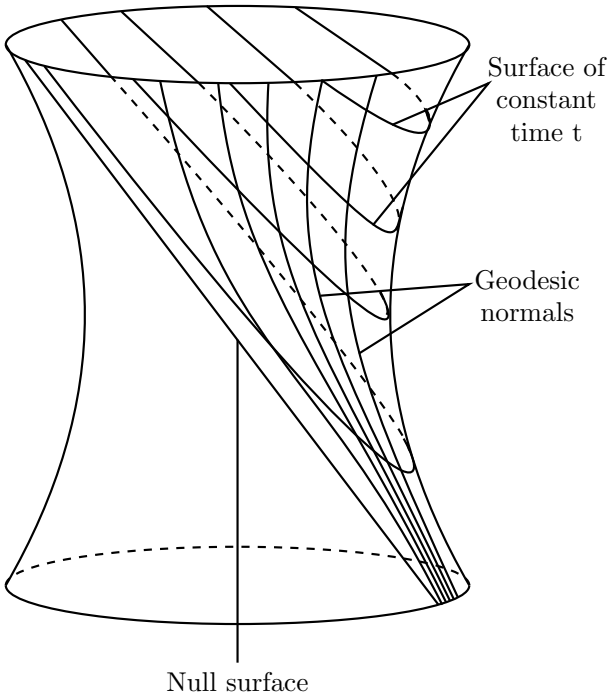


(a)

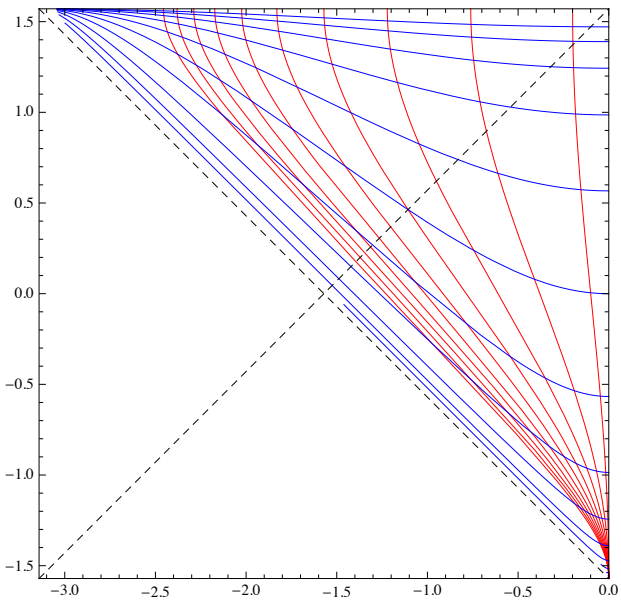



(b)

### 5.1.1 Flat coordinates




(c)




(d)

Figure 5.1: (c) de Sitter geodesic. (d) Flat slicing of de Sitter, drawn on the Penrose diagram. Blue curves are constant-$t$ flat slices, and red curves are the surfaces of constant-$r$. Intersections of the blue curves with the dashed line are the cross-sections of the cosmological horizon.

These are the coordinates $t, \mathbf{x}$, defined by

$$X^0 = l \sinh\left(\frac{ct}{l}\right) + \frac{1}{2l}x^2 e^{\frac{ct}{l}},$$
$$X^1 = l \cosh\left(\frac{ct}{l}\right) - \frac{1}{2l}x^2 e^{\frac{ct}{l}},$$
$$X^i = x^i e^{\frac{ct}{l}}, \quad i = 2, 3, 4.$$

where $x^2 = \delta_{ab}x^a x^b$. These coordinates do not cover the full de Sitter space, but only the patch

$$X^0 + X^1 = l e^{ct/l} > 0.$$

observable by observer at south pole ($\theta = \pi$) in penrose diagram. What it implies is that the space is coordinatized by a collection of plane in the ambient space with slope $-1$ and $X^d$ intercept given by $He^{t/H} > 0$. The flat slicing covers only half the Penrose diagram, so this metric by itself is past-geodesically incomplete. Timelike worldlines, unless they are specially chosen to sit at the South pole, will exit the flat slicing in the past, in finite affine time. In these coordinates, the metric reads

$$ds^2 = -c^2 dt^2 + e^{2ct/l} \sum_{i=1}^{3} dx_i^2.$$

This metric can be regarded as a FRW cosmology with an exponential function $a(t)$ by replacing $l = \frac{c}{H}$:

$$a(t) = e^{Ht}. \tag{5.1.2}$$

If we draw a diagonal plane through the embedding diagram, this is the 'upper triangle.' Similar coordinates can be chosen to cover just the 'lower triangle.'

### 5.1.2 Conformally flat coordinates

There is another coordinate which is conformal to flat Cartesian coordinates and covers more parts of de Sitter space than flat coordinates. Using familiar cartesian-like coordinates $(c\eta, x, y, z)$, the de Sitter hyperboloid is covered by:

$$x^0 = \frac{l^2 + s}{2c\eta}$$
$$x^1 = \frac{l^2 - s}{2c\eta}$$
$$x^2 = l\frac{x}{c\eta}$$
$$x^3 = l\frac{y}{c\eta}$$
$$x^4 = l\frac{z}{c\eta}$$

where $s = -c^2\eta^2 + x^2 + y^2 + z^2$ with $\eta, x, y, z \in (-\infty, \infty)$. Note that these coordinates still do not cover the full de Sitter space but only the patch

$$x^0 + x^1 \neq 0$$

In these coordinates, the de Sitter metric is

$$ds^2 = \frac{l^2}{c^2\eta^2}(-c^2 d\eta^2 + d\vec{x}^2)$$

where $\eta$ is usually referred to as conformal time. This is usually the preferred frame for the computation of cosmological correlators. In these coordinates the time $\eta$ is not a Killing vector, and the only manifest symmetries are translations and rotations of the $x^i$ coordinate.

### 5.1.3 Static coordinates

A very important aspect of dS space is that no single observer has access to the full spacetime. This is clear by just looking at the Penrose diagram. An important set of coordinates are those that describe the region accessible to a single observer. This is the intersection between the region of space that can affect the observer and the region that can be affected by them. In terms of embedding coordiantes, they are given by:

$$X^0 = l\sqrt{1-\frac{r^2}{l^2}}\sinh\left(\frac{ct}{l}\right),$$
$$X^1 = l\sqrt{1-\frac{r^2}{l^2}}\cosh\left(\frac{ct}{l}\right),$$
$$X^a = rn^a, \quad a = 2,3,4$$

where

$$0 \leq r < l$$

They only cover the region

$$x^1 > 0, \quad r^2 < l^2.$$

These are accelerating observers in the ambient space:

$$-(X^0)^2 + (X^1)^2 + (X^a)^2 = (l^2 - \cancel{r^2}) + \cancel{r^2} = \frac{1}{\alpha^2}$$

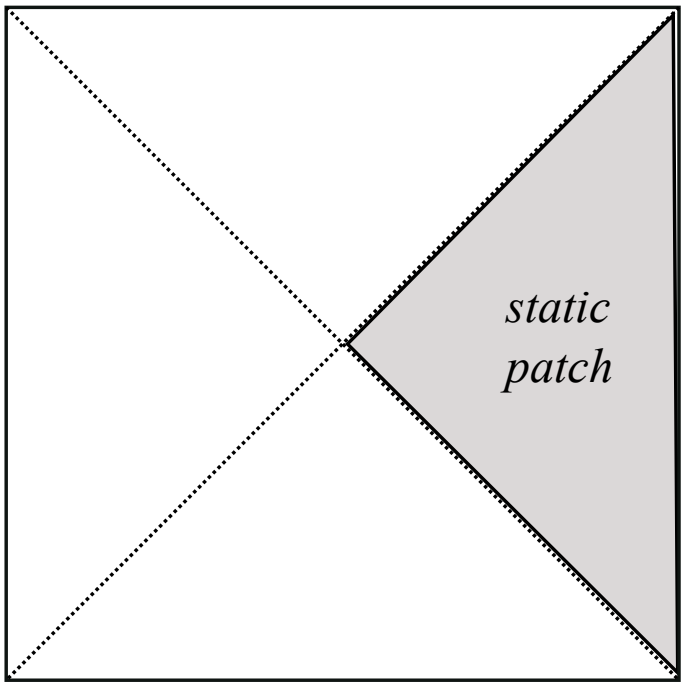



Figure 5.2: Static patch, on the Penrose diagram. This is the causal patch of an observer sitting at the north pole, i.e. $\theta = 0$ in global coordinates, i.e. $r = 0$ in static coordinates. The right edge of the diagram is $r_{\text{static}} = H$; the bifurcate Killing horizon is $r_{\text{static}} =$. The other three patches can also be covered by (independent) static coordinate systems, much like the four regions of the Penrose diagram for Schwarzschild black holes

In these coordinates, the metric reads

$$ds^2 = \left(1-\frac{r^2}{l^2}\right)c^2dt^2 - \frac{dr^2}{1-\frac{r^2}{l^2}} - r^2d\Omega_3^2.$$

Notice the presence of an explicit horizon at $r = l$, which makes manifest the presence of event horizons for observers in de Sitter space.

### 5.1.4 Conformal Structure

The conformal boundary of de Sitter spacetime is located at $\eta = 0$ and $\eta = \pi$, corresponding to past and future infinity. Unlike in Minkowski space, these infinities, $\mathcal{I}^-$ and $\mathcal{I}^+$, are **spacelike** in de Sitter. To analyze the causal structure, it is convenient to introduce a conformal time coordinate $T$, defined by

$$\cos T = \frac{1}{\cosh\left(\frac{\tau}{H}\right)}.$$

This maps the global time coordinate $\tau$ to a finite range,

$$-\tfrac{\pi}{2} \leq T \leq \tfrac{\pi}{2}.$$

The spatial sections of de Sitter space are 3-spheres. Writing the metric on $S^3$ as

$$d\theta^2 + \sin^2\theta \, d\Omega_2^2,$$

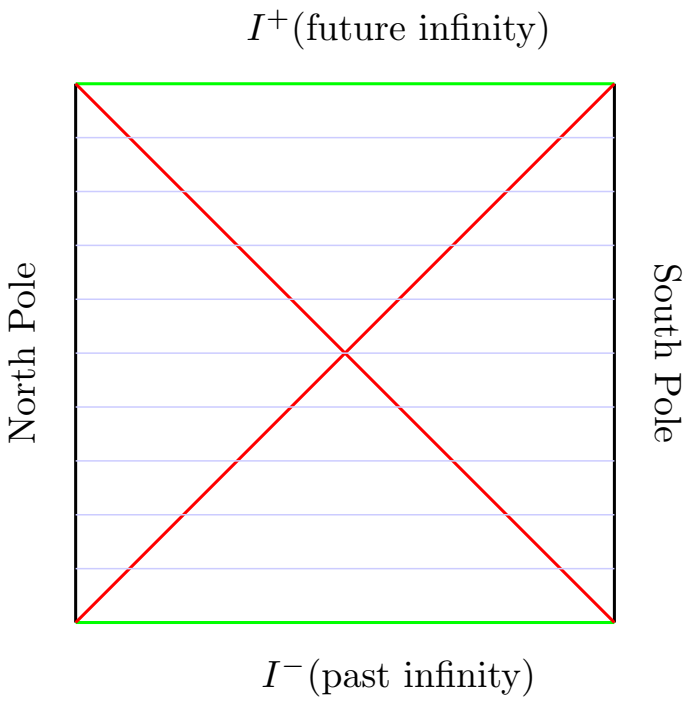
the full de Sitter metric becomes conformal to

$$ds^2 = \frac{1}{\cos^2 T}\Big( - dT^2 + d\theta^2 + \sin^2\theta \, d\Omega_2^2\Big),$$

with $\theta \in [0, \pi]$.

Suppressing the transverse 2-sphere $d\Omega_2$ and including the endpoints at $T = \pm\frac{\pi}{2}$ (past and future infinity), we obtain the Penrose diagram of de Sitter space in the $(T, \theta)$ plane. Each horizontal slice of this diagram (shown in blue) corresponds to a 3-sphere of radius

$$R(T) = \frac{1}{\cosh T}.$$



In this representation, each point along a horizontal line stands for a 2-sphere, except at the vertical boundaries $\theta = 0$ and $\theta = \pi$, where the geometry collapses to a single point. These are referred to as the North and South poles of the 3-sphere, and it is often convenient to imagine inertial observers located there.

Finally, the symmetry group of the de Sitter boundary is isomorphic to the conformal group of $\mathbb{R}^3$. This group includes rotations, translations, dilatations, and special conformal transformations. Since the boundary is spacelike, however, there is no notion of Lorentz boosts on the boundary.

## 5.2 Killing Vectors

We noted at the start of this chapter that de Sitter space is maximally symmetric space with 10 killing vector associated with translation (3), rotation (3), dilatation (1) and SCT (3). There are two ways to find these killing vector

- Either find the coordianate transformation such that metric is independent of one or more coordinates.
- Solve the killing equation in some coordinate system

Since finding all the relevant coordiante transformation is very much dependent on luck, we choose to solve the killing equation in conformal coordinate. The metric for de Sitter space in conformal coordinates is given as:

$$ds^2 = \frac{1}{(H\eta)^2}(-d\eta^2 + d\vec{x}^2) = \frac{1}{(H\eta)^2}\eta_{\mu\nu}dx^\mu dx^\nu$$

The killing equation would be given as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

It is easier to first focus on the case where $\Gamma^\lambda_{\mu\nu} = 0$, so will we start with that.

### 5.2.1 Translation

The first thing to note is that metric is already independent of $x^i$, therefore it has 3 killing vector associated with translation. We can show that they are $\xi^\mu = \delta^\mu_i$, and

$$\xi_\mu = g_{\mu\nu}\xi^\nu = g_{\mu\nu}\delta^\nu_i = g_{\mu i}$$

From killing equation:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} - 2\Gamma^\rho_{\mu\nu}\xi_\rho$$

Substituting $\xi_\rho = g_{\rho i}$, we get:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})g_{\rho i}$$

Now, using $g_{\rho i}g^{\rho\sigma} = \delta^\sigma_i$, the Killing condition simplifies to:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - (g_{i\mu,\nu} + g_{i\nu,\mu} - g_{\mu\nu,i}) = 0$$

Hence,

$$\xi^x = (0,1,0,0) \qquad \xi^y = (0,0,1,0) \qquad \xi^z = (0,0,0,1)$$

or more compactly

$$\boxed{\xi = \xi^i \partial_i = \partial_i}$$

To generate translation from them, we just exponentiate the generators:

$$e^{a\partial_x}x = x + a\partial_x x + 0 = x + a$$

### 5.2.2 Rotation

Next, we solve for $\mu \neq \nu$ case where $\Gamma^{\lambda}_{\mu\nu} = 0$:

$$\begin{aligned}\partial_x \xi_y + \partial_y \xi_x &= 0\\ \partial_x \xi_z + \partial_z \xi_x &= 0\\ \partial_y \xi_z + \partial_z \xi_y &= 0\end{aligned}$$

Since $\partial_i g^{\mu\nu} = 0$, we can rewrite them as

$$\begin{aligned}\partial_x \xi^y + \partial_y \xi^x &= 0\\ \partial_x \xi^z + \partial_z \xi^x &= 0\\ \partial_y \xi^z + \partial_z \xi^y &= 0\end{aligned}$$

These are same as killing vector for flat space rotation.

$$\xi = \epsilon^{(i)jk} x_j \partial_k$$

we can just exponentiate them and find rotation. We will do it for $i = 3$ which describes the rotation about z-axis

$$\xi = (0, -y, x, 0) \equiv -y\partial_x + x\partial_y$$

The relevant transformation is:

$$\begin{aligned}e^{\theta\xi} x &= \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n \left(-y\partial_x + x\partial_y\right)^n x\\ &= x + \theta \left(-y\partial_x + x\partial_y\right) x + \frac{\theta^2}{2!} \left(-y\partial_x + x\partial_y\right)^2 x + \frac{\theta^3}{3!} \left(-y\partial_x + x\partial_y\right)^3 x \ldots\\ &= x - \theta y - \frac{\theta^2}{2!} \left(-y\partial_x + x\partial_y\right) y - \frac{\theta^3}{3!} \left(-y\partial_x + x\partial_y\right) x \ldots\\ &= \left(1 - \frac{\theta^2}{2!} + \ldots\right) x - \left(\theta - \frac{\theta^3}{3!} + \ldots\right) y\\ &= \cos(\theta) x - \sin(\theta) y\end{aligned}$$

Similarly,

$$\begin{aligned}e^{\theta\xi} y &= \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n \left(-y\partial_x + x\partial_y\right)^n y\\ &= y + \theta \left(-y\partial_x + x\partial_y\right) y + \frac{\theta^2}{2!} \left(-y\partial_x + x\partial_y\right)^2 y + \frac{\theta^3}{3!} \left(-y\partial_x + x\partial_y\right)^3 y \ldots\\ &= y + \theta x + \frac{\theta^2}{2!} \left(-y\partial_x + x\partial_y\right) x - \frac{\theta^3}{3!} \left(-y\partial_x + x\partial_y\right) y \ldots\\ &= \left(1 - \frac{\theta^2}{2!} + \ldots\right) y + \left(\theta - \frac{\theta^3}{3!} + \ldots\right) x\\ &= \cos(\theta) y + \sin(\theta) x\end{aligned}$$

Represented in matrix form:

$$\begin{bmatrix}\eta'\\ x'\\ y'\\ z'\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}\eta\\ x\\ y\\ z\end{bmatrix}$$

### 5.2.3 Dilatation

Now to proceed further we need to write down the non vanishing connection terms:

$$\Gamma^x_{\eta x} = -\frac{1}{\eta} - \frac{H'}{H} \qquad \Gamma^y_{\eta y} = -\frac{1}{\eta} - \frac{H'}{H}$$
$$\Gamma^z_{\eta z} = -\frac{1}{\eta} - \frac{H'}{H} \qquad \Gamma^\eta_{xx} = -\frac{1}{\eta} - \frac{H'}{H}$$
$$\Gamma^\eta_{yy} = -\frac{1}{\eta} - \frac{H'}{H} \qquad \Gamma^\eta_{zz} = -\frac{1}{\eta} - \frac{H'}{H}$$
$$\Gamma^\eta_{\eta\eta} = -\frac{1}{\eta} - \frac{H'}{H}$$

More compactly, we can express them as:

$$\Gamma^i_{\eta i} = \Gamma^\eta_{\mu\mu} = -\frac{1}{\eta} + \epsilon H a(\eta) = -\frac{1}{\eta}\left(1 + \frac{\epsilon}{1-\epsilon}\right) = -\frac{1}{\eta(1-\epsilon)} \approx -\frac{1}{\eta} - \frac{\epsilon}{\eta}$$

Since, $\epsilon = 0$ for de Sitter space, we can drop those terms. Just a quick clarification—the conformal Killing vectors of quasi-de Sitter space match those of de Sitter space only at leading order in slow-roll. These should not be confused with the isometry-generating Killing vectors, which strictly preserve the metric, unlike conformal Killing vectors that preserve it only up to a conformal factor. The killing vectors being studied here are isometry generating killing vectors for de Sitter space not conformal killing vector. They just happen to be conformal killing vector of $\mathbb{R}^3$ and hence the association of name dilatation. The killing equation for such can be given as:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$
$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\Gamma^\lambda_{\mu\nu}\xi_\lambda = 0$$

for $\mu = \nu$, we have following four expressions:

$$\partial_x \xi_x = -\frac{1}{\eta}\xi_\eta \qquad \partial_y \xi_y = -\frac{1}{\eta}\xi_\eta$$
$$\partial_z \xi_z = -\frac{1}{\eta}\xi_\eta \qquad \partial_\eta \xi_\eta = -\frac{1}{\eta}\xi_\eta$$

We can solve the last equation as:

$$\frac{d\xi_\eta}{\xi_\eta} = -\frac{d\eta}{\eta} \quad \Rightarrow \quad \ln \xi_\eta = -\ln \eta + \ln f(x,y,z)$$

which leads to

$$\boxed{\xi_\eta(\eta, x, y, z) = \frac{f(x,y,z)}{\eta}}$$

Now we determine $f(x, y, z)$:

$$\xi_\eta = \frac{f(x,y,z)}{\eta} \Rightarrow \partial_\eta \xi_\eta = -\frac{1}{\eta}\xi_\eta = -\frac{f(x,y,z)}{\eta^2}$$

So,

$$\partial_x \xi_x = -\frac{f(x,y,z)}{\eta^2}; \qquad \partial_y \xi_y = -\frac{f(x,y,z)}{\eta^2}; \qquad \partial_z \xi_z = -\frac{f(x,y,z)}{\eta^2}$$

For now let us focus on the specific case where,

$$f(x,y,z) = \text{const} = A \Rightarrow \xi_\eta = \frac{A}{\eta}$$

We will determine the specific form of $f(x, y, z)$ later. Then:

$$\partial_x \xi_x = -\frac{A}{\eta^2} \Rightarrow \xi_x = -\frac{A}{\eta^2} x + \phi_1(y, z, \eta)$$

$$\partial_y \xi_y = -\frac{A}{\eta^2} \Rightarrow \xi_y = -\frac{A}{\eta^2} y + \phi_2(x, z, \eta)$$

$$\partial_z \xi_z = -\frac{A}{\eta^2} \Rightarrow \xi_z = -\frac{A}{\eta^2} z + \phi_3(x, y, \eta)$$

For now lets set $\phi_i = 0$ which describes rotation and fix coefficient $A$ such that in the component form:

$$\boxed{\xi^\mu = g^{\mu\nu}\xi_\nu = (\eta, x, y, z)}$$

is the killing vector associated with dilatation. With the basis, it looks like:

$$D = \xi^\mu \partial_\mu = x^\mu \partial_\mu$$

Exponentiating them

$$\begin{aligned}
e^{\lambda D} x = e^{\lambda x \cdot \partial_x} x &= \lim_{N\to\infty} \left(1 + \frac{\lambda}{N} x \cdot \partial\right)^N x \\
&= \lim_{N\to\infty} \underbrace{\left(1 + \frac{\lambda}{N} x \cdot \partial\right) \ldots \left(1 + \frac{\lambda}{N} x \cdot \partial\right)}_{N \text{ terms}} x \\
&= \lim_{N\to\infty} \underbrace{\left(1 + \frac{\lambda}{N} x \cdot \partial\right) \ldots \left(1 + \frac{\lambda}{N} x \cdot \partial\right)}_{N-1 \text{ terms}} \left(x + \frac{\lambda}{N} x\right) \\
&= \lim_{N\to\infty} \left(1 + \frac{\lambda}{N}\right) \underbrace{\left(1 + \frac{\lambda}{N} x \cdot \partial\right) \ldots \left(1 + \frac{\lambda}{N} x \cdot \partial\right)}_{N-1 \text{ terms}} x \\
&= e^\lambda x
\end{aligned}$$

or, more simply we can utilize $x \cdot \partial_x x = x$ and then

$$\begin{aligned}
e^{\lambda D} x &= \sum_{n=0}^{\infty} \frac{(\lambda x \cdot \partial_x)^n}{n!} x \\
&= 1 + \frac{\lambda}{1!} x \cdot \partial_x x + \frac{\lambda^2}{2!} (x \cdot \partial_x)(x \cdot \partial_x x) + \ldots \\
&= \left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots\right) x = e^\lambda x
\end{aligned}$$

### 5.2.4 Special conformal transformation

The remaining killing equation which lead to SCT are as follows:

$$\partial_i \xi_\eta + \partial_\eta \xi_i + \frac{2}{\eta} \xi_i = 0$$

Going back we found the solution to

$$\partial_\eta \, \xi_\eta \; = \; -\frac{1}{\eta} \, \xi_\eta,$$

as

$$\xi_\eta(\eta,x,y,z) \;=\; \frac{f(x,y,z)}{\eta} \quad \text{for some function } f(x,y,z).$$

This time we won't assume it to just be a constant. Therefore the killing equation to be solved becomes:

$$\frac{1}{\eta}\,\partial_i f(x,y,z) \;+\; \partial_\eta\,\xi_i \;+\; \frac{2}{\eta}\,\xi_i = 0,$$

which is a first-order linear ordinary differential equation in $\eta$ for each spatial component $\xi_i$. The homogeneous part $\partial_\eta\xi_i + (2/\eta)\xi_i = 0$ integrates immediately to $\xi_i^{(\mathrm{hom})}(\eta) = C_i(x,y,z)/\eta^2$. A particular solution of the inhomogeneous equation can be found by considering power series in $\eta$, defined as $\xi_i = A(x,y,z)\eta^\alpha$

$$\begin{aligned} A\alpha\eta^{\alpha-1} + 2A\eta^{\alpha-1} &= -\eta^{-1}\partial_i f(x,y,z) \\ (\alpha+2)A\eta^{\alpha-1} &= -\eta^{-1}\partial_i f(x,y,z) \implies \alpha = 0, A = -\frac{1}{2}\partial_i f(x,y,z) \end{aligned}$$

Therefore, $\xi_i^{(\mathrm{part})} = -\frac{1}{2}\,\partial_i f(x,y,z)$. Hence the full solution is

$$\begin{aligned} \xi_\eta &= \frac{f(x,y,z)}{\eta} \\ \xi_i(\eta,x,y,z) &= \frac{C_i(x,y,z)}{\eta^2} \;-\; \frac{1}{2}\,\partial_i f(x,y,z) \end{aligned}$$

Meanwhile, from the $i \neq j$ Killing equations:

$$\partial_i\xi_j + \partial_j\xi_i = \frac{1}{\eta^2}(\partial_i C_j + \partial_j C_i) - \partial_i\partial_j f = 0$$

we infer $\partial_i\partial_j f = 0$ and $\partial_i C_j + \partial_j C_i = 0$. From $\partial_i\partial_j f = 0$ one concludes $f$ is at most linear in $(x,y,z)$, so

$$f(x,y,z) = a_x\,x + a_y\,y + a_z\,z + A$$

with constant coefficients $a_i, A$. There are further constraints on $C_i$ as:

$$\begin{aligned} \partial_i\xi_i = \frac{1}{\eta^2}\partial_i C_i(x,y,z) - \frac{1}{2}\partial_i\partial_i f(x,y,z) &= -\frac{f(x,y,z)}{\eta^2} \\ \partial_i C_i(x,y,z) &= -f(x,y,z) \end{aligned}$$

Since we have already studied dilatation, where we studied the effect of $A \neq 0$, therefore let us set $A$ to zero,

$$C_i = -\int f(x,y,z)dx_i = -(a\cdot x)x_i + \frac{1}{2}a_i(x^i)^2 + g(x_j \forall j \neq i)$$

Then

$$\partial_i C_j + \partial_j C_i = 0 \implies C_i = -\Big(\sum_{j=1}^{3} a_j x^j\Big)x_i + \frac{1}{2}a_i\Big(\sum_{j=1}^{3} x^j x^j\Big)$$

Let choose $(a_x,a_y,a_z) = -\frac{2}{H^2}(b_x,b_y,b_z)$ so $\xi_i$ appears without fraction. Then

$$\xi_\eta(\eta,x,y,z) = -\frac{2b_i x^i}{H^2\eta} \qquad \xi_i(\eta,x,y,z) \qquad \begin{aligned} &= \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} - \frac{1}{2H^2}\partial_i\big(-2b_j x^j\big) \\ &= \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} + \frac{b_i}{H^2} \end{aligned}$$

Raising indices gives[4]

$$\xi^{\eta} = 2\,\eta\,(b_i x^i), \qquad \xi^i = (\eta^2 - \vec{x}^2)b^i + 2(b\cdot x)x^i$$

Since $x^\mu = (\eta, x^i)$ and $b^\mu = (0, b^i)$, this coincides exactly with

$$\xi^\mu = 2(b\cdot x)x^\mu - b^\mu(-\eta^2 + \vec{x}^2), \quad b^0 = 0$$

These are the generators of the isometry of de Sitter space not the conformal transformations of de Sitter space. We should be careful in regard to not interpret these killing vectors as conformal killing vectors. Since $b^\mu$ is spacelike with $b^0 = 0$ it does not generate SCT for $\mathbb{R}^4$ but for $\mathbb{R}^3$ on $\eta = 0$ hypersurface. We can see that these generators of de Sitter isometry has similar form as the expressions in section 1.4.11

- Translation
$$P_i = \partial_i$$
- Rotation
$$L_{ij} = x_i\partial_j - x_j\partial_i$$
- Dilatation
$$D = x^\mu\partial_\mu$$
- SCT
$$K^i = [2(b^j x_j)x^i - b^i(\eta_{kl}x^k x^l)]\partial_i$$

## 5.3 Ambient space

The isometries of de Sitter space are highly non-linear, which makes them difficult to study directly in intrinsic coordinates. A more convenient approach is to view them from the perspective of the higher-dimensional flat embedding space. The special relativity of higher dimension will apppear as general relativity on de Sitter space. The generators of $D+1$ dimensional lorentz transformations in such embedding space are given as:

$$J_{MN} = X_M\partial_N - X_N\partial_M$$

where $M, N = 0, 1, 2, \ldots D$. There are several ways to coordinatize the embedded de Sitter space, for this section we choose to work in flat slicing. The coordinate transformation between the two given as:

$$\left.\begin{aligned} X^0 &= \tfrac{\rho}{2(-\eta)}(1-s) \\ X^D &= \tfrac{\rho}{2(-\eta)}(1+s) \end{aligned}\right\} \implies \rho = -\eta(X^D + X^0)$$

$$X^i = \frac{\rho}{-\eta}x^i \implies x^i = \frac{-\eta}{\rho}X^i = \frac{X^i}{X^0 + X^D}$$

where

$$s = \eta^2 - \delta_{ij}x^i x^j$$

and $i = 1, 2, \ldots D-1$. We can note from below that $\rho$ =constant hypersurface are hyperbolic in nature.

$$g_{AB}X^A X^B = -(X^0)^2 + (X^D)^2 + \delta_{ij}X^i X^j = \frac{\rho^2 s^2}{\eta^2} + \frac{\rho^2}{\eta^2}\delta_{ij}x^i x^j = \rho^2$$

Inverting above relation,

$$\rho = \sqrt{g_{AB}X^A X^B} \implies \frac{\partial\rho}{\partial X^0} = -\frac{X^0}{\rho} = \frac{s-1}{2(-\eta)}$$

[4]indices of $b_i$ and $x_i$ are raised and lowered by $\eta_{\mu\nu}$

$$\eta = -\frac{\rho}{X^0 + X^D} \implies \frac{\partial \eta}{\partial X^0} = \frac{X_0 X_D + X_D^2 + X^i X_i}{(X^D + X^0)^2 \sqrt{g_{AB} X^A X^B}} = \frac{\frac{1+s}{2} + \delta_{ij} x^i x^j}{\rho}$$
$$x^i = \frac{X^i}{X^0 + X^D}$$

we can now use these to express the derivatives in the two coordinate system in the following manner:

$$\frac{\partial}{\partial X^0} = \frac{1}{2(-\eta)}(-1 + \eta^2 - \delta_{ij} x^i x^j)\frac{\partial}{\partial \rho} + \frac{1}{2\rho}(1 + \eta^2 + \delta_{ij} x^i x^j)\frac{\partial}{\partial \eta} + \frac{\eta}{\rho} x^i \partial_i$$
$$\frac{\partial}{\partial X^D} = \frac{1}{2(-\eta)}(1 + \eta^2 - \delta_{ij} x^i x^j)\frac{\partial}{\partial \rho} + \frac{1}{2\rho}(-1 + \eta^2 + \delta_{ij} x^i x^j)\frac{\partial}{\partial \eta} + \frac{\eta}{\rho} x^i \partial_i$$
$$\frac{\partial}{\partial X^i} = \frac{x_i}{-\eta}\frac{\partial}{\partial \rho} - \frac{x_i}{\rho}\frac{\partial}{\partial \eta} - \frac{\eta}{\rho}\partial_i$$

Metric in this coordinate

$$ds^2 = -(dX^0)^2 + (dX^D)^2 + \delta_{ij} dX^i dX^j = d\rho^2 + \rho^2 \frac{\left(-d\eta^2 + \delta_{ij} dx^i dx^j\right)}{\eta^2}$$

Then

$$\begin{aligned} P_i = J_{Di} - J_{0i} &= X_D \frac{\partial}{\partial X^i} - X_i \frac{\partial}{\partial X^D} - X_0 \frac{\partial}{\partial X^i} + X_i \frac{\partial}{\partial X^0} \\ &= (X_D - X_0)\frac{\partial}{\partial X^i} - X_i \left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right) \\ &= (X^D + X^0)\frac{\partial}{\partial X^i} - X^i \left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right) \\ &= \frac{\rho}{-\eta}\left(\frac{x^i}{-\eta}\frac{\partial}{\partial \rho} - \frac{x^i}{\rho}\frac{\partial}{\partial \eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta} x^i \left(\frac{1}{-\eta}\frac{\partial}{\partial \rho} - \frac{1}{\rho}\frac{\partial}{\partial \eta}\right) \\ &= \partial_i \end{aligned}$$

since $i = 1, 2, \ldots D-1$, there are $D-1$ momenta and SCT generator.

$$\begin{aligned} K_i = J_{Di} + J_{0i} &= X_D \frac{\partial}{\partial X^i} - X_i \frac{\partial}{\partial X^D} + X_0 \frac{\partial}{\partial X^i} - X_i \frac{\partial}{\partial X^0} \\ &= (X_D + X_0)\frac{\partial}{\partial X^i} - X_i \left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\ &= (X^D - X^0)\frac{\partial}{\partial X^i} - X^i \left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\ &= \frac{\rho s}{-\eta}\left(\frac{x^i}{-\eta}\frac{\partial}{\partial \rho} - \frac{x^i}{\rho}\frac{\partial}{\partial \eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta} x^i \left(\frac{s}{-\eta}\frac{\partial}{\partial \rho} + \frac{\eta^2 + \delta_{ij} x^i x^j}{\rho}\frac{\partial}{\partial \eta} + \frac{2\eta}{\rho} x^j \partial_j\right) \\ &= \frac{x^i s}{\eta}\frac{\partial}{\partial \eta} + s\partial_i + \frac{2\eta^2 - s}{\eta} x_i \frac{\partial}{\partial \eta} + 2x_i (x^j \partial_j) \\ &= (\eta^2 - \delta_{jk} x^j x^k)\partial_i + 2x_i \eta \partial \eta + 2x_i (x^j \partial_j) \\ &= 2x_i (x \cdot \partial) - (-\eta^2 + \vec{x}^2)\partial_i \end{aligned}$$

$$\begin{aligned} D = J_{D0} &= X_D \frac{\partial}{\partial X^0} - X_0 \frac{\partial}{\partial X^D} = X_D \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^D} \\ &= \frac{\rho}{2(-\eta)}(1+s)\left[\frac{1}{2(-\eta)}(-1+s)\frac{\partial}{\partial \rho} + \frac{1}{2\rho}(1 + \eta^2 + \delta_{ij} x^i x^j)\frac{\partial}{\partial \eta} + \frac{\eta}{\rho} x^i \partial_i\right] \end{aligned}$$

$$+\frac{\rho}{2(-\eta)}(1-s)\left[\frac{1}{2(-\eta)}(1+s)\frac{\partial}{\partial\rho}+\frac{1}{2\rho}(-1+\eta^2+\delta_{ij}x^ix^j)\frac{\partial}{\partial\eta}+\frac{\eta}{\rho}x^i\partial_i\right]$$
$$=-\eta\frac{\partial}{\partial\eta}-x^i\partial_i=-x\cdot\partial$$

This exercise showed us that there is a more transparent way to visualize the isometries of de Sitter space: by embedding it into a higher-dimensional ambient space where the symmetry algebra acts linearly. However, one important question remains: **what happens to the de Sitter radius?**

In particular, if we take the flat-space limit $l\to\infty$, how do the de Sitter generators reduce to the Poincaré generators, and which symmetries morph into which? To study this question more carefully, we will next examine how the de Sitter isometries can be mapped via a stereographic projection onto Minkowski space, which itself can be embedded in the same ambient space. This approach makes it straightforward to perform the Inönü–Wigner contraction and explicitly see how the symmetry algebra of de Sitter reduces to that of flat spacetime.

### Stereographic Projection of de Sitter Space

So far, the generators we wrote down have no explicit dependence on the de Sitter radius $l$. To introduce this dependence, we work in stereographic coordinates, where de Sitter space is projected conformally onto Minkowski space. We begin by recalling that de Sitter space can be realized as a hyperboloid embedded in $(d+1)$-dimensional Minkowski space:

$$-X_0^2+X_1^2+\cdots+X_{d-1}^2+X_d^2=l^2.$$

The normal to this surface could be given as:

$$n_A=\frac{\partial}{\partial x^A}\left(X^BX_B-l^2\right)=2X_A$$

Any tensor field $K$ which is tangent to this hypersurface has to satisfy the following transversality condition.

$$X^AK_A=0$$

Now, to perform the projection, let us consider the separation vector starting from $(0,-l)$ and ending at $(x^\mu,l)$ in $\mathbb{R}^{d,1}$. The components of the separation vector between the points could be given as:

$$\begin{aligned}X^A(\Omega(x))&=(0,0,\ldots,-l)+\Omega(x)[(x^\mu,l)-(0,0,\ldots,-l)]\\&=(0,0,\ldots,-l)+\Omega(x)(x^\mu,2l)\\&=(\Omega(x)x^\mu,2\Omega(x)l-l)\end{aligned}$$

The points on this line eventually passes through the hyperboloid $X_AX^A=l^2$, so it leads to the following condition over $\Omega(x)$:

$$\begin{aligned}g_{\mu\nu}\Omega(x)^2x^\mu x^\nu+l^2(2\Omega(x)-1)^2&=l^2\\\Omega(x)^2x^2+l^2(\cancel{1}-4\Omega(x)+4\Omega(x)^2)&=\cancel{l^2}\\\Omega(x)^2x^2-4\Omega(x)l^2+4\Omega(x)^2l^2&=0\\\Omega(x)^2(x^2+4l^2)-4\Omega(x)l^2&=0\\\Omega(x)[\Omega(x)(x^2+4l^2)-4l^2]=0\implies\Omega(x)&=\frac{4l^2}{x^2+4l^2}\end{aligned}$$

This newly found relationship between the points over de Sitter hypersurface and plane at $X^d=l$ is given by:

$$X^\mu=\Omega(x)x^\mu$$

$$X^d = 2\Omega(x)l - l = l\left(2\frac{1}{1+\frac{x^2}{4l^2}} - 1\right) = l\left(\frac{1-\frac{x^2}{4l^2}}{1+\frac{x^2}{4l^2}}\right)$$

Using this we define the stereographic projection by the following conformal transformation:[5]

$$X^\mu \equiv r^\mu = \Omega(x)x^\mu$$
$$X^4 \equiv r^4 = l\Omega(x)\left(1 - \frac{x^2}{4l^2}\right)$$
$$\Omega(x) = \frac{1}{1+\frac{x^2}{4l^2}}$$

Here, $r^a$ is the cartesian coordinate on de Sitter space and $x^a$ is the coordinate on projected minkowski space. We can note that $r_A r^A = l^2$ is imposed on de Sitter coordinates but no such condition is imposed on $x^2 = \eta_{\mu\nu}x^\mu x^\nu$. The inverse of transformation could be found by the considering that each hyperbolic hypersurface gets mapped onto their respective plane sitting the $X^d = l$:

$$x^\mu = \left(1 + \frac{l - X^4}{1 + X^4}\right)X^\mu = \left(\frac{2}{1+\frac{X^4}{\sqrt{X^\mu X_\mu + (X^4)^2}}}\right)X^\mu = \frac{1}{\Omega(x)}X^\mu$$
$$l = \sqrt{X^\mu X_\mu + (X^4)^2}$$

We will now define the following object:[6]

$$K^\mu_A = \frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\mu} = \frac{\delta^\mu_A}{\Omega} - r_A\frac{\partial}{\partial x_\mu}\left(\frac{1}{\Omega(x)}\right)$$
$$= \left(1 + \frac{x^2}{4l^2}\right)^2\frac{\partial r_A}{\partial x_\mu}$$

whose explicit form can be derived by using:

$$\frac{\partial r_\nu}{\partial x_\mu} = \Omega(x)\delta^\mu_\nu - x_\nu\Omega(x)^2\frac{x^\mu}{2l^2} \qquad \frac{\partial r_4}{\partial x^\mu} = -\Omega(x)^2\frac{x_\mu}{l}$$

This object has certain properties which as we will see, would make it a versatile tool at our disposal. But rather than using this straightforward method, we will derive their explicit form in somewhat lengthy but illuminating way which will reveal their true nature. These $K^\mu_A$ are found to satisfy the conformal killing equation

$$\frac{\partial}{\partial x^\mu}K^A{}_\nu + \frac{\partial}{\partial x^\nu}K^A{}_\mu = \delta^A_\mu\partial_\nu\left(\frac{1}{\Omega}\right) - \delta^A_\nu\partial_\mu\left(\frac{1}{\Omega}\right) - x^A\partial_\mu\partial_\nu\left(\frac{1}{\Omega}\right)$$
$$+ \delta^A_\nu\partial_\mu\left(\frac{1}{\Omega}\right) - \delta^A_\mu\partial_\nu\left(\frac{1}{\Omega}\right) - x^A\partial_\nu\partial_\mu\left(\frac{1}{\Omega}\right)$$
$$= -2x^A\partial_\nu\partial_\mu\left(\frac{1}{\Omega}\right) = 2f(x)g_{\mu\nu}$$

and

$$f(x)g_{\mu\nu} = -x^A\partial_\nu\partial_\mu\left(\frac{1}{\Omega}\right)$$

[5] pg 369 of Group Theoretical Concepts and Methods in Elementary Particle Physics by Feza Gürsey

[6] this insight was provided in A unified construction of Skyrme-type non-linear sigma models via the higher dimensional Landau models

$$f(x)g^{\mu\nu}g_{\mu\nu} = -x^A\partial_\mu\partial^\mu\left(\frac{1}{\Omega}\right) = \partial_\mu K^{A\mu}$$

$$f(x) = \frac{\partial_\mu K^{A\mu}}{d}$$

So, $K^A{}_\mu$ could be interpreted as killing vector as long as $\partial_\nu\partial_\mu\left(\frac{1}{\Omega}\right) \propto g_{\mu\nu}$. There is additional property that $K^A{}_\mu$ satisfies, namely transversality condition:[7]

$$r^A K_A{}^\mu\Big|_{r_A r^A = l^2} = 0 \implies x^\nu K_\nu{}^\mu + l\left(1 - \frac{x^2}{4l^2}\right)K_4{}^\mu = 0$$

Since we can write the general form of conformal killing vectors as following,

$$K^\mu_a = t^\mu_a + \epsilon_a x^\mu + \omega^{\mu\nu}_a x_\nu + \lambda^\mu_a x^2 - 2\lambda^\sigma_a x_\sigma x^\mu$$

Inserting above in transverality condition leads to:

$$x^\nu(t^\mu_\nu + \epsilon_\nu x^\mu + \omega^{\mu\sigma}_\nu x_\sigma + \lambda^\mu_\nu x^2 - 2\lambda^\sigma_\nu x_\sigma x^\mu) + l\left(1 - \frac{x^\nu x_\nu}{4l^2}\right)(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0$$

Or,

$$x^\nu t^\mu_\nu + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + \lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu) + l(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu)$$
$$- \frac{x^\nu x_\nu}{4l}(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0$$

It can be conveniently written as:

$$lt^\mu_4 + (x^\nu t^\mu_\nu + l\epsilon_4 x^\mu + l\omega^{\mu\nu}_4 x_\nu) + \left[-\frac{x^2}{4l}t^\mu_4 + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + l\lambda^\mu_4 x^2 - 2l\lambda^\sigma_4 x_\sigma x^\mu\right]$$
$$+ \left[\lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega^{\mu\nu}_4 x^2 x_\nu\right] + \left[-\frac{x^4}{4l}\lambda^\mu_4 + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda^\sigma_4\right] = 0$$

Let us set coefficient of $x^\nu$ to zero, order by order.

$$t^\mu_4 = 0$$
$$x^\nu t^\mu_\nu + l\epsilon_4 x^\mu + l\omega^{\mu\nu}_4 x_\nu = 0$$
$$-\frac{x^2}{4l}t^\mu_4 + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + l\lambda^\mu_4 x^2 - 2l\lambda^\sigma_4 x_\sigma x^\mu = 0$$
$$\lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega^{\mu\nu}_4 x^2 x_\nu = 0$$
$$-\frac{x^4}{4l}\lambda^\mu_4 + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda^\sigma_4 = 0$$

contracting the last four equations with $x_\mu$:

$$t^\mu_\nu x_\mu x^\nu + l\epsilon_4 x^2 = 0$$
$$\epsilon_\nu x^\nu x^2 + l\lambda^\mu_4 x_\mu x^2 - 2l\lambda^\sigma_4 x_\sigma x^2 = 0$$

[7] following the procedure outlined in section 2 of Gauge Theories on de Sitter space and Killing Vectors

$$\lambda^\mu_\nu x_\mu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 = 0$$
$$x^4 \lambda^\mu_4 = 0$$

Hence,

$$t^\mu_\nu x_\mu x^\nu + l\epsilon_4 x^2 = 0$$
$$\epsilon_\nu x^\nu x^2 = 0$$
$$-\lambda^\sigma_\nu x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 = 0$$

Since $t$ generates translation, we have $t^\mu_\nu = \delta^\mu_\nu$:

$$\epsilon_4 = -\frac{1}{l}$$
$$\lambda_{\mu\nu} = \frac{1}{4l^2} g_{\mu\nu}$$

Substituting them in Killing vector, we get:

$$K^\mu_\nu = \left(1 + \frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l^2}$$
$$= \delta^\mu_\nu + \frac{1}{4l^2}(x^2\delta^\mu_\nu - 2x_\nu x^\mu)$$
$$K^\mu_4 = -\frac{x^\mu}{l}$$

These satisfy few properties:

$$K^\mu_A K^{A\nu} = \left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\left[g^{\alpha\nu} + \frac{1}{4l^2}(x^2\, g^{\alpha\nu} - 2x^\nu x^\alpha)\right] + \frac{x^\mu x^\nu}{l^2}$$
$$= g^{\mu\nu} + \frac{1}{4l^2}(x^2 g^{\mu\nu} - \cancel{2x^\nu x^\mu}) + \frac{1}{4l^2}(x^2 g^{\mu\nu} - \cancel{2x^\nu x^\mu}) + \frac{1}{16l^4}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)(x^2\, g^{\alpha\nu} - 2x^\nu x^\alpha) + \cancel{\frac{x^\mu x^\nu}{l^2}}$$
$$= g^{\mu\nu} + \frac{x^2}{2l^2} g^{\mu\nu} + \frac{1}{16l^4}(x^4 g^{\mu\beta} - 2x^2 x^\mu x^\nu - 2x^2 x^\mu x^\nu + 4x^\alpha x_\alpha x^\mu x^\nu)$$
$$= \left(1 + \frac{x^2}{4l^2}\right)^2 g^{\mu\nu} = \frac{1}{\Omega(x)^2} g^{\mu\nu}$$
$$K^\mu_B K_{C\mu} = \left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\left[g_{\beta\mu} + \frac{1}{4l^2}(x^2\, g_{\beta\mu} - 2x_\mu x_\beta)\right]\delta^\alpha_B\delta^\beta_C + \frac{x^\mu x_\mu}{l^2}\delta^4_B\delta^4_C$$
$$- \frac{x_\mu}{l}\left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\delta^4_B\delta^\alpha_C - \left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\frac{x_\mu}{l}\delta^\alpha_B\delta^4_C$$
$$= \left(1 + \frac{x^2}{4l^2}\right)^2\left(g_{\alpha\beta} - \frac{r_\alpha r_\beta}{l^2}\right)\delta^\alpha_B\delta^\beta_C + \left(1 + \frac{x^2}{4l^2}\right)^2\left(1 - \frac{r_4^2}{l^2}\right)\delta^4_B\delta^4_C - \frac{1}{l}\left[x_\alpha - \frac{x^2 x_\alpha}{4l^2}\right](\delta^4_B\delta^\alpha_C + \delta^\alpha_B\delta^4_C)$$
$$= \left(1 + \frac{x^2}{4l^2}\right)^2\left(g_{BC} - \frac{r_B r_C}{l^2}\right)$$

It is precisely this property which makes it possible to use these killing vector as projection operator[8]

$$A_\mu = \frac{\partial r^B}{\partial x^\mu}\hat{A}_B = \Omega(x)^2 K^B{}_\mu \hat{A}_B$$

[8] hat tensors are on de Sitter and without hat ones are on Minkowski

$$K^{\mu}_{C}A_{\mu} = \Omega(x)^2 K^{B}{}_{\mu} K_{C}{}^{\mu} \hat{A}_B = \left(\delta^B_C - \frac{r^B r_C}{l^2}\right)\hat{A}_B$$
$$\hat{A}_B = K^{\mu}_{B} A_{\mu} \qquad \text{(where we used } r^B \hat{A}_B = 0\text{)}$$

It makes sense because $\frac{\partial r^A}{\partial x^\mu}$ forms a $d \times (d-1)$ matrix and as such does not have $\frac{\partial x^\nu}{\partial r^A}$ as its inverse. However, all hope is not lost, the inverse could be found but the procedure is slightly different and given as following:

$$\frac{\partial r^A}{\partial x^\mu}\frac{\partial r_A}{\partial x_\nu} = \Omega^4(x) K^A_\mu K^\nu_A = \Omega^2(x)\delta^\nu_\mu$$
$$\frac{\partial r^A}{\partial x^\mu}\frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\nu} = \delta^\nu_\mu$$
$$\left(\frac{\partial r^A}{\partial x^\mu}\right)^{-1} = \frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\nu} = K^\nu_A$$

We can define the derivative along de Sitter hyperspace independent of choice of slicing by projecting the derivatives of ambient space and killing off the remaining components of it:

$$\hat{\partial}_A = \frac{\partial}{\partial r^A} - \frac{r_A r_B}{l^2}\frac{\partial}{\partial r_B} \implies r^A \hat{\partial}_A = 0$$
$$\equiv K^\mu_A \frac{\partial}{\partial x^\mu}$$

using $r_A r^A = l^2$ and $\eta_{AB} = (-,+,+\ldots)$

$$\begin{aligned} J_{AB} &= r_A \frac{\partial}{\partial r^B} - r_B \frac{\partial}{\partial r^A} \\ &= r_A\left(\hat{\partial}_B + \frac{r_B r_C}{l^2}\frac{\partial}{\partial r_C}\right) - r_B\left(\hat{\partial}_A + \frac{r_A r_C}{l^2}\frac{\partial}{\partial r_C}\right) \\ &= (r_A\hat{\partial}_B - r_B\hat{\partial}_A) \\ &= (r_A K^\mu_B - r_B K^\mu_A)\partial_\mu \end{aligned}$$

in a more compact form:

$$r^A J_{AB} = l^2 K^\mu_B \partial_\mu$$

Then, for $A, B \neq 4$

$$\begin{aligned} L_{ab} &= \Omega(x)\left[x_a\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_b - \frac{x_a x_b x^\mu}{4l^2} - x_b\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_a + \frac{x_a x_b x^\mu}{4l^2}\right]\partial_\mu \\ &= (x_a\delta^\mu_b - x_b\delta^\mu_a)\partial_\mu \end{aligned}$$

These are spatial rotation on the hypersurface getting mapped to spatial rotation in the projected Minkowski space. Boosts in the ambient space and projected Minkowski plane could be given as

$$K_i = J_{0i}$$

Now we calculate the de Sitter Momenta

$$\begin{aligned} lP_\nu = J_{4\nu} &= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left\{\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l^2}\right\} + \frac{x_\nu x^\mu}{l}\right]\partial_\mu \\ &= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l} + \frac{x^2}{4l^2}\frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{l}\right]\partial_\mu \end{aligned}$$

$$
\begin{aligned}
&= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu + \frac{x^2}{4l^2}\frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{2l}\right]\partial_\mu \\
&= \left[l\left(1-\frac{x^2}{4l^2}\right)\delta^\mu_\nu + \frac{x_\nu x^\mu}{2l}\right]\partial_\mu \\
&= l\partial_\nu + \frac{1}{4l}\left[-x^2\delta^\mu_\nu + 2x_\nu x^\mu\right]\partial_\mu
\end{aligned}
$$

The zeroth component of this momenta (which was dilatation on the hypersurface) becomes our new Hamiltonian on the projected Minkowski plane:

$$H = \frac{J_{40}}{l}$$

The algebra on the projected Minkowski plane is:

$$
\begin{aligned}
[J_{\mu\nu}, J_{\lambda\rho}] &= \eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} + \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\nu\lambda}J_{\mu\rho} \\
[J_{\mu\nu}, P_\lambda] &= \frac{1}{l}[J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu \\
[P_\mu, P_\nu] &= \frac{1}{c^2l^2}[J_{4\mu}, J_{4\nu}] = \frac{1}{c^2l^2}J_{\mu\nu}
\end{aligned}
$$

We are now in a position to answer the question of how the de Sitter generators morph in the flat-space limit. As $l \to \infty$ the de Sitter momenta begin to commute and we recover Lorentz algebra. In this limit, the mixed generators behave as follows:

- $J_{D\mu}$ reduces to the generators of Minkowski translations ($D = 4$),
- $J_{0i}$ reduces to the generators of Minkowski boosts ($D - 1 = 3$).
- $J_{ab}$ stays the generator of spatial rotation ($\frac{(D-1)(D-2)}{2} = 3$).

That counts to $4 + 3 + 3 = 10$ generators and this is how they get mapped onto Minkowski plane. Physically, this corresponds to taking the cosmological constant $\Lambda \to 0$, which flattens the spacetime. This connection becomes evident when we recall the relation between the scalar curvature of de Sitter space, the cosmological constant, and the de Sitter radius:

$$R = \frac{d(d-1)}{l^2} = \frac{2d}{d-2}\Lambda = d(d-1)\left(\frac{H}{c}\right)^2$$

## 5.4 Wigner Innou Contraction

**Method 1**

We will first use the stereographic projection to see how the Newton Hook limit could be achieved. Under Wigner Innou contraction the rotations about these planes will appear as translation and they are generated by Momentum operator.

$$
\begin{aligned}
P_\mu &= \frac{J_{4\mu}}{l} \\
L_{ab} &= J_{ab} = x_a\partial_b - x_b\partial_a \\
-K_a &= L_{a0} = \frac{J_{a0}}{c}
\end{aligned}
$$

where $a = 1, 2, 3$ and $\mu = 0, 1, 2, 3$ with mostly plus metric signature:

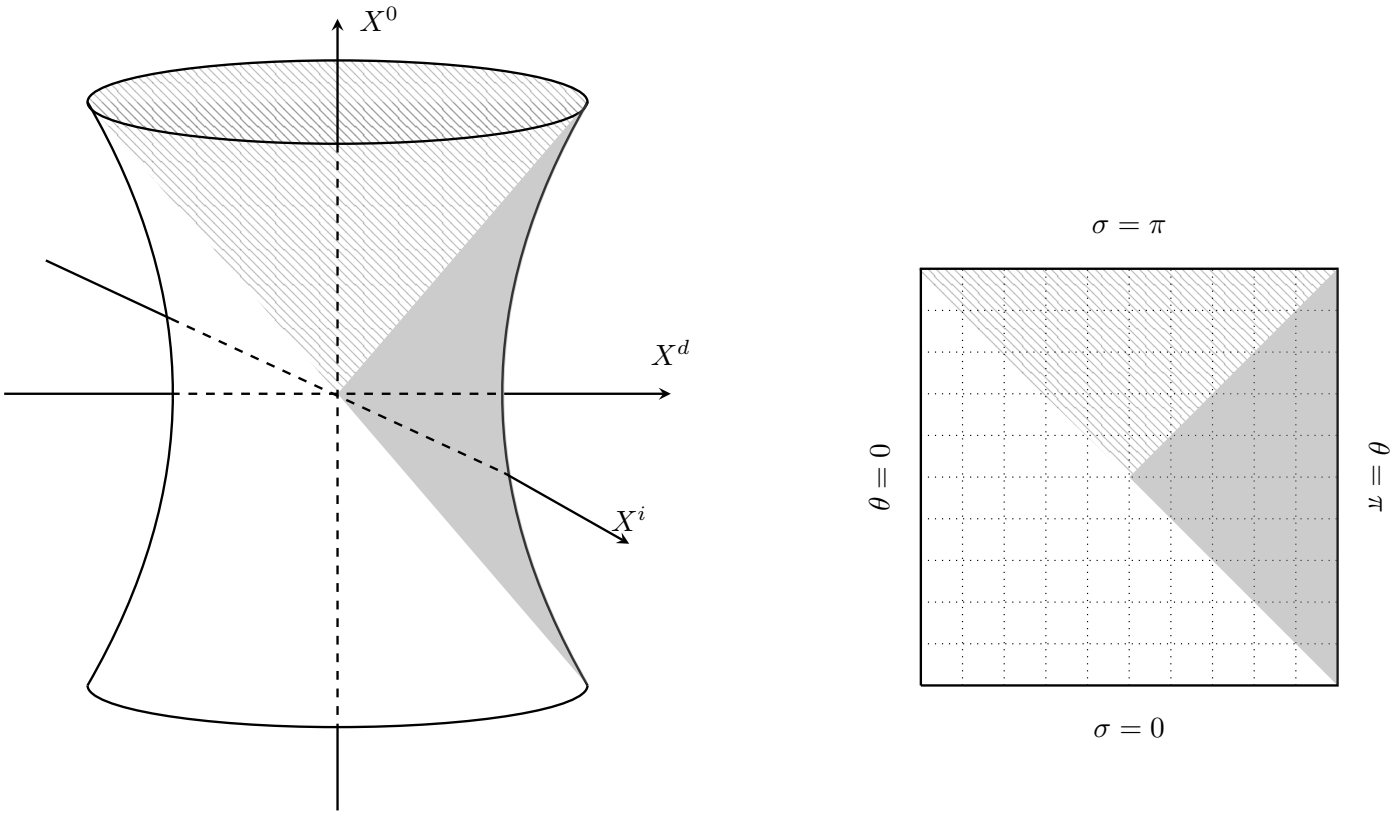



Figure 5.3: To take the $l \to \infty$ limit, imagine placing a plane at $X^d = l$. The plane will be coordinatized by $X^\mu$ and the rotation in $X^\mu X^d$ plane will appear as translation. That's why we split the generators of the ambient space into 6 lorentz generator + 4 translation. Taking the limit basically implies that we are confining ourselves to origin of the plane near the $X^d = l$ plane.

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} - \eta_{AD}J_{BC} + \eta_{BD}J_{AC} - \eta_{BC}J_{AD}$$

which leads to

$$\begin{aligned}
[J_{\mu\nu}, J_{\lambda\rho}] &= \eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} + \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\nu\lambda}J_{\mu\rho} \\
[J_{\mu\nu}, P_\lambda] &= \frac{1}{l}[J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu \\
[P_\mu, P_\nu] &= \frac{1}{c^2l^2}[J_{4\mu}, J_{4\nu}] = \frac{1}{c^2l^2}J_{\mu\nu}
\end{aligned}$$

Then, the algebra becomes:[9]

$$\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{ad}L_{be} + \delta_{be}L_{ad} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\
[L_{0b}, L_{0e}] &= \frac{1}{c^2}[J_{0b}, J_{0e}] = \eta_{00}\frac{1}{c^2}J_{be}, \\
[L_{ab}, P_d] &= \frac{1}{l}[J_{ab}, J_{4d}] = \delta_{bd}\frac{J_{a4}}{l} - \delta_{ad}\frac{J_{b4}}{l} = \delta_{ad}P_b - \delta_{bd}P_a \\
[L_{a0}, P_b] &= \frac{1}{cl}[J_{a0}, J_{4b}] = -\frac{1}{cl}\delta_{ab}J_{04} = \frac{1}{c}\delta_{ab}P_0,
\end{aligned}$$

[9]This form of the analysis was suggested in section 2.4.2 of de Sitter Relativity: Foundation and some physical implications

$$
\begin{aligned}
[L_{a0}, P_0] &= \frac{1}{cl}[J_{a0}, J_{40}] = \eta_{00}\frac{1}{cl}J_{a4} = \frac{1}{c}P_a, \\
[L_{ab}, P_0] &= \frac{1}{l}[J_{ab}, J_{40}] = 0, \\
[P_a, P_b] &= \frac{1}{l^2}[J_{4a}, J_{4b}] = \frac{\eta_{44}}{l^2}J_{ab} = \frac{1}{l^2}L_{ab}, \\
[P_a, P_0] &= \frac{1}{l^2}[J_{4a}, J_{40}] = \frac{\eta_{44}}{l^2}L_{a0} = -\frac{c}{l^2}K_a \\
[P_0, P_0] &= \frac{1}{l^2}[J_{40}, J_{40}] = 0.
\end{aligned}
$$

We find that unless we scale $P_0$ by $c$. The resulting algebra will be ill defined as $[L_{a0}, P_0] \to 0$ and it will not match with Galilean group where $[L_{a0}, P_0] = P_a$.

$$
\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be}L_{ad} - \delta_{ad}L_{be} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\
[L_{0b}, L_{0e}] &= -\frac{1}{c^2}L_{be}, \\
[L_{ab}, P_d] &= \delta_{ad}P_b - \delta_{bd}P_a \\
[L_{a0}, P_b] &= \frac{1}{c^2}\delta_{ab}P_0, \\
[L_{a0}, P_0] &= P_a, \\
[L_{ab}, P_0] &= 0, \\
[P_a, P_b] &= \frac{1}{\tau^2 c^2}L_{ab}, \\
[P_a, P_0] &= \frac{1}{\tau^2}L_{a0}, \\
[P_0, P_0] &= 0.
\end{aligned}
$$

where $\tau = \frac{l}{c}$ is kept constant during the process of taking the limit $l \to \infty$ and $c \to \infty$:

$$
\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be}L_{ad} - \delta_{ad}L_{be} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\
[L_{0b}, L_{0e}] &= 0 \\
[L_{ab}, P_d] &= \delta_{ad}P_b - \delta_{bd}P_a \\
[L_{a0}, P_b] &= 0, \\
[L_{a0}, P_0] &= P_a, \\
[L_{ab}, P_0] &= 0, \\
[P_a, P_b] &= 0 \\
[P_a, P_0] &= -\frac{1}{\tau^2}K_a, \\
[P_0, P_0] &= 0.
\end{aligned}
$$

which is same as

$$
\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk}J_k, && [J_i, P_j] = \epsilon_{ijk}P_k, && [J_i, K_j] = \epsilon_{ijk}K_k, \\
&[P_i, P_j] = 0, && [K_i, K_j] = 0, && [K_i, P_j] = 0, \\
&[H, J_i] = 0, && [H, P_i] = \frac{c^2\Lambda}{3}K_i, && [H, K_i] = P_i,
\end{aligned}
$$

for

$$J_i = \frac{1}{2}\epsilon_{ijk}J_{jk} \qquad\qquad P_0 = H$$

**Method 2**

Alternatively, we can derive the contraction by considering the Lie algebra of the Killing vector fields in the ambient space in the form

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk}J_k, && [J_i, P_j] = \epsilon_{ijk}P_k, && [J_i, K_j] = \epsilon_{ijk}K_k,\\
&[P_i, P_j] = \epsilon_{ijk}J_k, && [K_i, K_j] = -\epsilon_{ijk}J_k, && [K_i, P_j] = -\delta_{ij}H,\\
&[H, J_i] = 0, && [H, P_i] = K_i, && [H, K_i] = P_i,
\end{aligned}$$

Under the Wigner-Innou contraction to be considered here, we only have to examine Lie brackets which doesn't involve $J_i$. Therefore, we only have to worry about the following 5 commutators:

$$\begin{aligned}
&[P_i, P_j] = \epsilon_{ijk}J_k, && [K_i, K_j] = -\epsilon_{ijk}J_k, && [K_i, P_j] = -\delta_{ij}H,\\
&[H, P_i] = K_i, && [H, K_i] = P_i,
\end{aligned}$$

We will consider the following rescaling as suggested in section 2 of covariant formulation of newton-hooke particle and its canonical analysis[10]. The ambient space operators are scaled as following:

$$\tilde{P}_i = \frac{P_i}{cl} = \frac{J_{4i}}{cl} \qquad \tilde{P}_0 \equiv \tilde{H} = \frac{H}{l} = \frac{J_{40}}{l} \qquad \tilde{K}_i = \frac{K_i}{c} = \frac{J_{0i}}{c}$$

With this rescaling, the algebra becomes:

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk}J_k, && [J_i, \tilde{P}_j] = \epsilon_{ijk}\tilde{P}_k, && [J_i, \tilde{K}_j] = \epsilon_{ijk}\tilde{K}_k,\\
&[\tilde{P}_i, \tilde{P}_j] = \frac{1}{c^2l^2}\epsilon_{ijk}J_k && [\tilde{K}_i, \tilde{K}_j] = -\frac{1}{c^2}\epsilon_{ijk}J_k && [\tilde{K}_i, \tilde{P}_j] = -\frac{1}{c^2}\delta_{ij}\tilde{H}\\
&[\tilde{H}, J_i] = 0, && [\tilde{H}, \tilde{P}_i] = \frac{1}{l^2}K_i, && [\tilde{H}, \tilde{K}_i] = \tilde{P}_i,
\end{aligned}$$

Now define the parameter $l = \frac{c^2\Lambda}{3}$ so that in the limit $c \to \infty$ and $\Lambda \to 0$ with $c^2\Lambda$ fixed, the commutators contract to

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk}J_k, && [J_i, P_j] = \epsilon_{ijk}P_k, && [J_i, K_j] = \epsilon_{ijk}K_k,\\
&[P_i, P_j] = 0, && [K_i, K_j] = 0, && [K_i, P_j] = 0,\\
&[H, J_i] = 0, && [H, P_i] = \frac{c^2\Lambda}{3}K_i, && [H, K_i] = P_i,
\end{aligned}$$

It is often convenient to consider the central extension of Newton Hooke' group by redefining $\tilde{H} = \frac{H}{l} - mc^2$. The only commutator this modifies is

$$[\tilde{K}_i, \tilde{P}_j] = \lim_{c\to\infty}\frac{1}{c^2l}[K_i, P_j] = -\lim_{c\to\infty}\frac{1}{c^2}\delta_{ij}\frac{H}{l} = -\lim_{c\to\infty}\left(m + \frac{\tilde{H}}{c^2}\right)\delta_{ij} = -m\delta_{ij}$$

where $m$ appearing as central charge is the mass of the particle.

[10] Boosts are scaled down by $c$ for Galilean contraction.

**Method 3**

We embed $d$-dimensional spacetime into a $(d+1)$-dimensional space with coordinates $X^A = (X^0, X^i, X^D)$, $i = 1, \ldots, d-1$, and define the Lorentz generators

$$J_{AB} = X_A \partial_B - X_B \partial_A.$$

**Translation generators**

Introduce a large parameter $l$ and set $X^D = l$, $X^\mu = x^\mu$. Then

$$J_{D\mu} = X_D \partial_\mu - X_\mu \partial_D = l\partial_\mu - x_\mu \partial_D.$$

Divide by $l$ and take $l \to \infty$:

$$\frac{1}{l} J_{D\mu} = \partial_\mu - \frac{x_\mu \partial_D}{l} \xrightarrow{l\to\infty} \partial_\mu.$$

Thus the generators corresponding to translations after contraction are

$$P_\mu = \partial_\mu.$$

**Spatial rotations**

The spatial rotations remain unchanged:

$$J_{ij} = x_i \partial_j - x_j \partial_i.$$

**Galilean boosts**

Set $X^0 = ct$, $X^i = x^i$, then

$$J_{0i} = X_0 \partial_i - X_i \partial_0 = ct\partial_i - x_i \frac{1}{c} \partial_t.$$

Divide by $c$ and take $c \to \infty$:

$$B_i = \lim_{c\to\infty} \frac{1}{c} J_{0i} = t\partial_i.$$

**Dilatation**

The dilatation generator in embedding space is

$$D = J_{D0} = X_D \partial_0 - X_0 \partial_D = l\partial_0 - ct\partial_D.$$

Divide by $l$ and take $l \to \infty$:

$$D \xrightarrow{l\to\infty} \partial_0.$$

Note that the contraction here took place in the ambient space, not on the hypersurface, so we did not concern ourselves with

$$\begin{aligned} P_i &= J_{Di} - J_{0i} \\ &= \frac{1}{l} \left[X_D \partial_i - X_i \partial_D\right] - \frac{1}{c} \left[X_0 \partial_i - X_i \partial_0\right] \\ &= \frac{1}{l} \left[l\partial_i - x_i \partial_D\right] - \frac{1}{c} \left[ct\partial_i + x_i \frac{1}{c} \partial_t\right] \\ &= (1-t)\partial_i \end{aligned}$$

and

$$K_i = J_{Di} + J_{0i}$$

$$
\begin{aligned}
&= \frac{1}{l}\left[X_D\partial_i - X_i\partial_D\right] + \frac{1}{c}\left[X_0\partial_i - X_i\partial_0\right] \\
&= \frac{1}{l}\left[l\partial_i - x_i\partial_D\right] + \frac{1}{c}\left[ct\partial_i + x_i\frac{1}{c}\partial_t\right] \\
&= (1+t)\partial_i
\end{aligned}
$$

because these combination do not survive in the traditional sense. Now, to find the algebra, we compute the commutators of the above representations with each other, since the algebra is independent of the choice of representation and the point in spacetime at which it is evaluated.

## 5.5 Action

We are given the action of a relativistic point particle in a de Sitter background,

$$S = -mc\int d\tau\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

In the static patch, the metric components take simpler familiar form:

$$g_{00} = -f(r), \quad g_{rr} = \frac{1}{f(r)}, \quad f(r) = 1 - \Lambda r^2, \quad r^2 = x^i x^i$$

Then the action reduces to:

$$S = -mc\int d\tau\sqrt{f(r)\dot{t}\dot{t} - \frac{1}{f(r)}\dot{r}^2}$$

$$S = \frac{m}{2}\int d\tau\left(\frac{\dot{x}^i\dot{x}^i}{\dot{t}} + \frac{\epsilon x^i x^i\dot{t}}{R^2}\right) - mc^2\int d\tau\dot{t}$$

# Chapter 6

# Ward Identities in de Sitter space

The infinitesimal conformal transformation which respects the de Sitter isometry will look like

$$\begin{aligned} \text{dilation:} \quad & \eta \to \eta(1+\lambda), && \mathbf{x} \to \mathbf{x}(1+\lambda), \\ \text{SCT:} \quad & \eta \to \eta(1-2\mathbf{b}\cdot\mathbf{x}), && \mathbf{x} \to \mathbf{x} - 2(\mathbf{b}\cdot\mathbf{x})\mathbf{x} + (\mathbf{x}^2-\eta^2)\mathbf{b}, \end{aligned}$$

We consider the following quantity:

$$\begin{aligned} & e^{i\lambda D}\langle\ldots\rangle = \langle\ldots\rangle \implies D\langle\ldots\rangle = 0 \\ & e^{i\vec{b}\cdot\vec{K}}\langle\ldots\rangle = \langle\ldots\rangle \implies \vec{b}\cdot\vec{K}\langle\ldots\rangle \equiv \mathbf{b}\cdot\mathbf{K}\ \langle\ldots\rangle\ = 0 \end{aligned}$$

↑ function, not an operator

Since, at late time ($\eta \to 0$) we will be decomposing our fields like:[1]

$$\Phi = \sum_{\{\Delta\}} \eta^{\Delta} O_{\Delta}(\vec{x})$$

where $\Delta$ is the scaling dimension of $O_\Delta$. We deduce how the generators act on the boundary operator $O_\Delta$ by using the fact that scaling dimension of $\Phi$ is zero in de sitter space. Under $x \to x' \equiv \lambda x$, $\partial'_\mu = \lambda^{-1}\partial_\mu$ and $\phi'(x') = \lambda^{-\Delta}\phi(x)$:[2]

$$\begin{aligned} S &= \int d^4x' \frac{(\partial'_0\phi')^2 - (\nabla'\phi')^2}{(H\eta')^2} \\ &= \lambda^4 \int d^4x \frac{\lambda^{-2(1+\Delta)}[(\partial_0\phi)^2 - (\nabla\phi)^2]}{\lambda^2(H\eta)^2} \implies \Delta = 0 \end{aligned}$$

If we don't wish to use the weyl scaling, just the coordinate transformation then:

$$\begin{aligned} S &= \int d^4x' \sqrt{g'} g'^{\mu\nu}\partial'_\mu\phi\partial'_\nu\phi = \int d^4x' \frac{(\partial'_0\phi')^2 - (\nabla'\phi')^2}{(H\eta')^2} \\ &= \int d^4x \sqrt{g} g'^{\mu\nu}\partial'_\mu\phi\partial'_\nu\phi = \int d^4x \lambda^2 \frac{\lambda^{-2(1+\Delta)}[(\partial_0\phi)^2 - (\nabla\phi)^2]}{(H\eta)^2} \implies \Delta = 0 \end{aligned}$$

Let us now come back to dilatation and study how it acts on $O$:

$$D\Phi = -x\cdot\partial\Phi$$

[1] $\eta = \lim_{t\to\infty} -\frac{e^{-Ht}}{H} = 0$

[2] note that we used the weyl scaling to remove the $\lambda^2$ factor from metric under change of coordinate. So this basically amounts to substituting the coordinate transformation in the metric.

$$
\begin{aligned}
&= -x\cdot\partial\sum_{\{\Delta\}}\eta^{\Delta}O_{\Delta}\\
&= \sum_{\{\Delta\}}(-\eta\partial_{\eta}-\vec{x}\partial_{\vec{x}})\eta^{\Delta}O_{\Delta}\\
&= \sum_{\{\Delta\}}\eta^{\Delta}\underbrace{(-\Delta-\vec{x}\partial_{\vec{x}})}_{D}O_{\Delta}\\
&= D\sum_{\{\Delta\}}\eta^{\Delta}O_{\Delta}\\
b\cdot K\Phi &= b^{\mu}[-2x_{\mu}\eta\partial_{\eta}-2x_{\mu}\vec{x}\cdot\partial_{\vec{x}}-(\eta^2-|\vec{x}|^2)\partial_{\mu}]\Phi\\
&= [-2(\vec{b}\cdot\vec{x})\eta\partial_{\eta}-2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}}-(\eta^2-|\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]\Phi\\
&= \sum_{\{\Delta\}}\underbrace{[-2(\vec{b}\cdot\vec{x})\Delta-2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}}-(\eta^2-|\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]}_{b\cdot K}\eta^{\Delta}O_{\Delta}
\end{aligned}
$$

in the limit $\eta\to 0$

$$= [-2(\vec{b}\cdot\vec{x})\Delta-2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}}+|\vec{x}|^2\vec{b}\cdot\partial_{\vec{x}}]\Phi$$

In momentum space, the above operators take the following form:

$$
\begin{aligned}
D &: (3-\eta\partial_{\eta})+k^i\partial_{k_i},\\
\mathbf{b}\cdot\mathbf{K} &: (3-\eta\partial_{\eta})\,2b^i\partial_{k_i}-\mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i}+2k^i\partial_{k_i}b^j\partial_{k_j}.
\end{aligned}
$$

or, simply

$$
\begin{aligned}
D &: (3-\Delta)+k^i\partial_{k_i},\\
\mathbf{b}\cdot\mathbf{K} &: (3-\Delta)\,2b^i\partial_{k_i}-\mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i}+2k^i\partial_{k_i}b^j\partial_{k_j}.
\end{aligned}
$$

There are two ways to derive this expression, we will discuss both of them. The first is based on using Fourier Transform:

$$f(\vec{x})=\int d^3x e^{i\vec{x}\cdot\vec{k}}f(\vec{k})$$

In our case

$$
\begin{aligned}
D\underbrace{\int d^3k e^{i\vec{x}\cdot\vec{k}}O_{\Delta}(\vec{k})}_{O_{\Delta}(\vec{x})} &= -\left(\Delta+x^j\frac{\partial}{\partial x^j}\right)\int d^3k e^{i\vec{x}\cdot\vec{k}}O_{\Delta}(\vec{k})\\
&= \int d^3k\left[-\Delta-x^j\frac{\partial}{\partial x^j}\right]e^{i\vec{x}\cdot\vec{k}}O_{\Delta}(\vec{k})\\
&= \int d^3k[-\Delta e^{i\vec{x}\cdot\vec{k}}-x^j e^{i\vec{x}\cdot\vec{k}}(ik_j)]O_{\Delta}(\vec{k})
\end{aligned}
$$

we have to get rid of $x^j$, so we consider:

$$
\begin{aligned}
&= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}}-\left(-i\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right)ik_j\right]O_{\Delta}(\vec{k})\\
&= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}}-\left(\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right)k_j\right]O_{\Delta}(\vec{k})
\end{aligned}
$$

integrating by parts the second term

$$= \int d^3k e^{i\vec{x}\cdot\vec{k}} \left[ -\Delta + \left( \frac{\partial}{\partial k_j} \right) k_j \right] O_\Delta(\vec{p})$$
$$= \int d^3k e^{i\vec{x}\cdot\vec{k}} \left( -\Delta + \cancel{\frac{\partial k_j}{\partial k_j}}^{3} + k_j \frac{\partial}{\partial k_j} \right) O_\Delta(k)).$$

Thus the action of the dilatation generator in momentum space is

$$DO_\Delta(\vec{k}) = \left( 3 - \Delta + k^j \frac{\partial}{\partial k^j} \right) O_\Delta(\vec{k})$$

The second way is to replace the following in the expression in coordinate space (it can be seen operating the corresponding derivative operator on $e^{ix^\mu k_\mu}$ and the corresponding integration by parts).

$$x_\mu \to -i \frac{\partial}{\partial k^\mu}$$
$$\frac{\partial}{\partial x_\mu} \to -ik^\mu$$

Substituting in

$$DO_\Delta(\vec{x}) = -\left( \Delta + x^j \frac{\partial}{\partial x^j} \right) O_\Delta(\vec{x})$$

we get

$$DO_\Delta(\vec{k}) = -\left[ \Delta - i \frac{\partial}{\partial k^j} \left( -ik^j \right) \right] O_\Delta(\vec{k}) = -\left( \Delta - 3 - k^j \frac{\partial}{\partial k^j} \right) O_\Delta(\vec{k})$$

## 6.1 Conformal Ward Identity in momentum space

The following is taken from "Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex". We start by assuming that the generator of conformal transformation annihilates the correlation function.

$$D \langle O_1(\vec{x}_1) O_2(\vec{x}_2) \dots O_n(\vec{x}_n) \rangle = 0$$
$$e^{ix_n \cdot P} D \langle O_1(\vec{x}_1) O_2(\vec{x}_2) \dots O_n(\vec{x}_n) \rangle e^{-ix_n \cdot P} = 0$$
$$\underbrace{e^{ix_n \cdot P} D e^{-ix_n \cdot P}}_{D + \vec{x}_n \cdot \partial_{\vec{x}_n}} \underbrace{e^{ix_n \cdot P} \langle O_1(\vec{x}_1) O_2(\vec{x}_2) \dots O_n(\vec{x}_n) \rangle e^{-ix_n \cdot P}}_{\langle O_1(\vec{x}_1 - \vec{x}_n) O_2(\vec{x}_2 - \vec{x}_n) \dots O_n(0) \rangle} = 0$$

now, in Fourier space:

$$\langle O_1(\vec{x}_1 - \vec{x}_n) O_2(\vec{x}_2 - \vec{x}_n) \dots O_n(0) \rangle = \int d^d k_1 \dots d^d k_n e^{\sum_{j=1}^{n-1} ik_j \cdot (x_j - x_n) + 0}$$
$$\times \delta^d \left( \sum_j^n k_j \right) \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle$$
$$= \int d^d k_1 \dots d^d k_n e^{\sum_{j=1}^{n-1} ik_j \cdot x_j - ix_n \cdot (\sum_{j=1}^{n-1} k_j)}$$
$$\times \delta^d \left( k_n + \sum_j^{n-1} k_j \right) \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle$$

$$\equiv \int d^dk_1 \dots d^dk_n e^{\sum_{j=1}^n ik_j\cdot x_j}$$
$$\times \delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle$$

From first and third equality, we observe that the replacement we need to make are:

$$(x_j - x_n)_\mu \to -i\frac{\partial}{\partial k_j^\mu}$$
$$\frac{\partial}{\partial x_\mu} \to -ik^\mu$$

Thus,

$$\underbrace{(D + \vec{x}_n\cdot\partial_{\vec{x}_n})}_{-(\sum_{j=1}^n \Delta_j + x_j\cdot\partial_{x_j}) + x_n\partial_{x_n}}\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle = 0$$
$$-\left[\sum_j^n \Delta_j + \sum_j^{n-1}(x_j-x_n)\cdot\partial_{x_j}\right]\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle = 0$$
$$-\underbrace{\left[\sum_j^n \Delta_j + \sum_j^{n-1}\left(-i\frac{\partial}{\partial k_j^\mu}\right)\cdot(-ik_j^\mu)\right]}_{-[\Delta-(n-1)d-\sum_{j=1}^{n-1}k_j\cdot\partial_{k_j}]}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\delta^d\left(\sum_j^n k_j\right) = 0$$

The alternate way to do the same is by explicitly doing it. We then will have to use

$$\int dx f(x)\partial_x\delta(x-a) = -\int dx\partial_x f(x)\delta(x-a) = -\partial_x f(a)$$

Consider the following integral:

$$I_{\alpha\beta} = \int d^dk\left[\frac{\partial}{\partial k^\alpha}\delta^d(k^\mu)\right]k_\beta$$
$$= -\int d^dk\delta^d(k^\mu)\underbrace{\frac{\partial k_\beta}{\partial k^\alpha}}_{g_{\alpha\beta}} = -g_{\alpha\beta} \tag{6.1.1}$$

However, we also know that:

$$\int d^dk\delta^d(k^\mu) = 1$$
$$\int d^dk\delta^d(k^\mu)\frac{k^2}{k^2} = 1$$
$$\int d^dk\delta^d(k^\mu)\frac{g^{\alpha\beta}k_\alpha k_\beta}{k^2} = 1$$
$$\int d^dk\frac{\delta^d(k^\mu)}{k^2}k_\alpha k_\beta = \frac{1}{d}g_{\alpha\beta} \tag{6.1.2}$$

From Schur's lemma, the only object invariant under all the isometry of minkowski is metric. So the integral $I_{\alpha\beta}$ should eitherway be proportional to $g_{\alpha\beta}$. From (6.1.1) and (6.1.2), we get

$$\frac{\partial}{\partial k^\alpha}\delta^d(k^\mu) = -\frac{d}{k^2}k_\alpha\delta^d(k^\mu)$$

We can only use this identity if the function paired with $\delta$ is a constant, this is not pointwise identity. Notice that

$$\int d^dP \partial_\alpha \delta^d f(P) = -\partial_\alpha f(0)$$
$$-d\int d^dP P^\alpha \frac{\delta^d(P)}{P^2} f(P) = 0$$

However since the integration measure paired with fourier transform does not contain the variable inside the delta function. We utilise the identity to derive:

$$k^\alpha \frac{\partial}{\partial k^\alpha}\delta^3(\vec{k}) = -\frac{3}{k^2}k^\alpha k_\alpha \delta^3(\vec{k}) = -3\delta^3(\vec{k})$$

Alternatively, we can derive it using the integral form:

$$\int_{-\infty}^{\infty} d^dP\delta^d(P) = 1 = \int_{-\infty}^{\infty} d^d[\lambda P]\delta^d(\lambda P) = \lambda^d \int_{-\infty}^{\infty} d^dP\delta^d(\lambda P) \implies \delta^d(\lambda P) = \lambda^{-d}\delta^d(P)$$

From this, we get

$$\lambda\frac{\partial}{\partial\lambda}\delta^d(\lambda P) = \lambda\frac{\partial\lambda P^\alpha}{\partial\lambda}\frac{\partial}{\partial\lambda P^\alpha}\delta^d(\lambda P) = \lambda P^\alpha\frac{\partial}{\partial\lambda P^\alpha}\delta^d(\lambda P)$$

$$\begin{aligned}
P^\alpha\frac{\partial}{\partial P^\alpha}\delta^d(\lambda P) &= P^\alpha\frac{\partial\lambda P^\beta}{\partial P^\alpha}\frac{\partial}{\partial\lambda P^\beta}\delta^d(\lambda P) = \lambda P^\alpha\frac{\partial}{\partial\lambda P^\alpha}\delta^d(\lambda P) = \lambda\frac{\partial}{\partial\lambda}\delta^d(\lambda P)\\
&= -d\lambda\lambda^{-d-1}\delta^d(P) = -d\lambda^{-d}\delta^d(P) = -d\delta^d(\lambda P)\\
P^\alpha\frac{\partial}{\partial P^\alpha}\delta^d(\vec{P}) &= P^\alpha\frac{\partial P}{\partial P^\alpha}\frac{\partial}{\partial P}\delta^d(\vec{P}) = \frac{P^\alpha 2P_\alpha}{2\sqrt{P^\alpha P_\alpha}}\frac{\partial}{\partial P}\delta^d(\vec{P}) = P\frac{\partial}{\partial P}\delta^d(\vec{P}) = -d\delta^d(\vec{P})
\end{aligned}$$

Then, we have:

$$\begin{aligned}
&-\sum_{j=1}^{n}\left(\Delta_j + x_j^\alpha\frac{\partial}{\partial x_j^\alpha}\right)\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle\\
&\qquad= -\sum_{j=1}^{n}\left(\Delta_j + x_j^\alpha\frac{\partial}{\partial x_j^\alpha}\right)\int\prod_{l=1}^{n}d^dk_l\delta^d(\textstyle\sum_{i=1}^n\vec{k}_i)e^{ik_lx_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\\
&\qquad= -\int\prod_{l=1}^{n}d^dk_l\delta^d(\textstyle\sum_{i=1}^n\vec{k}_i)e^{ik_lx_l}\left(\sum_{j=1}^{n}\Delta_j - \underbrace{\sum_{j=1}^{n}d}_{nd} - \sum_{j=1}^{n}k_j^\alpha\frac{\partial}{\partial k_j^\alpha}\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle + \delta'_{\text{term}}
\end{aligned}$$

where

$$\delta'_{\text{term}} = \sum_{j=1}^{n}\int\prod_{l=1}^{n}d^dk_l\left[k_j^\alpha\frac{\partial}{\partial k_j^\alpha}\delta^d(\textstyle\sum_{i=1}^n\vec{k}_i)\right]e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle$$

defining $P = \sum_{i=1}^n\vec{k}_i$

$$\begin{aligned}
&= \sum_{j=1}^{n}\int\prod_{l=1}^{n}d^dk_l\left[k_j^\alpha\frac{\partial}{\partial P^\alpha}\delta^d(P)\right]e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\\
&= \int\prod_{l=1}^{n}d^dk_l\left[P^\alpha\frac{\partial}{\partial P^\alpha}\delta^d(P)\right]e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle
\end{aligned}$$

$$= -d \int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{ik_l x_l} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle$$

Thus, finally

$$-\sum_{j=1}^{n} \left( x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + \Delta_j \right) \langle O_1(\vec{x}_1) O_2(\vec{x}_2) \dots O_n(\vec{x}_n) \rangle = -\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n k_i) e^{ik_l x_l}$$
$$\left( \sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n} k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \right) \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle \quad (6.1.3)$$

To find the ward identity we one less momenta, let us perform the following transformation $(k_1, k_2 \dots k_n) = (\tilde{k}_1, \tilde{k}_2, \dots P)$, with $\tilde{k}$ and $P$ are treated as independent variables[3]

$$\left.\begin{aligned} k_i^\mu &= \tilde{k}_i^\mu, \\ k_n^\mu &= P_k^\mu - \sum_{i=1}^{n-1} k_i^\mu = P_k^\mu - q^\mu \end{aligned}\right\} \quad \left.\begin{aligned} \tilde{k}_i^\mu &= k_i^\mu, \\ P_k^\mu &= \sum_{i=1}^{n-1} k_i^\mu + k_n^\mu \end{aligned}\right.$$

The derivative term transforms as:

$$\frac{\partial}{\partial k_i^\mu} = \frac{\partial \tilde{k}^\alpha}{\partial k_i^\mu} \frac{\partial}{\partial \tilde{k}_i^\alpha} + \frac{\partial P_k^\alpha}{\partial k_i^\mu} \frac{\partial}{\partial P_k^\alpha} = \frac{\partial}{\partial \tilde{k}_i^\mu} + \frac{\partial}{\partial P_k^\alpha}$$
$$\frac{\partial}{\partial k_n^\mu} = \frac{\partial \tilde{k}^\alpha}{\partial k_n^\mu} \frac{\partial}{\partial \tilde{k}^\alpha} + \frac{\partial P_k^\alpha}{\partial k_n^\mu} \frac{\partial}{\partial P_k^\alpha} = \frac{\partial}{\partial P_k^\mu}$$

We can also see that

$$\begin{aligned} \sum_{i=1}^{n} k_i^\alpha \frac{\partial}{\partial k_i^\alpha} &= \sum_{i=1}^{n-1} \tilde{k}_i^\alpha \frac{\partial}{\partial k_i^\alpha} + (P-q)^\alpha \frac{\partial}{\partial P_k^\alpha} \\ &= \sum_{i=1}^{n-1} \tilde{k}_i^\alpha \frac{\partial}{\partial \tilde{k}_i^\alpha} + \sum_{i=1}^{n-1} \tilde{k}_i^\alpha \frac{\partial}{\partial P_k^\alpha} + (P-q)^\alpha \frac{\partial}{\partial P_k^\alpha} \\ &= \sum_{i=1}^{n-1} \tilde{k}_i^\alpha \frac{\partial}{\partial \tilde{k}_i^\alpha} + P^\alpha \frac{\partial}{\partial P_k^\alpha} \end{aligned}$$

We only need to make following replacement in the derivation:

$$\sum_{j=1}^{n} k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle = \sum_{j=1}^{n-1} \tilde{k}_j^\alpha \frac{\partial}{\partial \tilde{k}_j^\alpha} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle + P^\alpha \frac{\partial}{\partial P^\alpha} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle$$
$$\delta'_{\text{term}} = \sum_{j=1}^{n} \left[ k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) \right] e^{ik_l \cdot x_l} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle$$

[3]There are two ways to deal with this kind of issue. We either eliminate one of the variable using the constraint i.e. we set $k_n = -\sum_{i=1}^{n-1} k_i$, hence we treat the rest of them as independent and $k_n$ as fixed. This allows us to write $\frac{\partial}{\partial k_n}$ in terms of remaining derivs and we make the replacement in every equation.

$$\frac{\partial F(k_1, k_2, k_3)}{\partial k_1} = \frac{\partial F(k_1, k_2, -k_1 - k_2)}{\partial k_1} = \frac{\partial F}{\partial k_1} + \frac{\partial(-k_1 - k_2)}{\partial k_1} \frac{\partial F}{\partial(-k_1 - k_2)} = \frac{\partial F(k_1, k_2, k_3)}{\partial k_1} - \frac{\partial F(k_1, k_2, k_3)}{\partial k_3}$$

This could be re-interpreted in the alternate approach as constructing a coordinate system adapted to the constraint such that the constraint surface becomes $P = \text{const}$ hypersurface. In this coordinate, every coordinate is independent of each other, therefore we recover the same set of good properties.

$$= \left[\sum_{j=1}^{n-1} k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) + P^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle$$
$$= \left[P^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle$$

The rest of the procedure as same as before. The derivative terms in new coordinate is given in terms of old coordinate as:

$$\frac{\partial}{\partial k_i^\mu} = \frac{k_{i\mu}}{k_i}\frac{\partial}{\partial k_i} - \frac{k_{n\mu}}{k_n}\frac{\partial}{\partial k_n}$$
$$\sum_{i=1}^{n-1} k_i^\alpha \frac{\partial}{\partial k_i^\alpha} = \sum_{i=1}^{n-1} k_i^\alpha\left[\frac{k_{i\alpha}}{k_i}\frac{\partial}{\partial k_i} - \frac{k_{n\alpha}}{k_n}\frac{\partial}{\partial k_n}\right]$$
$$= \sum_{i=1}^{n-1} k_i\frac{\partial}{\partial k_i} - \left(\sum_{i=1}^{n-1} k_i^\alpha\right)\frac{k_{n\alpha}}{k_n}\frac{\partial}{\partial k_n}$$
$$= \sum_{i=1}^{n-1} k_i\frac{\partial}{\partial k_i} + k_n\frac{\partial}{\partial k_n} = \sum_{i=1}^{n} k_i\frac{\partial}{\partial k_i}$$

Before we proceed, there is another identity we'd like to derive which would be very helpful. [4]

$$\partial_\alpha\partial_\beta\delta^d(k^\mu) = \partial_\alpha\left[-\frac{d}{k^2}\delta^d(k^\mu)k_\beta\right]$$
$$= -d\left(\frac{\partial}{\partial k^\alpha}k^{-2}\right)\delta^d(k^\mu)k_\beta - \frac{d}{k^2}[\partial_\alpha\delta^d(k)]k_\beta - \frac{d}{k^2}\delta^d(k^\mu)\partial_\alpha k_\beta$$
$$= \frac{2d}{k^3}\frac{k_\alpha}{k}\delta^d(k^\mu)k_\beta + \frac{d^2}{k^4}\delta^d(k^\mu)k_\alpha k_\beta - \frac{d}{k^2}\delta^d(k^\mu)g_{\alpha\beta}$$

$$= \frac{d(d+2)}{k^4}\delta^d(k^\mu)k_\alpha k_\beta - \frac{d}{k^2}\delta^d(k^\mu)g_{\alpha\beta}$$

We will now discuss the same for SCT. We perform the same substitution

$$-iK = -2x_\mu\Delta - 2x_\mu\underbrace{\vec{x}\cdot\partial_{\vec{x}}}_{-i\frac{\partial}{\partial k^\alpha}(-ik^\alpha)} + |\vec{x}|^2\partial_\mu$$
$$= 2i\Delta\frac{\partial}{\partial k^\mu} - 2i\frac{\partial^2}{\partial k^\mu\partial k^\alpha}k^\alpha + i\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}k_\mu$$
$$= 2i\Delta\frac{\partial}{\partial k^\mu} - \underbrace{2i\frac{\partial}{\partial k^\mu}\left(d + k^\alpha\frac{\partial}{\partial k^\alpha}\right)}_{2i\left(d\frac{\partial}{\partial k^\mu} + \frac{\partial}{\partial k^\mu} + k^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}\right)} + \underbrace{i\frac{\partial}{\partial k^\alpha}\left(\delta_\mu^\alpha + k_\mu\frac{\partial}{\partial k_\alpha}\right)}_{2i\frac{\partial}{\partial k^\mu} + ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}}$$
$$= 2i\Delta\frac{\partial}{\partial k^\mu} - 2id\frac{\partial}{\partial k^\mu} - 2ik^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha} + ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}$$

[4] In the third line, we have used

$$\frac{\partial k}{\partial k^\alpha} = \frac{\partial\sqrt{k^\alpha k_\alpha}}{\partial k^\alpha}$$
$$= \frac{1}{2\sqrt{k^\alpha k_\alpha}}\left[\frac{\partial k^\alpha}{\partial k^\alpha}k_\alpha + k_\alpha\frac{\partial k^\alpha}{\partial k^\alpha}\right] = \frac{k^\alpha}{k}$$

$$= i\left[2(\Delta-d)\frac{\partial}{\partial k^\mu} - 2k^\alpha\frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu\frac{\partial^2}{\partial k^\alpha \partial k_\alpha}\right]$$

Then,

$$\begin{aligned}
K^\mu\delta^d(\textstyle\sum_{i=1}^n p_i^k) &= \sum_{j=1}^n\left[k_j^\mu\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\alpha}} - 2k_j^\alpha\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\mu}} + 2(\Delta_j-d)\frac{\partial}{\partial k_{j\mu}}\right]\delta^d(\overbrace{\textstyle\sum_{i=1}^n k_i^\mu}^{P^\mu=\sum_{i=1}^n k_i^\mu}) \\
&= \sum_{j=1}^n\left[k_j^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha} - 2k_j^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu} + 2(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \left[\left(\sum_{j=1}^n k_j^\mu\right)\frac{\partial^2}{\partial P^\alpha\partial P_\alpha} - 2\left(\sum_{j=1}^n k_j^\alpha\right)\frac{\partial^2}{\partial P^\alpha\partial P_\mu} + 2\sum_{j=1}^n(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \left[P^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha} - 2P^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu} + 2(\Delta-nd)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \frac{2d}{P^2}\delta^d(P)P^\mu - 2\frac{d^2+d}{P^2}\delta^d(P)P^\mu + \frac{2d(\Delta-nd)}{P^2}\delta^d(P)P^\mu \\
&= 2d\left[nd-d-\Delta\right]\frac{\delta^d(P)P^\mu}{P^2} \\
&= -2[(n-1)d-\Delta]\frac{\partial\delta^d(P)}{\partial P^\mu}
\end{aligned}$$

In the fourth line, we used $\sum_{j=1}^n\Delta_j=\Delta$. Alternatively,

$$\begin{aligned}
K^\mu\delta^d(P) &= \sum_{j=1}^n\left[k_j^\mu\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\alpha}} - 2k_j^\alpha\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\mu}} + 2(\Delta_j-d)\frac{\partial}{\partial k_{j\mu}}\right]\delta^d(P) \\
&= \left[P^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha} - 2P^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu} + 2(\Delta-nd)\frac{\partial}{\partial P_\mu}\right]\delta^d(P)
\end{aligned}$$

Using

$$\begin{aligned}
\left[\frac{\partial}{\partial P^\alpha}\frac{\partial}{\partial P_\alpha}, P^\mu\right]f = [\partial^2, P^\mu]f &= \partial^2(P^\mu f) - P^\mu(\partial^2 f) \\
&= \underbrace{(\partial^2 P^\mu)}_{=0}f + 2(\partial^\alpha P^\mu)(\partial_\alpha f) + \cancel{P^\mu\partial^2 f} - \cancel{P^\mu(\partial^2 f)} \\
&= 2(\partial^\alpha P^\mu)(\partial_\alpha f) = 2(\partial_\alpha f)
\end{aligned} \tag{6.1.4}$$

and

$$P^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}\delta^d(P) = \frac{\partial^2[P^\mu\delta^d(\vec{P})]}{\partial P^\alpha\partial P_\alpha} - 2\frac{\partial}{\partial P^\alpha}\delta^d(\vec{P})$$

Assuming there's and integral outside, $P^\mu\delta^d(\vec{P})=0$

$$P^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}\delta^d(P) = -2\frac{\partial}{\partial P^\alpha}\delta^d(\vec{P}) \tag{6.1.5}$$

and

$$P\cdot\frac{\partial}{\partial P}\delta^d(\vec{P}) = -d\delta^d(\vec{P})$$

$$\frac{\partial}{\partial P^\mu}\left(P^\alpha\frac{\partial}{\partial P^\alpha}\delta^d(\vec{P})\right) = -d\frac{\partial}{\partial P^\mu}\delta^d(\vec{P})$$

$$P^\alpha \frac{\partial}{\partial P^\alpha P^\mu}\delta^d(\vec{P}) = -(d+1)\frac{\partial}{\partial P^\mu}\delta^d(\vec{P}) \tag{6.1.6}$$

Hence, using (6.1.5) and (6.1.6)

$$\begin{aligned} K^\mu \delta^d(P) &= \left[P^\mu \frac{\partial^2}{\partial P^\alpha \partial P_\alpha} - 2P^\alpha \frac{\partial^2}{\partial P^\alpha \partial P_\mu} + 2(\Delta - nd)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\ &= \left[-2 + 2(d+1) + 2(\Delta - nd)\right]\frac{\partial}{\partial P_\mu}\delta^d(P) \\ &= 2\left[\Delta - (n-1)d\right]\frac{\partial}{\partial P_\mu}\delta^d(P) \end{aligned}$$

Now, when $K$ operates on the correlation function, it produces three kinds of terms,

- All operators in $K$ acting purely on $\langle O_1(p_1)\dots O_n(p_n)\rangle$
- All operators in $K$ acting purely on $\delta^d(\sum_{i=1}^n p_i)$
- Operators acting on both $\langle O_1(p_1)\dots O_n(p_n)\rangle$ and $\delta^d(\sum_{i=1}^n p_i)$

We will consider the action of

$$k_j^\mu \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\alpha}} - 2k_j^\alpha \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\mu}}$$

then, they will operate like:

$$2\frac{\partial\delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\alpha}\underbrace{\left[k_j^\mu \frac{\partial}{\partial k_{j\alpha}} - k_j^\alpha \frac{\partial}{\partial k_{j\mu}}\right]}_{iL_{\mu\alpha}}\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle - 2\frac{\partial\delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\mu}k_j^\alpha\frac{\partial}{\partial k_j^\alpha}\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

First part vanishes due to rotational invariance. Therefore the extra terms would be:

$$\delta'_{\text{terms}} = -2\frac{\partial\delta^d(P)}{\partial P}\left[(n-1)d - \Delta + \sum_{j=1}^n k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\right]\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

The above vanishes from (6.1.3). Therefore the SCT ward identity is given as:

$$K_\mu\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle = -\left[2(\Delta - d)\frac{\partial}{\partial k^\mu} - 2k^\alpha\frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha}\right]\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle \tag{6.1.7}$$